∞-Ground states in the plane

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based on joint work with Peter Lindqvist

Archipelagic perspectives on mathematics, physics

and perceptible spectra of reality

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Outline

What will I talk about?

The ∞-eigenvalue equation:

$$\max\left\{\lambda - \frac{|\nabla u|}{u},\ \underbrace{\sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}}_{\Delta_\infty u}\right\} = 0$$

Arises as the Euler-Lagrange equation of the Rayleigh quotient

$$\frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$

The eigenvalue problem for the Laplacian

The problem is a highly nonlinear version of the eigenvalue problem for the Laplacian:

Minimizers of

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \quad \text{with } u = 0 \text{ on } \partial\Omega$$

satisfy

$$\Delta u + \lambda u = 0$$

where λ is the minimum.

Δ_∞: The infinity Laplacian

The infinity Laplacian

$$\Delta_\infty u := \langle \nabla u, D^2 u \nabla u \rangle = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

Solutions of

$$\Delta_\infty u = 0$$

are called ∞-harmonic functions.

Discovered by Gunnar Aronsson in the 60's in connection to Lipschitz extensions.

Dirichlet's principle

If u minimizes

$$\int_\Omega |\nabla u|^2, \quad \text{among functions coinciding on } \partial\Omega,$$

then u is harmonic and

$$\Delta u = 0 \quad \text{in } \Omega.$$

Δ_p – The p-Laplacian

If u_p ($p \geq 2$) minimizes

$$\int_\Omega |\nabla u|^p, \quad \text{among functions coinciding on } \partial\Omega,$$

then u_p is p-harmonic and

$$\Delta_p u_p = \operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0 \quad \text{in } \Omega$$

Δ_∞ via the p-Laplacian

As $p \to \infty$

$$\|\nabla u\|_{L^p(\Omega)} \to \|\nabla u\|_{L^\infty(\Omega)},$$

$$\Delta_p u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\Delta_\infty u \to \Delta_\infty u$$

Reasonable that $u_p \to u$ where u minimizes

$$\|\nabla u\|_{L^\infty(\Omega)}, \quad \text{among functions coinciding on } \partial\Omega$$

and solves $\Delta_\infty u = 0$ in Ω.

Aronsson 66. Bhattacharya, DiBenedetto and Manfredi 89.

Lipschitz extensions and Δ_∞

Let Ω be open and bounded, $g : \partial\Omega \to \mathbb{R}$ be Lipschitz and

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x,y\in\partial\Omega} \frac{|g(x)-g(y)|}{|x-y|}.$$

and

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \|\nabla g\|_{L^\infty(\partial\Omega)}$$

This was first proved by Aronsson for C^2 functions.

Properties of the ∞-Laplace equation

- Classical solutions is not a good notion of solutions. One should use *viscosity solutions*.
- Existence and uniqueness of solutions of the Dirichlet problem on bounded domains, Aronsson 67, Jensen 93.
- Solutions vs minimizers of $\|\nabla u\|$, Aronsson 67, Crandall-Evans-Gariepy 2001
- Differentiability in any dimension, Evans-Smart 2011
- $C^{1,\alpha}$-regularity in the plane, Savin-Evans 2008
- $C^2 + u_{xx}u_{yy} - u_{xy}^2 \neq 0 \Rightarrow C^\infty$, Aronsson 67.

Some ∞-harmonic functions

- Cones: $|x - x_0|$ for $x \neq x_0$
- Aronsson's function $x^{\frac{4}{3}} - y^{\frac{4}{3}}$. It is merely $C^{1,1/3}$ which is believed to be the optimal regularity of solutions.
- Any C^1 solution of the eikonal equation $|\nabla u| =$constant. Note that
$$\frac{1}{2}\Delta_\infty u = \langle \nabla u, \nabla |\nabla u|^2 \rangle.$$
- The distance function to a set is ∞-harmonic wherever it is C^1.

A warning!

The functional for Δ_∞ is not additive (as the one for Δ) so we ask that it is a minimizer on any subdomain, otherwise the set of minimizers can be large.

Example: A stadium minus a point with boundary data identically equal to one on the boundary of the stadium and 0 at the removed point.

The eigenvalue problem for finite p

Friedrichs's inequality (for $p > 1$ and Ω bounded):

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

for smooth functions vanishing on $\partial\Omega$.

