Stockholm University

Non-Hermitian Phases and Topology

What can we learn

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Table of Contents

Non-hermitian Hamiltonians

Hermitian Hamiltonians

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$$H = \mathrm{H}^{\dagger}$$

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Non-hermitian Hamiltonians

What happens if we remove the condition that the Hamiltonian must be hermitian?

- $E_i \in \mathbb{C}$
- $H \left|\psi_i\right\rangle = E_i \left|\psi_i\right\rangle$, the eigenvectors is not necessarily orthogonal.

The real world is hermitian, so why consider non-hermitian quantum mechanics?

Why consider something so silly?

- Effective description for open systems

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- A lot of recent experiments
  - Enginer systems?

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Why consider something so silly?

- Effective description for open systems
- A lot of recent experiments
  - Enginer systems?
- It is interesting from both a mathematics and physics perspective
  - What concepts changes when $H \neq \mathrm{H}^\dagger$
  - What is still the same?

Biorthogonal quantum mechanics

Before we start looking at topological phases, we need a a way around the problem that the eigenvectors don't form a basis

Left eigenvectors and right eigenvectors

- $H \left|\psi_{\mathrm{i}}^{\mathrm{R}}\right\rangle = E_i \left|\psi_{\mathrm{i}}^{\mathrm{R}}\right\rangle$
- $\mathrm{H}^{\dagger} \left|\psi_{\mathrm{i}}^{\mathrm{L}}\right\rangle = E_i^{*} \left|\psi_{\mathrm{i}}^{\mathrm{L}}\right\rangle \Leftrightarrow \left\langle\psi_{\mathrm{i}}^{\mathrm{L}}\right| H = \left\langle\psi_{\mathrm{i}}^{\mathrm{L}}\right| E_i$

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Biorthogonalization

$$\left\langle\psi_i^L \middle| \psi_j^R\right\rangle = \delta_{ij}$$

Can be used to diagonalize operators

$$M = \sum \lambda_i \left|\psi_{\mathrm{i}}^{\mathrm{R}}\right\rangle \left\langle\psi_{\mathrm{i}}^{\mathrm{L}}\right|$$

General about topological phases and restrictions

Symmetry protected topological phases

- Two gapped states of matter are in the same topological phase of matter if you can continously deform the Hamiltonian from one to the other without closing the energygap or break the symmetry.

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  - What the invariant is depends on the symmetries and dimension of the system

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- Phase-transitions when the energygap closes

Topological insulator

- Boring bulk
  - Just an insulator?
  - But they are topologically different

SPT

Topological insulator

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  - But they are topologically different
- Interesting things happens if we have a boundary
  - Bulk-boundary correspondence
  - "zero-modes at the boundary"

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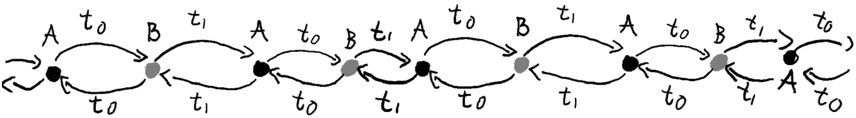
Some restrictions

- Fermionic systems
- Non-interacting systems (not necessary)
- Translational invariance (not necessary)

Hermitian SSH model



$$H = t_0 \sum_m \left(|m, B\rangle \langle m, A| + h.c. \right) + t_1 \sum_m \left(|m+1, A\rangle \langle m, B| + h.c. \right)$$

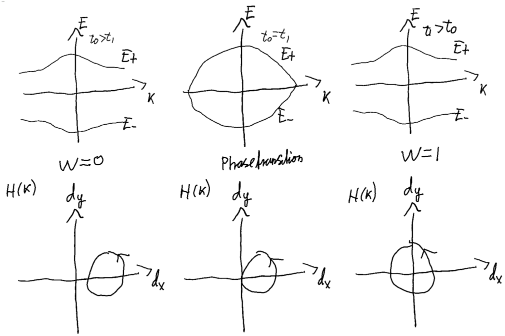
- Translation symmetry $\rightarrow$ momentum space (Blochs theorem)

$$H(k) = \begin{pmatrix} 0 & t_0 + t_1 e^{-ik} \\ t_0 + t_1 e^{ik} & 0 \end{pmatrix} = (t_0 + t_1)\cos(k)\sigma_x + t_1 \sin(k)\sigma_y, \quad k \in S^1$$

