# Some Heuristics For Mathematical Crack Growth

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# What is a free boundary problem?

Given a boundary value problem (possibly some Euler-Lagrange equations)

$$\begin{aligned} &PDE(x, t, u, \nabla u, D^2 u) = f(x, t) && \text{for } x \in D \\ &u(x, t) = g(x, t) && \text{on } \partial D. \end{aligned}$$

Find and describe a given set (implicitly) defined by the solution, for instance a level set.

# An example: the Obstacle problem.

**The Dirichet problem:** Minimize

$$\int_D |\nabla u(x)|^2 dx \tag{1}$$

among all functions $u \in W^{1,2}(D)$ such that $u = g$ on $\partial D$.

**The Obstacle Problem:** Minimize (1) under the extra condition that

$$u(x) \geq \phi(x) \quad \text{for a.e. } x \in D,$$

where $\phi$ is a given obstacle (say i $C^2(D)$) such that $\phi(x) \leq g(x)$ on $\partial D$.

# Why is it important?

The "holy grail" of the theory is to prove that the free boundary $\partial\{x;\ u(x) = \phi(x)\}$ is $C^1$ a.e.

**Six reasons why:**

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2. For instance: hat making
3. Other *“big mathematicians”* are doing it.
4. What about Riemann?
5. Gives me access to grants.
6. You are naive!

# The problem of this talk.

We will study a play version of Griffith's crack growth: Given a domain (reference body) $D \subset \mathbb{R}^3$ and boundary data $\mathbf{g}(x, t)$ find a pair $(\mathbf{u}, \Omega(t))$, $\mathbf{u}$ is a function and $\Omega(t)$ a "$2-$dimensional set" such that:

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Don't worry, we will simplify this!

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- **Restrained ("no curvature") problem:** Assume that $\Omega$ is contained in a given set. Not as many problems!

# Our simplified problem.

Let $D = B_1^+(0) \in \mathbb{R}^3$ and $g(x, t)$ be given boundary data. Minimize, for each $t \geq 0$,

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- $u(x_1, x_2, 0, t) = 0$ for $(x_1, x_2, 0) \notin \Omega(t)$.

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In particular we would want to find some sort of "Euler-Lagrange equations" for how the speed of $\Gamma(t)$ is determined by $g(x, t)$ and $\Omega(t)$.

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- "Jumps" in the crack growth.

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4. Regularity makes it possible to attack the problem by classical calculus. (More or less.)
5. Discuss some issues/open questions.

# Growth of solutions - a heuristic proof.

## Theorem

*If* $(u, \Omega)$ *is a minimizer of*

$$\int_{D \setminus \Omega(t)} |\nabla u(x)|^2 dx + \mathcal{H}^2(\Omega(t))$$

*and* $0 \in \Gamma$ *then*

$$c \leq \frac{\|u\|_{L^2(B_r^+(0))}}{r^{\frac{n+1}{2}}} \sim \frac{\|\nabla u\|_{L^2(B_r^+(0))}}{r^{\frac{n-1}{2}}} \leq C.$$

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**Consequences:**

1. Better growth estimates around free boundary points.

$$u(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin\left(\frac{\varphi}{2}\right) + o(r^{1/2})$$

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**Theorem**

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**Consequences:** This is a regularity result bit it allows us to differentiate the solution i a rigorous way which implies that we can derive the growth equations rigorously in many cases.

# The equations for growth.

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4. *when* $c_{1/2} = \sqrt{\frac{2}{\pi}}$ *and* $\nu(x) = 0$

# Open questions.

- Behavior at singular points.

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