THE BOLTZMANN HALDANE EQUATION FOR ANYONS.

SOME MATHEMATICAL RESULTS.

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ABSTRACT

The talk will discuss a Boltzmann equation (BE) for anyons and kinetic gases with Haldane statistics in general.

The focus will be on mathematical results and aspects of their proofs.

PLAN OF THE TALK.

- **The Boltzmann Haldane equation (BHE)**
- **The space-homogeneous BHE**
- **The anyon BE in a slab**
- **The BHE in higher dimensions**
- **On the proofs**

The quantum BHE is a kinetic equations of Boltzmann type for quasi-particles with Haldane fractional exclusion statistics, such as anyons in two dimensions. This exclusion statistics interpolates between the fermion and boson quantum behaviours. The collisions in an elastic collision operator of Boltzmann type, are pair collisions preserving mass, first moments and energy.

For two particles with pre-collisional velocities (v, v_*) in $\mathbb{R}^d$, the velocities after collision are denoted (v', v'_*). The density functions in the corresponding variables are denoted f, f_* respectively f', f'_*.

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One collision operator Q_α for Haldane statistics was introduced by Bhaduri, Bhalerao, and Murthy (1996),

$$Q_\alpha(f) = \int_{I\!R^d \times S^{d-1}} B(v - v_*, \omega)\Big(f' f'_* F_\alpha(f) F_\alpha(f_*) - f f_* F_\alpha(f') F_\alpha(f'_*)\Big) dv_* d\omega.$$

The filling factor F_α, accounts for the influence of the final state of the pair, and is given by

$$F_\alpha(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha) f)^{1-\alpha}, \quad 0 < \alpha < 1,$$

i.e. a weighted geometric mean of the usual filling factors for bosons and fermions.

The limiting cases represent boson ($\alpha = 0$) and fermion statistics ($\alpha = 1$).
For $0 < \alpha < 1$, F_α is α-Hölder continuous ($|g(x) - g(y)| \leq C|x - y|^\alpha$) but lacks Lipschitz continuity ($|g(x) - g(y)| \leq C|x - y|$).

The filling factor $F_\alpha(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha) f)^{1-\alpha}$ requires strong L^1 limits, not the weak L^1 limits usually applied to the classical Boltzmann equation (BE).

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The Cauchy problem for the BHE on a torus $[0,1]^k$**,** $k \in \{1,2,3\}$ **is**

$$\partial_t f + \bar{\mathbf{v}} \cdot \nabla_x f = Q_\alpha(f), \quad \big(t, x, v = (v_1, v_2, v_3)\big) \in [0,T] \times [0,1]^k \times \mathbb{R}^3), \tag{1.1}$$

$$f(0,x,v) = f_0(x,v), \tag{1.2}$$

where

$\bar{\mathbf{v}} = (v_1)($ **resp.** $\bar{\mathbf{v}} = (v_1, v_2),$ **resp.** $\bar{\mathbf{v}} = v)$ **for** $k = 1$ $($ **resp.** $k = 2,$ **resp.** $k = 3).$

Notice that $0 \leq f \leq \frac{1}{\alpha}$ **is formally preserved by (1.1), since the gain (resp. loss) term vanishes for** $f = \frac{1}{\alpha}$ **(resp.** $f = 0$**), making the** Q_α**-term non-positive (non-negative).**

With polar angle $cos\theta = \omega \cdot \frac{v-v_*}{|v-v_*|}$**, the measurable kernel** $B(|v-v_*|, \omega)$ **will be written** $B(|v-v_*|, \theta)$**. We have been using the same cross sections as for the classical BE.**

The BHE for the limiting cases, representing boson (fermion) statistics, $\alpha = 0$ ($\alpha = 1$), was introduced by Nordheim in 1928.

In the space-dependent fermion case ($\alpha = 1$), general existence results were obtained by J. Dolbeault (1994), P.-L. Lions (1994) and X. Lu (2008).

That case is simplified by the cancellation of quartic terms,

while the density is kept bounded for fermions.

For $0 < \alpha < 1$, however, there is no cancellation in the collision term.

General existence results for the space-homogeneous isotropic boson ($\alpha = 0$) large data case have been obtained by X.Lu (2000, 2004), M. Escobedo-J.L. Velazquez (2015).