The associated Rayleigh quotient:

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p}{\displaystyle\int_\Omega |u|^p}$$

The eigenvalue equation for finite p

The eigenvalue equation:
Minimizers of

$$\frac{\displaystyle\int_\Omega |\nabla u|^p}{\displaystyle\int_\Omega |u|^p}, \quad u \in W_0^{1,p}(\Omega)$$

satisfy

$$\Delta_p u + \lambda_p |u|^{p-2} u = 0$$

Terminology: A *ground state* is a minimizer of the Rayleigh quotient.

Known results

- Ground states equivalent to solutions.
- The ground state is unique up to a multiplicative constant. Thelin (balls), Sakaguchi (convex domains), Anane ($C^{2,\alpha}$-domains), Lindqvist (any).
- The ground state is log-concave (convex domains). (Sakaguchi generalized the Brascamp-Lieb Theorem)
- The first eigenvalue is isolated and there is a well-defined second eigenvalue.
- **Unknown** if the eigenvalues are countable ($p \neq 2$).

Uniqueness proof

Proof for $p = 2$: (Belloni-Kawohl, 2002, general p):
Consider two extremals with u, v with

$$\|u\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)} = 1.$$

Put

$$w = \left(\frac{u^2 + v^2}{2}\right)^{\frac{1}{2}}.$$

Then

$$\int_\Omega w^2 dx = 1.$$

Uniqueness cont

By the convexity of $x \mapsto x^2$

$$\int |\nabla w|^2 = \frac{1}{2}\int_\Omega (u^2+v^2)\left(\frac{u^2}{u^2+v^2}\frac{\nabla u}{u} + \frac{v^2}{u^2+v^2}\frac{\nabla v}{v}\right)^2$$
$$\leq \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2,$$

so that

$$\lambda_2 \leq \int_\Omega |\nabla w|^2 \leq \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2 \leq \lambda_2.$$

The strict convexity of $x \mapsto x^2$ forces $w = u = v$.

The ∞-eigenvalue problem

The Rayleigh quotient

$$\left(\inf_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}\right)^{\frac{1}{p}} \to \inf_u \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}} := \lambda_\infty$$

The eigenvalue equation:

$$\max\left\{\lambda_\infty - \frac{|\nabla u|}{u}, \underbrace{\sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}}_{\Delta_\infty u}\right\} = 0$$

First studied by Juutinen, Lindqvist, Manfredi.

Some known results

- $\lambda_p^{\frac{1}{p}} \to 1/R := \lambda_\infty$, where R is the radius of the largest ball that can be inscribed in Ω.
- There are many more minimizers than solutions.
- The first egenfunction is unique in a large class of domains including the ball and stadiums, Yu. Unknown in general.
- There is a counter example to uniqueness in a non-convex, dumbbell shaped domain with at least three linearly independent ground states. Hynd-Smart-Yu.

We call a non-negative solution of the equation (with $\lambda = 1/R$) a *ground state*.

Examples/non-examples

- In a ball (or a stadium) the ground state is the distance function.
- In the square, the distance function is *not* a ground state, however
$$v \leq u \leq d$$
where v is the ∞-harmonic function which is 1 at the midpoint and zero on the boundary.

Variational ground states in convex domains

To state our results, we assume

- $\Omega \subset \mathbb{R}^2$ is a convex polygon
- u is a limit of p-eigenvalues (*variational* ground state)

Then (Yu and Sakaguchi):

- $\ln u$ is concave
- u is ∞-harmonic outside a closed set Υ with zero measure
- the set of singular points of u is contained in Υ
- $|\nabla u| > \lambda_\infty u$ outside Υ
- on Υ we have "$|\nabla u| = \lambda_\infty u$"
- u attains its max exactly on the set where the distance function attains its max (the high ridge)

Streamlines for smooth ∞-harmonic functions

A streamline $\alpha = \alpha(t)$, is a solution of

$$\frac{d\alpha}{dt} = \nabla u(\alpha(t)).$$

If u is ∞-harmonic

$$\frac{d}{dt}|\nabla u(\alpha(t))|^2 = 2\Delta_\infty u(\alpha(t)) = 0.$$

Hence,

$$|\nabla u(\alpha(t))| = \text{the speed} = \text{constant}.$$

Requires second order derivatives! In general not true.

Main result

Let H be the *high ridge*. Streamlines starting at a corner are called *attracting* streamlines and streamlines starting at the point where $|\nabla u|$ attains a max between two corners is called a *median*. The convexity implies that the normal derivative $(= |\nabla u|)$ is monotone along the half-edges corner-median.