Hermitian SSH model 2

$$H(k) = \begin{pmatrix} 0 & t_0 + t_1 e^{-ik} \\ t_0 + t_1 e^{ik} & 0 \end{pmatrix} \Rightarrow E_\pm(k) = \pm|t_0 + t_1 e^{-ik}|$$

- Two energy bands with an energy gap $E_+ - E_-$
- This model has two different phases



AZ-symmetry classes 1

Definition (Particle-hole symmetry(PHS))

A Hamiltonian H has PHS if there exist a unitary $\mathcal{C}$ such that

$$\mathcal{C}\, H^*(k)\, \mathcal{C}^{-1} = -H(-k)$$

Definition (Chiral symmetry(CS))

A Hamiltonian H has CS if there exist a unitary Γ such that

$$\Gamma\, H(k)\, \Gamma^{-1} = -H(k)$$

Definition (Time reversal (TRS))

A Hamiltonian H has TRS if there exist a unitary $\mathcal{T}$ such that

$$\mathcal{T}\, H^*(k)\, \mathcal{T}^{-1} = +H(-k)$$

AZ-symmetry classes 2

Symmetry Class	Time reversal symmetry	Particle hole symmetry	Chiral symmetry
A	No	No	No
AIII	No	No	Yes
AI	Yes, $T^2 = 1$	No	No
BDI	Yes, $T^2 = 1$	Yes, $C^2 = 1$	Yes
D	No	Yes, $C^2 = 1$	No
DIII	Yes, $T^2 = -1$	Yes, $C^2 = 1$	Yes
AII	Yes, $T^2 = -1$	No	No
CII	Yes, $T^2 = -1$	Yes, $C^2 = -1$	Yes
C	No	Yes, $C^2 = -1$	No
CI	Yes, $T^2 = 1$	Yes, $C^2 = -1$	Yes

AZ-symmetry classes 3

Symmetry Class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AI	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$	
AII	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$
CII		$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
C			$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
CI				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Hermitian SSH model 3

The SSH model is in the BDI class

$$H(k) = \begin{pmatrix} 0 & t_0 + t_1 e^{-ik} \\ t_0 + t_1 e^{ik} & 0 \end{pmatrix}$$

- $\mathbb{Z}$ Topological number
- We have to extend the model to get higher winding numbers
- Chiral symmetry $\Gamma = \sigma_z$

Hermitian SSH model 4

More generally for a 1D system with chial symmetry $\Gamma = \sigma_z$

$$H(k) = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$$

- $w = \int \frac{dk}{2\pi} q^{-1} \partial_k q \in \mathbb{Z}$
- If the system has a boundary it will have w zero modes at the edge
  - Bulk-boundary correspondence
  - Everything is still fine with more bands

K-theory classification

Table of classification How did we get to the classification we saw earlier?

Main idea without any details

1. Kitaev 2009
2. K-theory give you information to differentiate vector bundles
3. Construct vector bundles of eigenstates over Brillouin torus.
4. Symmetry constraints

Some natural questions bout non-hermitian topological phases

- What do we mean by an energy-gap when the energies are complex?
- AZ-symmetry classes?
- Classification of topological phases
- Bulk-boundary correspondence?

$H \neq H^\dagger \Rightarrow H^* \neq H^T$

$$\mathcal{T}_\pm \, H^*(k) \, \mathcal{T}_\pm^{-1} = \pm \, H(-k), \quad \mathcal{T}_\pm \, \mathcal{T}_\pm{}^* = \pm 1$$
$$\mathcal{C}_\pm \, H^T(k) \, \mathcal{C}_\pm^{-1} = \pm \, H(-k), \quad \mathcal{C}_\pm \, \mathcal{C}_\pm{}^* = \pm 1$$
$$\Gamma \, H^\dagger(k) \, \Gamma^{-1} = - \, H(k), \quad \Gamma^2 = 1$$
$$\mathcal{S} \, H(k) \, \mathcal{S}^{-1} = - \, H(k), \quad \mathcal{S}^2 = 1$$

Symmetries in non-hermitian topological phases 2

What happens to time reversal?

