Quantized Hall conductance for interacting systems

My version of Hastings and Michalakis work

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Overview

1. The quantum Hall effect
2. The problem
3. Tools
4. Parallel transport
5. Constant curvature

Adiabatic curvature

aka Berry's curvature

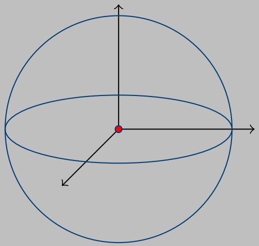
Bundle of projections, 3D, $|\mathbf{x}| \neq 0$

$$P(\mathbf{x}) = \frac{1}{2|\mathbf{x}|}\begin{pmatrix} |\mathbf{x}| + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & |\mathbf{x}| - x_3 \end{pmatrix}$$

- $P^2 = P, \quad P = P^*, \quad P_\perp = \mathbb{1} - P$

Curvature

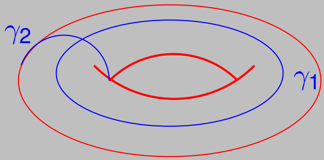
- $\Omega = i\ \big(P(\partial_j P)(\partial_k P)P\big)\ dx^j \wedge dx^k$
- $\omega = Tr\,\Omega = \dfrac{\mathbf{x} \cdot d\mathbf{x} \wedge d\mathbf{x}}{4|\mathbf{x}|^3}$
- Chern: $Ch(P) = \dfrac{1}{2\pi}\displaystyle\int \omega = 1$

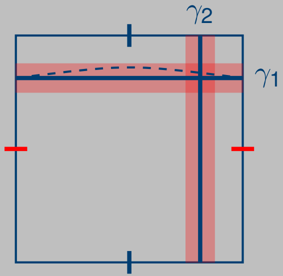


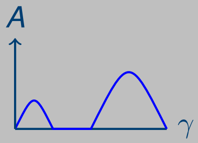
Magnetic fields fluxes and loops on torus

What data determine A uniquely

- Magnetic field: $B = dA$
- Fluxes: $\int_{\gamma_j} A = \phi_j$
- Gauge freedom: $A \mapsto A + d\Lambda$
- Free to deform $A(\gamma_j)$







Parameter space: Flux torus

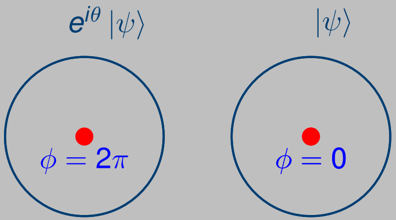
Aharonov-Bohm periods

Flux quantum $\frac{h}{e} = 2\pi$

Classically: ϕ gauge invarinat

Quantumly: $\phi \mod 2\pi$ gauge invariant

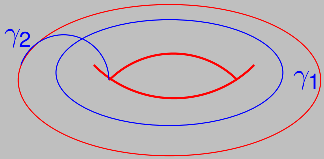
- $e^{-i\theta}(-i\partial_\theta)e^{i\theta} = -i\partial_\theta + 1$
- $\int 1 d\theta = 2\pi$



Quantum Hall effect

Flux controlled Hamiltonians

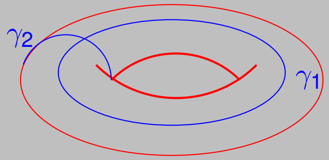
- $H(A) \geq 0$
- $H(A) \underbrace{P(A)}_{\text{projection}} = 0$
- emf$= \dot{\phi}_1 = -\int_{\gamma_1} \dot{A} \cdot dx$
- Current operator: $I \cdot d\phi = -dH$



Hall conductance=Adiabatic curvature$= \omega$

The problem

Quantized adiabatic curvature



Chern =Flux average conductance

$$\frac{1}{2\pi}\int_{\text{Flux torus}} \omega \in \mathbb{Z}$$

Quantized conductance=Quantized curvature

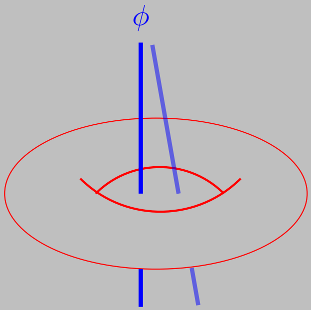
$$\omega(\phi_1,\phi_2) \xrightarrow[\text{large systems}]{?} \frac{n}{2\pi} d\phi_1 \wedge d\phi_2$$

Flat metric on flux torus

Curvature depends on connection

Conductance, in general, depends on flux

- Curvature: $\omega_P = i Tr\, \Omega_P$
- ϕ dependent unitaries $Q = U_\phi P U_\phi^*$
- $\omega_Q - \omega_P = i \underbrace{d\, Tr(PU^* dU) \approx 0}_{\text{in cohomology}}$
- Chern: $Ch(P) = \frac{1}{2\pi}\int \omega_P \in \mathbb{Z}$