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Set

$$f^\sharp(t,x,v) = f(t, x + t\bar{v}, v), \quad (t,x,v) \in \mathbb{R}_+ \times [0,1]^k \times \mathbb{R}^3, \quad \bar{v} = (v_1, \cdots, v_k) \in \mathbb{R}^k. \tag{1.3}$$

f **is called a STRONG SOLUTION to the BHE (1.1) on the time interval** I**, if**

$$f^\# \in \mathcal{C}^1(I; L^1([0,1]^k \times \mathbb{R}^3)),$$

and

$$\frac{d}{dt} f^\sharp = \big(Q_\alpha(f)\big)^\sharp, \quad \textbf{on } I \times [0,1]^k \times \mathbb{R}^3.$$

The INITIAL VALUE $f_0(x,v)$ **is integrable, periodic in x, with range in** $[0, \frac{1}{\alpha}]$**.**

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THE SPACE-HOMOGENEOUS BHE.

Theorem 2.1 (Arkeryd 2010)

Consider the space-independent BHE with velocities in $I\!R^d$***,*** $d \geq 2$

and hard potential kernels with

$$0 < B(z, \theta) \leq C|z|^\beta |\sin\theta\cos\theta|^{d-1}, \quad (z, \theta) \in \mathbb{R}_+ \times [-\frac{\pi}{2}, \frac{\pi}{2}],$$

where $0 < \beta \leq 1$***,*** $d > 2$ ***or*** $0 < \beta < 1$***,*** $d = 2$***.***

Let the initial value f_0 ***have finite mass and energy.***

If

$$0 < f_0 \leq \frac{1}{\alpha} \qquad \textbf{\textit{and}} \qquad \text{ess sup}(1 + |v|^s) f_0 < \infty \quad \textbf{\textit{for}} \quad s = d - 1 + \beta,$$

then the initial value problem has a strong solution in the space of functions continuous from $t \geq 0$ ***into*** $L^1 \cap L^\infty$***, which conserves mass and energy,***

and

$$\text{ess} \sup_{v \in \mathbb{R}^d,\, t \leq t_0} |v|^{s'} f(t, v) \quad \textbf{\textit{is bounded, where}} \ s' = \min\{s, \frac{2\beta(d+1)+2}{d}\}.$$

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THE ANYON BOLTZMANN EQUATION IN A SLAB $0 \leq x \leq 1$.

The kernel $B(|v - v_*|, \theta)$ is here assumed measurable with

$$0 \leq B \leq B_0. \tag{2.1}$$

Head on and grazing collisions are damped,

here for convenience for some $\gamma, \gamma', c_B > 0$,

$$B(|v - v_*|, \theta) = 0 \quad \textbf{for} \quad |\cos\theta| < \gamma', \quad \textbf{for} \quad 1 - |\cos\theta| < \gamma', \quad \textbf{and for} \quad |v - v_*| < \gamma, \tag{2.2}$$

$$\int B(|v - v_*|, \theta) d\theta \geq c_B > 0 \quad \textbf{for } |v - v_*| \geq \gamma. \tag{2.3}$$

The initial value $f_0(x, v)$, periodic in x, is assumed to be a measurable function with values in $]0, \frac{1}{\alpha}]$, and such that

$$(1 + |v|^2) f_0 \in L^1([0,1] \times \mathbb{R}^2), \int \sup_{x \in [0,1]} f_0(x, v) dv = c_0 < \infty,$$

$$\inf_{x \in [0,1]} f_0(x, v) > 0, \quad \textbf{a.a. } v \in \mathbb{R}^2. \tag{2.4}$$

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With v_1 denoting the component of v in the slab x-direction, consider for functions periodic in x, the initial value problem

$$\partial_t f(t,x,v) + v_1 \partial_x f(t,x,v) = Q_\alpha(f)(t,x,v), \tag{2.5}$$

$$f(0,x,v) = f_0(x,v), \quad (t,x,v) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}^2. \tag{2.6}$$

Theorem 2.2 (Arkeryd, Nouri 2015)

Assume (2.1)-(2.2)-(2.3).