- The hermitian Time reversal symmetry, $\mathcal{T}\, H^{*}(k)\, \mathcal{T}^{-1} = H(-k)$, has two non-hermitian counterparts
  1. $\mathcal{T}_{+}\, H^{*}(k)\, \mathcal{T}_{+}^{-1} = H(-k)$ called *TRS*
  2. $\mathcal{C}_{+}\, H^{T}(k)\, \mathcal{C}_{+}^{-1} = H(-k)$ called *TRS*†
- This gives us a lot more symmetry classes

Symmetries in non-hermitian topological phases 3

Symmetry class		TRS ($\mathcal{T}_+$)	PHS ($\mathcal{C}_-$)	TRS† ($\mathcal{C}_+$)	PHS† ($\mathcal{T}_-$)	CS (Γ)
Complex AZ	A	0	0	0	0	0
	AIII	0	0	0	0	1
Real AZ	AI	+1	0	0	0	0
	BDI	+1	+1	0	0	1
	D	0	+1	0	0	0
	DIII	−1	+1	0	0	1
	AII	−1	0	0	0	0
	CII	−1	−1	0	0	1
	C	0	−1	0	0	0
	CI	+1	−1	0	0	1
Real AZ†	AI†	0	0	+1	0	0
	BDI†	0	0	+1	+1	1
	D†	0	0	0	+1	0
	DIII†	0	0	−1	+1	1
	AII†	0	0	−1	0	0
	CII†	0	0	−1	−1	1
	C†	0	0	0	−1	0
	CI†	0	0	+1	−1	1

Figure: 10 Altland-Zirnbauer classes to 38 Bernard-LeClair classes. From Kawabata et al. 2019

Symmetries in non-hermitian topological phases 3

Complex AZ	A	0	0	0	0	0
	AIII	0	0	0	0	1
Re						0
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						0
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Real AZ†	AI†	0	0	+1	0	0
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Where are the other classes?

Symmetries in non-hermitian topological phases 3

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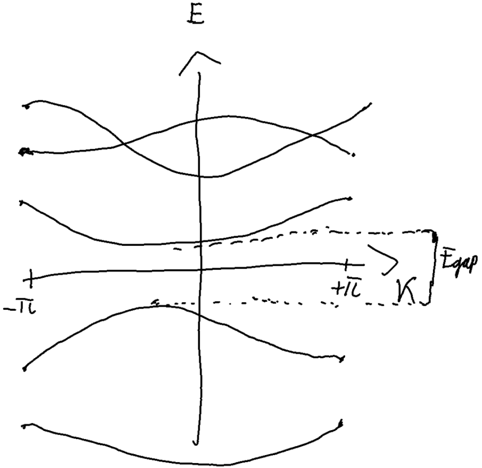
	BDI†	0	0	+1	+1	1
	D†	0	0	0	+1	0

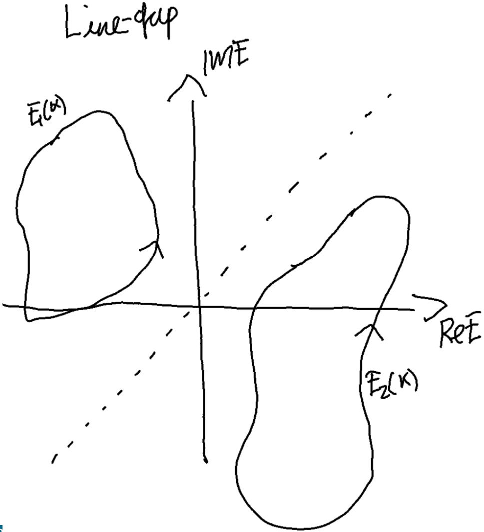
Where are the other classes?