Theorem 1: Υ lies in set of attracting streamlines. Streamlines starting at other points cannot meet before joining an attracting streamlines or H. Along such streamlines, the speed $|\nabla u|$ is constant.

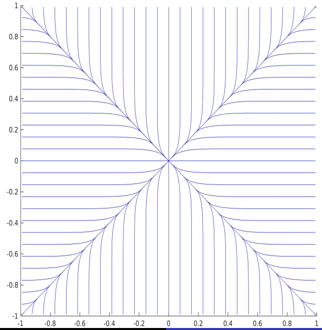
Consquences

Cor 1: The arc of a streamline from the boundary to an attracting streamline (or H) is either convex or concave. The length of this arc is $u/|\nabla u|$ at the hitting point. In particular, all arcs with the first meeting point in Υ have unit length.

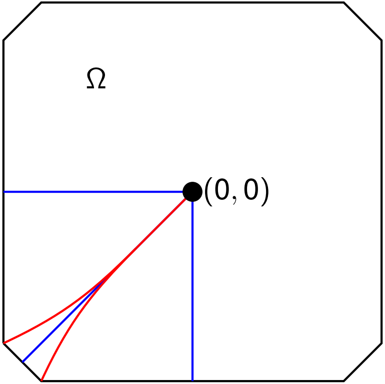
Cor 2: A median is a straight line segment until it joins some attracting streamline (or H).

Example 1: The square

Let Ω be a square and K the center. The attracting streamlines are the four half-diagonals. All streamlines meet at a diagonal, except the four segments along the coordinate axes.



Example 2: A truncated square



The speed

The speed along a streamline α

$$\left|\frac{d\alpha(t)}{dt}\right| = |\nabla u_\infty(\alpha(t))|$$

is non-decreasing outside Υ.

A fundamental inequality

Assume that $D \subset\subset \Omega \setminus \Upsilon$ and $p > 2$. Then

$$\boxed{\oint_{\partial D} |\nabla u|^{p-2} \langle \nabla u, \mathbf{n} \rangle \, ds \;\leq\; 0}$$

where **n** is the outer normal.

Idea: If for u_p, then the inequality follows from that

$$\oint_{\partial D} |\nabla u_p|^{p-2} \langle \nabla u, \mathbf{n} \rangle \, ds = \int_D \Delta_p u_p \, dx \leq 0,$$

since $\Delta_p u_p \leq 0$.

Consequence: The gradient is non-increasing along streamlines as long as they do not meet.

Consequence: the speed is sometimes non-increasing

Sketch of proof:
Assume that

- the points x_1 and x_2 are on the same level curve $u = a$,
- the points y_1 and y_2 both are on the higher level curve $u = b > a$,
- ascending streamlines join x_1 with y_1 and x_2 with y_2 but do not meet before.

Then

$$\|\nabla u\|_{\infty,\overline{y_1 y_2}} \leq \|\nabla u\|_{\infty,\overline{x_1 x_2}},$$

that is, the lower level curve has the larger gradient.

The main idea

The idea is to exploit that the speed is non-decreasing and that sometimes, when the streamlines do not meet, it is also non-increasing, so that it is constant along suitable arcs of streamlines.

Note also that since $\log u$ is concave, $|\nabla u|/u$ is decreasing along streamlines. So once a streamline hits the set Υ, the rest of the streamline also lies in Υ.

Things I don't know/thoughts:

- When is a ground state ∞-harmonic also across the attracting streamlines?
- Can we prove uniqueness using this in some simple cases?
- General (smooth) convex sets?
- We only use *log*-concavity, not that we have a limit of u_ps.

Thank you for listening!

Some references:

- Extension of functions satisfying Lipschitz conditions, Aronsson 66.
- On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Aronsson 67.
- Nonuniqueness of infinity ground states, 2013, Hynd-Smart-Yu.
- The ∞-Eigenvalue problem, 1999, Juutinen-Lindqvist-Manfredi
- Infinity-Harmonic Potentials and Their Streamlines, L.-Lindqvist 2019
- The Gradient Flow of Infinity-Harmonic Potentials, joint, L.-Lindqvist 2020
- On ∞-Ground States in the Plane, L.-Lindqvist 2021 (preprint)
- Some properties of the Ground state of the infinity Laplacian, 2007, Yu.