There exists a strong solution $f \in \mathcal{C}([0,\infty[; L^1([0,1]\times\mathbb{R}^2))$ ***of*** (2.5)-(2.6) ***with***

$$0 < f(t,\cdot,\cdot) < \frac{1}{\alpha} \quad \textit{for} \quad t > 0.$$

There is $t_m > 0$ ***such that for any*** $T > t_m$***, there is*** $\eta_T > 0$ ***so that***

$$f(t,\cdot,\cdot) \leq \frac{1}{\alpha} - \eta_T, \quad t \in [t_m, T].$$

The solution is unique and depends continuously in $\mathcal{C}([0,T]; L^1([0,1]\times\mathbb{R}^2))$ ***on the initial*** L^1***-datum.***

It conserves the mass, momentum and energy.

REMARKS.

The results are restricted to a slab $a \leq x \leq b$, the technical reason being that the proof uses the so called Bony functional not available in higher space dimensions.

That is also why our method only would give local in time results in higher dimensions, if the same assumptions as in the anyon case, are kept on the collision kernel B.

The results are, however, global in time under the supplementary assumption of very soft potential for large relative velocities, which will be next result.

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THE BOLTZMANN HALDANE EQUATION IN HIGHER DIMENSIONS.

The earlier assumptions on B are kept.

The initial datum $f_0(x,v)$ is assumed to be a measurable function periodic in $x \in \mathbb{R}^3$, with values in $[0, \frac{1}{\alpha}]$, and such that

$$(1+|v|^2)f_0(x,v) \in L^1([0,1]^k \times \mathbb{R}^3), \tag{2.7}$$

$$\int \sup_{x\in[0,1]^k} f_0(x,v)dv = c_0 > 0, \tag{2.8}$$

$$\int \sup_{x\in[0,1]^k} |v|^2 f_0(x,v)dv = \tilde{c}_0 > 0, \tag{2.9}$$

for any subset X of $\mathbb{R}^3$ of positive measure, $\int_X \inf_{x\in[0,1]^k} f_0(x,v)dv > 0. \quad (2.10)$

Theorem 2.3 [Arkeryd, Nouri 2019]

Under these assumptions on B and f_0 and the supplementary assumption of collision kernels very soft at large relative velocities,

$$B(u,\theta) = B_1(u)B_2(\theta) \quad \textbf{with} \quad |B_1(u)| \leq c|u|^{-3-\eta}, \tag{2.11}$$

for some $\eta > 0$ and B_2 bounded,

there exists a unique periodic in x, strong solution

$$f \in \mathcal{C}^1([0,\infty[; L^1([0,1]^k \times \mathbb{R}^3)),$$

of

$$\partial_t f + \bar{\mathbf{v}} \cdot \nabla_x f = Q_\alpha(f), \quad \big(t, x, v = (v_1, v_2, v_3)\big) \in [0,\infty[\times[0,1]^k \times \mathbb{R}^3,$$
$$f(0,x,v) = f_0(x,v),$$

for $k \in \{1,2,3\}$.

For $T > 0$ the solution continuously depends in $\mathcal{C}([0,T]; L^1([0,1]^k \times \mathbb{R}^3))$ on the initial L^1-datum. The solution f conserves mass, momentum and energy.

REMARK.

The restriction to a very soft kernel for large relative velocities, seems physically reasonable, since for very large energies the quasi-particle modelling anyhow has to be different.

ABOUT THE PROOFS.

The first step in the proofs, is to introduce an approximation scheme with added Lipschitz continuity.

This is done by replacing in the filling factor $(1-\alpha f)^{\alpha}$ **with**

$\frac{(1-\alpha f)}{(\frac{1}{j}+1-\alpha f)^{1-\alpha}}$**, which is Lipschitz for** $0 \leq f \leq \frac{1}{\alpha}$**,**

and introducing a velocity cut-off $\chi_j(v, v_*) = 1$ **if** $v^2 + v_*^2 \leq j^2$**, else** $= 0$.

The approximating sequence of equations with the new Lipschitz continuous filling factors, can be solved using contraction mappings, with solutions bounded by $\frac{1}{\alpha}$**.**

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Then the bulk of the proof is to show that the approximating solutions $(f_j)_{j\in\mathbb{N}}$ converge in norm in L^1. Our approach is;

1) in the space homogeneous case to extend a method from the classical BE,

2) for anyons in a slab, build the proof directly on an estimate of the Bony functional

$$\int |v-v_*|^2 Bff_*F(f')F(f'_*)dvdv_*d\theta dx,$$

which is only available in one space dimension.