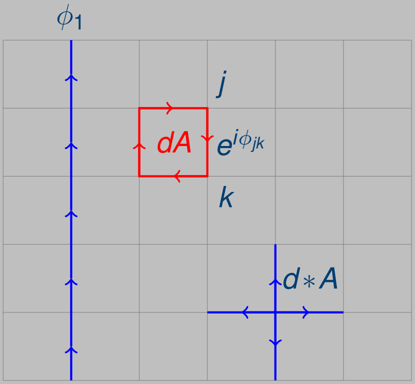


$$Ch(P) = Ch(Q)$$
$$\omega(P) \neq \omega(Q)$$

Setting

Lattice gauge theory

- Torus $\mathbb{Z}^2/(\ell\mathbb{Z})^2$
- Gauge field $e^{i\phi_{jk}}$
- Bonds
- B flux through plaquette
- $d * A$: Star



Setting

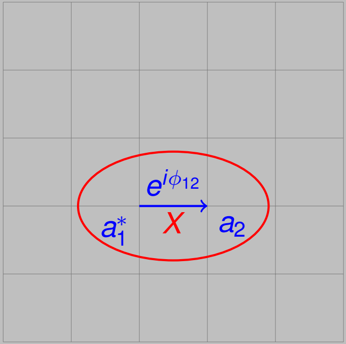
Large lattice system, Local Hamiltonian, Gauge invariance

- Large torus $\mathbb{Z}^2/(\ell\mathbb{Z})^2, \ell \gg 1$
- Particles on vertices
- Local Hamiltonian

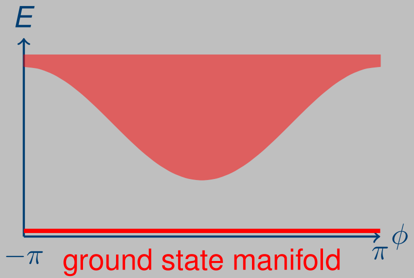
$$H = \sum_X \Phi(X), \quad |\Phi| \leq 10$$

- Gauge invariant

$$\Phi(\{1,2\}) = e^{i\phi_{12}} a_1^* a_2$$



Gap

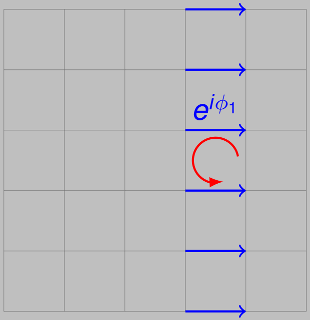


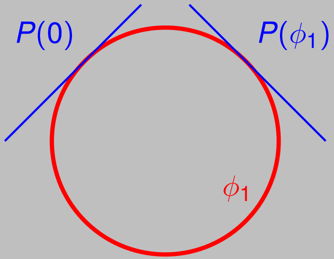
Finite gap, large system

Time and energy scale: gap=1

Controlling a flux while Keeping B fixed

Moving in ground state manifold





Changing ϕ: Not gauge transformation

Gauge invariant: $\oint A = \phi$

Tools

Generator of parallel transport, Lieb-Robinson

Heisenberg: $\tau_t(O_X) = e^{iHt} O_X e^{-iHt}$

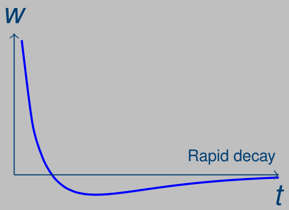
Generator of parallel transport

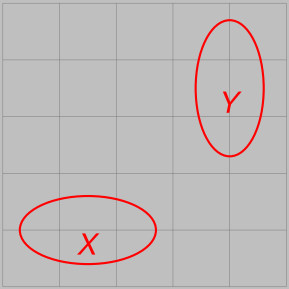
$$T = \int_{-\infty}^{\infty} w(t)\tau_t(dH)dt$$

Rapidly converging

Lieb-Robinson

$$\left\| \tau_t^H(O_X) - \tau_t^{H+\Phi(Y)}(O_X) \right\| \le e^{-|Y-X|+|t|}$$





A useful identity

Orthogonal Projections

- $P^2 = P, \quad P = P^*$
- $P_\perp = \mathbb{1} - P$

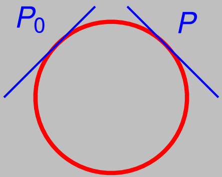
dP purely off diagonal

$$dP = \begin{pmatrix} 0 & P(dP)P_\perp \\ P_\perp(dP)\,P & 0 \end{pmatrix}$$

Motion in ground state manifold

T: Generator of parallel transport

- Intertwining: $PU = UP_0$
- Generator: $T = -i(dU)U^*$
- Commutator equation: $dP = -i[T, P]$
- Determines: $T = \begin{pmatrix} * & -iPdP\,P_\perp \\ iP_\perp dP\,P & * \end{pmatrix}$



Parallel transport: $PTP = 0$

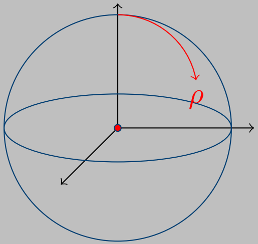
$$T = \begin{pmatrix} 0 & -idP \\ idP & * \end{pmatrix}$$