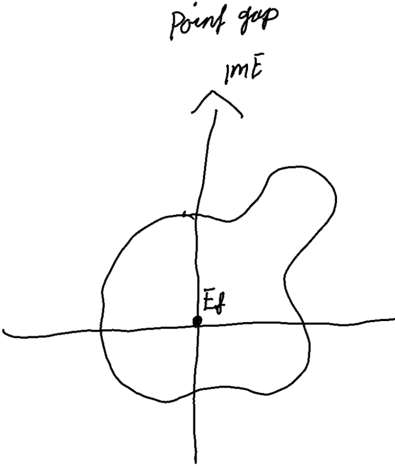
1. The chiral symmetry, Γ, also splits into two different kinds of symmetries. The other kind is not shown here is called Sub lattice symmetry.
2. It was long believed that there were actually 42 classes.Bernard and LeClair 2002
3. Kawabata et al. 2019 showed that there were some overcounting.

Hermitian energy gap



Line-gap and point-gap





General idea for classifying

Hermitization

- Map the system to a hermitian system in a "good" way
  - Then we can use the classification from hermitian systems directly
  - We have to distinguish between point-gap and line gap

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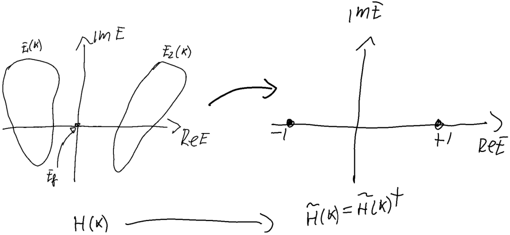
What do we mean by classifying nh topological phases?

- A littlebit unclear
- Don't change symmetry class
- Don't close gap
- Should not matter if you add more bands far away from the gap
- don't change "topological" properties

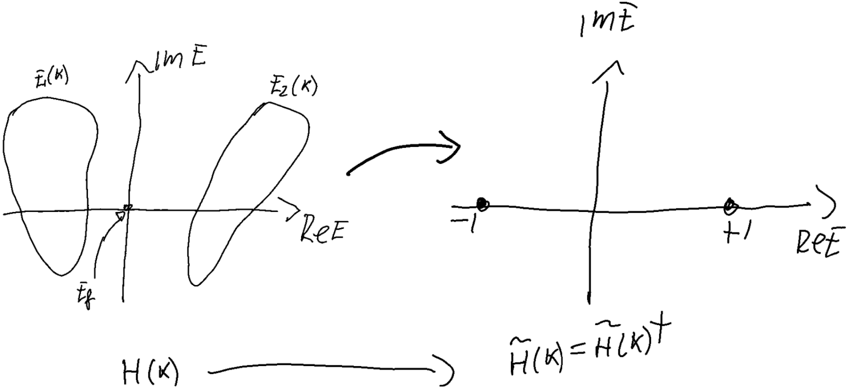
Theorem (Hermitian flattening for line gaps)

If a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ has a line gap in the real (imaginary) part of its complex spectrum [real (imaginary) gap], it can be continuously deformed into a Hermitian (an anti-Hermitian) matrix while keeping the line gap and its symmetry



Two theorems 2

- We send the system to a hermitian system with the same classification



Two theorems 3

Theorem (unitary flattening for point gaps)

If a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ has a point gap, it can be continuously deformed into a unitary matrix $U(\boldsymbol{k})$ while keeping the point gap and its symmetry.

Two theorems 3

Theorem (unitary flattening for point gaps)

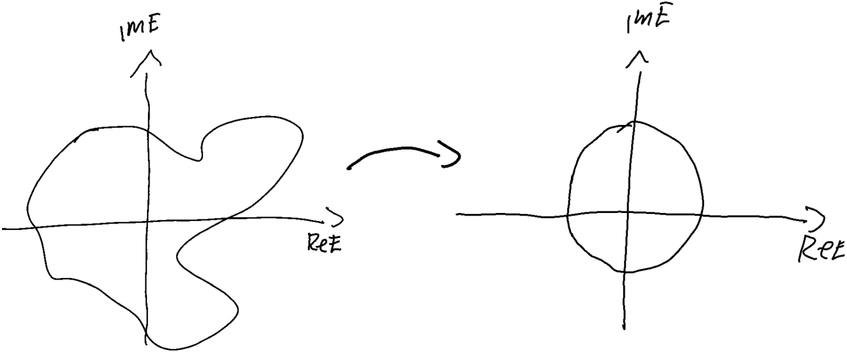
If a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ has a point gap, it can be continuously deformed into a unitary matrix $U(\boldsymbol{k})$ while keeping the point gap and its symmetry.