3) Additional velocity dimensions beyond 2 complicate the arguments.

A delicate step in the proofs concerns the initial layer $0 \le t \le t_m$,

and how the solutions afterwards stay uniformly away from $\frac{1}{\alpha}$, i.e. for some $\epsilon > 0$,

the approximations f_k satisfy $f_k \le \frac{1}{\alpha} - \epsilon$.

I will end my talk with a discussion of the initial layer for the third theorem.

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Lemma 3.1 There is a time $t_m > 0$, such that for $V > 0$ and $T \geq t_m$ there is a distance μ_V to

$\frac{1}{\alpha}$ and a rate of decrease $b_V > 0$, such that for the approx. index $j \geq j_T$,

the approximate solutions satisfy

$$f_j^\sharp(t,\cdot,v) \leq \frac{1}{\alpha} - b_V t, \quad t \in [0, t_m], \quad |v| \leq V,$$
$$f_j^\sharp(t,\cdot,v) \leq \frac{1}{\alpha} - \mu_V, \quad t \in [t_m, T], \quad |v| \leq V.$$

Proof.

Recall that the collision kernel decreases as $|v - v_*|^{-3-\epsilon}$ at infinity.

Write the approximate collision terms as

$$Q_j(f_j) = F_j(f_j)\tilde{\nu}_j(f_j) - f_j \nu_j(f_j).$$

where

$$\tilde{\nu}_j(f_j) := \int B\chi_j f'_j f'_{*j} F_j(f_j) dv_* d\omega, \quad \nu_j(f_j) := \int B\chi_j f_{*j} F_j(f'_j) F_j(f'_{*j}) dv_* d\omega,$$

The approximate solutions are bounded by $\frac{1}{\alpha}$, and $Q(f,f) < 0$ for $f = \frac{1}{\alpha}$.

We shall first find a uniform bound on the set $\{v_*; f(t, x+t\bar{v}, v') > \frac{1}{2}$ or $f(t, x+t\bar{v}, v'_*) > \frac{1}{2}\}$.

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From the equation in exponential form

$$f_j^{\#}(t) = (f_0 \wedge (\frac{1}{\alpha} - \frac{1}{j}) \exp(-\int_0^t \nu_j^{\#} f_j(s) ds) + \int_0^t \exp(-\int_s^t \nu_j^{\#}(f_j(\tau)) d\tau)(Q_j^+(f_j, f_j))^{\#}(s) ds, \quad k \geq 1,$$

it follows by computation, that there is $c_1(T) > 0$ such that

$$M_j(T) = \int \sup_{(s,x)\in[0,T]\times[0,1]} f_j^{\sharp}(s,x,v) dv \leq c_1(T), \quad j \in \mathbb{N}^*, \tag{3.1}$$

and from (3.1) that $\nu_j(f_j)^{\sharp}$ and $\tilde{\nu}_j(f_j)^{\sharp}$ are bounded from above uniformly w.r. to j.

Denote for convenience f_j by f, and a bound from above of $(\tilde{\nu}_j(f)^{\sharp})_{j\in\mathbb{N}}$ by $c_2(T)$.

By definition,

$$\nu_j(f)^{\sharp}(t,x,v) = \int B\chi_j f(t, x + t\bar{v}, v_*) F_j(f(t, x + t\bar{v}, v')) F_j(f(t, x + t\bar{v}, v'_*)) dv_* d\omega.$$

one gets that

$$f(t, x + t\bar{v}, v_*) \geq c_3(T) f_0(x, v_*) > 0, \quad \textbf{a.a.} \ (t, x, v, v_*) \in [0,T] \times [0,1]^k \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{3.2}$$

for some constant $c_3(T) > 0$.