Curvature and Parallel transport

$PTP = 0$

Curvature: $\omega = iTr\,(PT \wedge TP)$

$$P(dP)\wedge(dP)P = -P[T,P]\wedge[T,P]P = PT\wedge TP$$



T dictates motion in P

$\rho = U\rho_0 U^* \implies d\rho = -i[T,\rho]$

Kato's choice of parallel transport

$PTP = P_\perp TP_\perp = 0$

Kato

$$T = \begin{pmatrix} 0 & -idP \\ idP & 0 \end{pmatrix}$$

$$dP \wedge dP = \Omega_P + \Omega_{P_\perp} = \underbrace{(PdP) \approx 0}_{\text{in cohomology}}$$

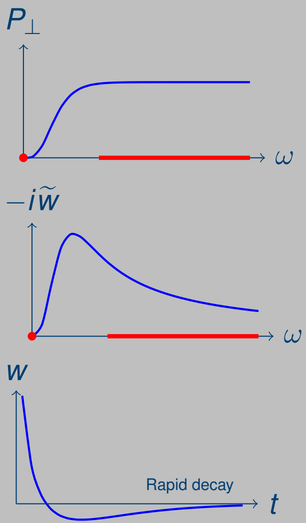
Hastings parallel transport

$T(H, dH)$ is self-adjoint, local and parallel transports

Pick $H\widetilde{w}(H) = iP_\perp$

$$T = \int w(t)\tau_t(dH)dt = \begin{pmatrix} 0 & -idP \\ idP & * \end{pmatrix}$$

$$\begin{aligned} PT &= P\int w(t)e^{iHt}(dH)e^{-iHt}dt \\ &= P\int w(t)(dH)e^{-iHt}dt \\ &= P(dH)\widetilde{w}(H) \\ &= -P(dP)\underbrace{H\widetilde{w}(H)}_{=iP_\perp} = -iPdP \end{aligned}$$



QM scales

c has no-business in NR QM

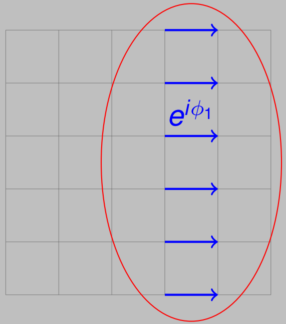
Fine structure constant

$$\alpha = \frac{e^2}{\hbar c}[esu] = \frac{e^2}{4\pi\epsilon_0 \hbar c}[mks] \approx \frac{1}{137}$$

velocity	$c\alpha$	$\frac{e^2}{\hbar}$	$c\alpha$
energy	$mc^2\alpha^2$	$\frac{me^4}{\hbar^2}$	O([ev])
time	$\frac{\hbar}{\text{energy}}$	$\frac{\hbar^3}{me^4}$	$O(10^{-17}[s])$
length	time $\times\, c\alpha$	$\frac{\hbar^2}{me^2}$	$O($ Å$)$

T is localized not too far from dH

LR + fast decay of w

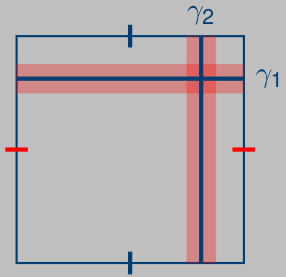


Localization of observables

- dH
- $T = \int w(t)\tau_t(dH)dt$

Localization: Adiabatic curvature

Intersection of current and emf loops



Adiabatic curvature, Ω, localized

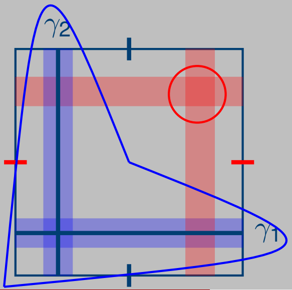
- $T\left(\partial_{\phi_1} H\right)$ localized near γ_2
- $T\left(\partial_{\phi_1} H\right) T\left(\partial_{\phi_2} H\right)$ localized near intersection

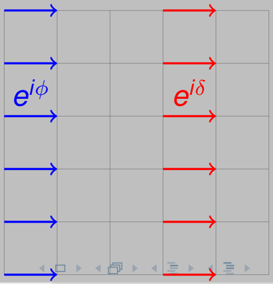
Constant curvature

Consequence of: Gauge invariance+Localization of T and Ω

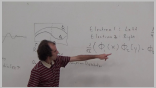
A function and its derivative are independent

- $T(\partial_\phi H)$ near blue strips, $T(\partial_\delta H)$ near red
- $\Omega_\delta = T(d_\delta H) \wedge T(d_\delta H)$ near red intersection ($\bigcirc$).
- $\Omega_\delta(\phi) \approx \Omega_\delta(\phi')$ since $T(d_\phi H)$ is far ($\heartsuit$)





Acknowledgments



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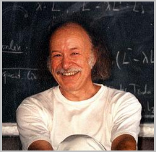

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