Remark

The classification for point gaps is therefor equivalent with the classification of unitary matrices with the right symmetries.
For the actual classification one uses the flattened Hermitian hamiltonian

$$\tilde{H}(\boldsymbol{k}) := \begin{pmatrix} 0 & U(\boldsymbol{k}) \\ U^\dagger(\boldsymbol{k}) & 0 \end{pmatrix}.$$

Two theorems 4



$$H(K) \longrightarrow U(K) \text{ - unitary}$$

$$\Big\updownarrow \; 1\text{-}1$$

$$\tilde{H}(K) = \begin{pmatrix} 0 & U(K) \\ U^{\dagger}(K) & 0 \end{pmatrix} = \tilde{H}^{\dagger}(K)$$

Two theorems 5

$$\tilde{H}(\boldsymbol{k}) := \begin{pmatrix} 0 & U(\boldsymbol{k}) \\ U^{\dagger}(\boldsymbol{k}) & 0 \end{pmatrix}.$$

The hermitization of the point gap has an extra chiral symmetry

- $\Sigma \tilde{H}(k) \Sigma = -\tilde{H}(k)$ with $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Two theorems 5

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The hermitization of the point gap has an extra chiral symmetry

- $\Sigma\tilde{H}(k)\Sigma = -\tilde{H}(k)$ with $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Hermitian systems in odd dimensions with a chiral symmetry has a well defined winding number.

$$w_{2n+1} = \frac{n!}{(2\pi i)^{n+1}(2n+1)!} \int_{BZ} tr(U^{-1}dU)^{2n+1} \in \mathbb{Z}$$

Two theorems 6

1D

$$w_1 = \int_{S^1} \frac{dk}{2\pi i} tr\left(H^{-1}\frac{dH}{dk}\right)$$

Two theorems 6

1D

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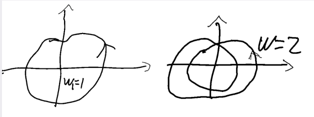
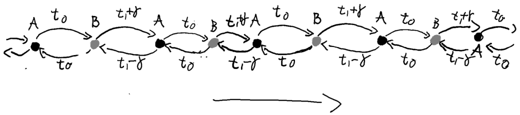
1D, 1 band

- For 1 band it tells us how the band winds around origo.

$$w_1 = \int_{S^1} \frac{dk}{2\pi i} \partial_k lnE(k)$$

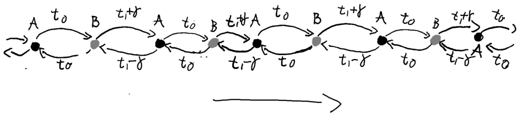


Non-hermitian SSH 1



Different hopping in different directions

$$
\begin{aligned}
H(k) = &\begin{pmatrix} 0 & t_0 + (t_1+\gamma)e^{-ik} \\ t_0 + (t_1-\gamma)e^{ik} & 0 \end{pmatrix} \\
= &(t_0+t_1)\cos(k)\sigma_x + (t_1\sin(k) - i\gamma)\sigma_y, \quad k \in S^1
\end{aligned}
$$



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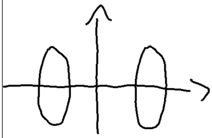
$$= (t_0+t_1)\cos(k)\sigma_x + (t_1\sin(k) - i\gamma)\sigma_y, \quad k \in S^1$$

- Gap closings at $E_+ = E_-$, that is, $t_1 = \pm(t_0 \pm \gamma)$
- Symmetry class, AI with SLS
  - $\mathbb{Z}$ for line gap
  - $\mathbb{Z} \oplus \mathbb{Z}$ for point gap

Non-hermitian SSH 2

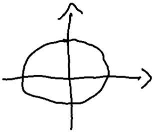
$-(t_0+\gamma) < t_1 < t_0-\gamma$

LG



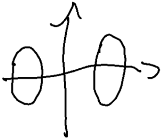
$t_0-\gamma < t_1 < t_0+\gamma$

PG



$t_0+\gamma < t_1$

LG



Why an extra $\mathbb{Z}$ for point gap?