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It follows from a change of variables $v_* \to v'$ ($v_* \to v'_*$) with Jacobian bdd by $(\gamma')^{-4}$,

and (3.1) that

$$\int f(t,x+t\bar{v},v')dv_* <\frac{c_1(T)}{(\gamma')^4} \quad \textbf{and} \quad \int f(t,x+t\bar{v},v'_*)dv_* < \frac{c_1(T)}{(\gamma')^4}\,,$$
$$\textbf{a.a.} \;\; (t,x,v,\theta,\varphi) \in [0,T]\times[0,1]^k\times\mathbb{R}^3\times[0,2\pi]\times[0,\pi], \quad |\cos\theta|>\gamma'.$$

Consequently, the measure of the set

$$Z_{(j,t,x,v,\theta,\varphi)} := \{v_*; f(t,x+t\bar{v},v') > \frac{1}{2} \quad \textbf{or} \quad f(t,x+t\bar{v},v'_*) > \frac{1}{2}\} \tag{3.3}$$

is bounded by $\frac{2c_1(T)}{(\gamma')^4}$, uniformly for (x,v,θ,φ) with $|\cos\theta|>\gamma'$, $t\in[0,T]$, $j\in\mathbb{N}^*$.

We can now estimate quantitatively the rate of decrease of a solution when close to $\frac{1}{\alpha}$.

Take j_T so large that $\frac{4}{3}\pi j_T^3$ is at least twice this uniform bound of measure.

Notice that here j_T only depends on T, $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Denote by $\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)$ the ball of radius $\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}$.

It follows from (3.2) and the definition of j_T that

$$\nu_j(f)^\sharp(t,x,v)$$

$$\geq \int_{\mathcal{S}^2}\int_{\mathcal{B}\left(0,\left(\frac{3c_1(T)}{\pi(\gamma')^4}\right)^{\frac{1}{3}}\right)\cap Z^c_{(j,t,x,v,\theta,\varphi)}} B\chi_j f(t,x+t\bar{v},v_*)F_j(f(t,x+t\bar{v},v'))$$

$$F_j(f(t,x+t\bar{v},v'_*))dv_*d\omega$$

$$\geq c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\int_{\mathcal{S}^2}\int_{\mathcal{B}\left(0,\left(\frac{3c_1(T)}{\pi(\gamma')^4}\right)^{\frac{1}{3}}\right)\cap Z^c_{(j,t,x,v,\theta,\varphi)}} B(|v-v_*|,\theta)\inf_{x\in[0,1]^k} f_0(x,v_*)dv_*d\omega,$$

$$j\geq j_T,\quad \textbf{a.a. } (t,x,v)\in[0,T]\times[0,1]^k\times\{v\in\mathbb{R}^3;|v|<V\}.$$

Using a median property for the restriction of $v\to\inf_{x\in[0,1]^k} f_0(x,v)$ to the ball $\mathcal{B}\big(0,\big(\frac{3c_1(T)}{\pi(\gamma')^4}\big)^{\frac{1}{3}}\big)$, there are two disjoint sets Ω_1 and Ω_2 of equal volume, such that

$$\inf_{x\in[0,1]^k} f_0(x,v_1)\leq \inf_{x\in[0,1]^k} f_0(x,v_2)\qquad \textbf{for a.a. } v_1\in\Omega_1,\quad v_2\in\Omega_2.$$

Set $\Gamma = V + (\frac{3c_1(T)}{\pi(\gamma')^4})^{\frac{1}{3}}$ **and** $c_\Gamma = \int_{\mathcal{S}^2} \inf_{u\in[\gamma,\Gamma]} B(u,\theta)d\omega$**.**

For $j \geq j_T$ **and a.a.** $(n,t,x,v) \in \mathcal{S}^2 \times [0,T] \times [0,1]^k \times \{v \in \mathbb{R}^3; |v| < V\}$**,**

$$\int_{\mathcal{B}\left(0,\left(\frac{3c_1(T)}{\pi(\gamma')^4}\right)^{\frac{1}{3}}\right)\cap Z^c_{(j,t,x,v,\theta,\varphi)}} B(|v-v_*|,\theta) \inf_{x\in[0,1]^k} f_0(x,v_*)dv_*$$

$$\geq \inf_{u\in[\gamma,\Gamma]} B(u,\theta) \inf_{\overline{\Omega}\subset\mathcal{B}\left(0,\left(\frac{3c_1(T)}{\pi(\gamma')^4}\right)^{\frac{1}{3}}\right);\,|\overline{\Omega}|=\frac{2c_1(T)}{(\gamma')^4}} \int_{\overline{\Omega}} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_*$$

$$= \inf_{u\in[\gamma,\Gamma]} B(u,\theta) \int_{\Omega_1} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_*.$$

Hence, using the properties (2.3) of the cross-section, for $j \geq j_T$ **and a.a.** $(t,x,v) \in [0,T] \times [0,1]^k \times \{v \in \mathbb{R}^3; |v| < V\}$**,**

$$\begin{aligned} \nu_j(f)^\sharp(t,x,v) &\geq c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\Big(\int_{\mathcal{S}^2} \inf_{u\in[\gamma,\Gamma]} B(u,\theta)d\omega\Big)\int_{\Omega_1} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_* \\ &\geq c_\Gamma c_3(T)(1-\frac{\alpha}{2})^{2\alpha}\int_{\Omega_1} \inf_{x\in[0,1]^k} f_0(x,v_*)dv_*. \end{aligned} \quad (3.4)$$

which by assumption on f_0 **is a positive bound from below,** $c_4(T)$**, of**

$$\big(\nu_j(f)^\sharp(t,x,v)\big)_{j\geq j_T} \quad \textbf{on} \quad [0,T]\times[0,1]^k\times\{v\in\mathbb{R}^3; |v|<V\}.$$

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The functions defined on $]0,\frac{1}{\alpha}]$ **by** $x\to\frac{F_j(x)}{x}$ **are uniformly bounded from above w.r.t.** j **by**

$$x\to\alpha^{\alpha-1}\frac{(1-\alpha x)^{\alpha}}{x},$$

that is continuous and decreasing to zero at $x=\frac{1}{\alpha}$**.**

We can explicitly compute $\tilde{\mu}_V\in]0,\frac{1}{\alpha}[$**, so that**

$$x\in[\frac{1}{\alpha}-\tilde{\mu}_V,\frac{1}{\alpha}]\quad\textbf{implies}\quad\frac{F_j(x)}{x}\leq\frac{c_4(T)}{4c_2(T)},\quad j\geq j_T.$$

Consequently, for $j\geq j_T$ **and** $|v|<V$**,**

$$\begin{aligned} f^\sharp(t,x,v)\in[\frac{1}{\alpha}-\tilde{\mu}_V,\frac{1}{\alpha}]\quad\Rightarrow\quad D_tf^\sharp(t,x,v)&=\big(F_j(f^\sharp)\tilde{\nu}_j^\sharp-\frac{1}{2}f^\sharp\nu_j^\sharp\big)(t,x,v)-\frac{1}{2}f^\sharp\nu_j^\sharp(t,x,v)\\ &<-\frac{1}{2}f^\sharp\nu_j^\sharp(t,x,v)\\ &<-\frac{\alpha c_4(T)}{4}:=-b_V. \end{aligned}\tag{3.5}$$

This gives a maximum time $t_1=\frac{\tilde{\mu}_V}{b}$ **for** $f^\#$ **to reach** $\frac{1}{\alpha}-\tilde{\mu}_V$ **from an initial value**

$$f_0(x,v)\in]\frac{1}{\alpha}-\tilde{\mu}_V,\frac{1}{\alpha}].$$

On this time interval $D_tf^\sharp\leq -b_V$**.**

If $t_1\geq T$**, then at** $t=T$ **the value of** $f^\#$ **is bounded from above by** $\frac{1}{\alpha}-b_VT:=\frac{1}{\alpha}-\mu'_V$

with $0<\mu'\leq\tilde{\mu}_V$**.**

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$$t_m = \min\{t_1, T\}, \quad \mu_V = \min\{\tilde{\mu}_V, \mu'_V\}.$$

For any (x, v) **with** $|v| < V$**, if** $f(0, x, v) < \frac{1}{\alpha} - \mu_V$ **were to reach** $\frac{1}{\alpha} - \mu_V$ **at** (t, x, v) **with** $t \leq t_m$**, then** $D_t f^{\#}(t, x, v) \leq -b_V$**, which excludes such a possibility.**

It follows that

$$f^{\sharp}(t, x, v) \leq \frac{1}{\alpha} - \mu_V \text{ **for** } j \geq j_T, (t, x) \in [t_m, T] \times [0, 1]^k, \quad |v| < V,$$
$$f^{\sharp}(t, x, v) \leq \frac{1}{\alpha} - b_V t \quad \text{ **for** } j \geq j_T, (t, x) \in [0, t_m] \times [0, 1]^k, |v| < V.$$

The previous estimates leading to the definition of t_m **are independent of** $j \geq j_T$**.**

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