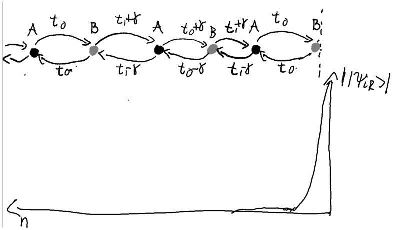
- Point gap hermitization:

$$\tilde{H}(\boldsymbol{k}) := \begin{pmatrix} 0 & H(\boldsymbol{k}) \\ H^\dagger(\boldsymbol{k}) & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & t_0 + (t_1+\gamma)e^{-ik} \\ 0 & 0 & t_0 + (t_1-\gamma)e^{ik} & 0 \\ 0 & t_0 + (t_1-\gamma)e^{-ik} & 0 & 0 \\ t_0 + (t_1+\gamma)e^{ik} & 0 & 0 & 0 \end{pmatrix}$$

- Two different chiral symmetries, $\Sigma = 1 \otimes \sigma_z$ and $S = \sigma_z \otimes 1$
- This gives us two different winding numbers

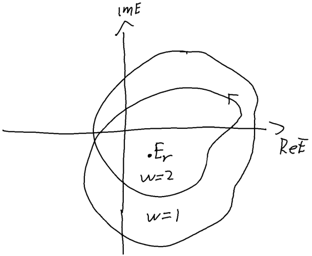
What is the non-hermitian skin effect?



A non-hermitian phenomena

- States pile up at the boundary for OBC
- Breaks Bulk-boundary correspondence
  - Topological invariant no longer tell you the number of edge modes
- The SSH model has the skin effect for both line gap and point gap

Topological origin of the skin effect in one slide



Point gap ⇔ skin effect

- Okuma et al. 2020
- Zhang, Yang, and Fang 2020
- Why do we have skin effect even when we have a line gap?
  - We have regions with different winding numbers!
- For obc the regions with non-zero winding number will be filled up with states
- These states are skin states

Other classification schemes

Some potential issues

- To simple to divide systems with an energy gap into point gap or line gap
- Does the classification make sense when talking about the reference energies that are relevant for the skin effect?

Other classification schemes

Some potential issues

- To simple to divide systems with an energy gap into point gap or line gap
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Homotopy based approaches

- There has been some recent work trying to study the topology of the bands more directly
  - Li and Mong 2021
  - Wojcik et al. 2020
  - Hu and Zhao 2021

Braids classifying 1D non-hermitian systems

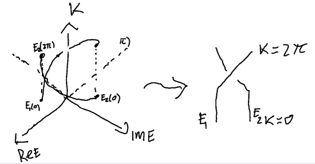
1D systems w.o. symmetry are just braids? Hu and Zhao 2021

- Set of n bands, $E_i(k) \in \mathbb{C}$
- Bands do not touch, $E_i(k) \neq E_j(k)$

Braids classifying 1D non-hermitian systems

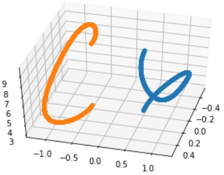
1D systems w.o. symmetry are just braids? Hu and Zhao 2021

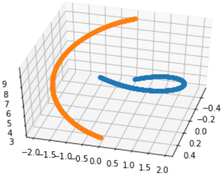
- Set of n bands, $E_i(k) \in \mathbb{C}$
- Bands do not touch, $E_i(k) \neq E_j(k)$
- Braids in $\mathbb{C} \times S^1$

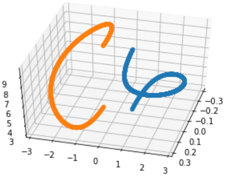


- Phase transition when bands "touch"
- Consequence: The classification is just equivalent to the classification of knots

SSH braids in three different phases







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- Only systems without symmetries
  - Unclear how to generalize to symmetry protected phases
- Sensitive to the number of bands
- Bands far away from the energy gap cannot cross each other

References

General references non-hermitian

1. Bergholtz, Budich, and Kunst 2021
2. Ashida, Gong, and Ueda 2020

Classification

1. Bernard and LeClair 2002
2. Gong et al. 2018
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Anomaly and qft picture of SPT and non-hermitian systems