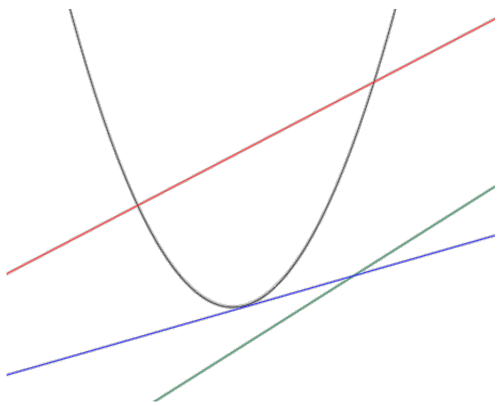
KTH Teknikvetenskap

Topics in Applied Algebraic Geometry
Lecture 4: Introduction to The Chow ring and the intersection product



1 Products

Consider the morphism

$$\times : Z_k(X) \times Z_l(Y) \to Z_{k+l}(X \times Y)$$

defined as $[Z] \times [V] = [Z \times V]$.

Proposition 1.1. *There are morphisms*

$$\times : A_k(X) \times A_l(Y) \to A_{k+l}(X \times Y)$$

Bevis. The morphism above is well defined over A_*. Assume $Z \in Z_k(X)$, $Z = \partial W$ and consider $[Z \times V]$, for $V \in Z_l(Y)$. Then $[Z \times V] = p^*(Z) \in B_*(X \times Y)$, for $p : X \times V \to X$. It follows that $[Z] \times [V] \sim 0$ for all V. □

Example 1.2. Consider the quadric hypersurface $Q = im(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3)$. It is irreducible and thus $A_2(Q) = \mathbb{Z}$. considering to the above map

$$A_1(\mathbb{P}^1) \times A_1(\mathbb{P}^1) \to A_2(\mathbb{P}^1 \times \mathbb{P}^1)$$

we confirm that $[Q] = [\mathbb{P}^1 \times \mathbb{P}^1]$. There also maps:

$$A_0(\mathbb{P}^1) \times A_1(\mathbb{P}^1) \to A_1(\mathbb{P}^1 \times \mathbb{P}^1), A_1(\mathbb{P}^1) \times A_0(\mathbb{P}^1) \to A_1(\mathbb{P}^1 \times \mathbb{P}^1)$$

showing that $[p_i \times \mathbb{P}^1] \sim [p_j \times \mathbb{P}^1]$ for any $p_i \neq p_j \in \mathbb{P}^1$ and $[\mathbb{P}^1 \times q_i] \sim [\mathbb{P}^1 \times q_j]$ for any $q_i \neq q_j \in \mathbb{P}^1$. Finally consider the map $A_0(\mathbb{P}^1) \times A_0(\mathbb{P}_1) \to A_0(Q)$ mapping the classes $([p], [q])$ to $[(p, q)]$.

More generally we can prove the following.

Definition 1.3. We say that a variety X has a *stratification* if it is a disjoint union of irreducible, locally closed, $X = \cup U_i$ such that $\overline{U_i} \cap U_j \neq \emptyset \Rightarrow U_j \subseteq \overline{U_i}$

If $U_i \cong \mathbb{C}^k$ for some k then we say that X har an *affine stratification*. Let $Y_i = \overline{U_i}$ then

$$U_i = Y_i \setminus \bigcup_{Y_j \subset Y_i} Y_j$$

Example 1.4. $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \ldots \subset \mathbb{P}^n$ is an affine stratification of $\mathbb{P}^n$ with $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$.

Proposition 1.5. *If X is an affine stratification then $A(X)$ is generated by the classes of the its closed strata.*

Example 1.6. Let us look again at Q. Let $H_0 \subset H_1 \subset \mathbb{P}^1, T_0 \subset T_1 \subset \mathbb{P}^1$ be the affine stratifications of the two factors. There is an induces affine stratification of Q given by $T_i \times H_j$. In fact: $U_{i,j} = T_i \times H_j \setminus (T_{i-1} \times H_j \cup T_i \times H_{j-1}) \cong \mathbb{C}^1 \times \mathbb{C}^j$. It follows that:

$$A_0(Q) =< [p, q] >, A_1 =< \pi_1^*(p) > \oplus < \pi_2^*(q) >, A_2(Q) =< [Q] >$$

Example 1.7.

$$A(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[H, T]/(H^{n+1}, T^{m+1}).$$

2 Intersection product

When the variety is smooth the Chow group can be endowed with a product that corresponds to geometric intersection. We will assume from now on that the varieties and subvarieties are always smooth.

Definition 2.1. Let A, B be two subvarieties of a variety X. We say that A and B *intersect trasversally* at a point $p \in A \cap B$ if $T_pA + T_pB = T_pX$.

We say that A and B are *generically transversal* if they intersect transversally at a generic point of each component of their intersection. This implies that their intersection has the expected codimension i.e each component has codimension $\operatorname{codim}(A) + \operatorname{codim}(B)$.

If A, B are generically transversally and irreducible then we define:

$$[A] \cdot [B] = [A \cap B].$$

This induces a map: $A_{n-k}(X) \times A_{n-l}(X) \to A_{n-(k+l)}(X)$.

One can generalize this notions to cycles in the following way. Two cycles $\alpha = \sum m_i A_i, \beta = \sum n_j B_j$ *intersect transversally* (resp. are generically transversal) if A_i, B_j intersect transversally (resp. are generically transversal). One can in this case define:

$$\alpha \cdot \beta = \sum_{i,j} m_i n_j [A_i \cap B_j].$$

THEOREM 2.2. *[FU, 11.4](Moving Lemma) Let X be a smooth (quasi) projective variety.*

1. *Let $[\alpha] \in A(X), B \in Z(X)$, then there exists $A \in [\alpha]$ which is generically transversal to B.*

2. *If $\alpha, \beta \in Z(X)$ are generically transversal then $[\alpha \cdot \beta] \in A(X)$ only depends on the class $[\alpha], [\beta] \in A(X)$.*

The moving lemma gives the existence of a well defined product on $A_*(X)$ for which the graded group $A^*(X)$ becomes an *associative commutative graded ring*.

3 Examples

Example 3.1.

$$A^*(\mathbb{P}^n) = \mathbb{Z}[x]/x^{n+1}.$$

$$A^*(\mathbb{P}^{n_1} \times \ldots \times (\mathbb{P}^{n_k}) = \mathbb{Z}[x_1, \ldots, x_k]/(x_1^{n_1}, \ldots, x_k^{n_k}).$$

Corollary 3.2. *Let $V, W \subset \mathbb{P}^n$ be two subvarieties of complementary dimension. Then*

$$V \cup W = \deg(V) \cdot \deg(W).$$

Corollary 3.3. *(Bezout's theorem) Let $V_1, \dots, V_k \subset \mathbb{P}^n$ be subvarieties of codimension c_i with $\sum c_i \leq n$ and intersecting generically transversally. Then*

$$\deg(V_1 \cup \ldots \cap V_k) = \Pi_i \deg(V_i)$$

Example 3.4. *The Veronese embedding.* Consider the map $v_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$, defines by

$$(x_0 : \ldots : x_n) \mapsto (\ldots, x^I, \ldots)$$

where x^I ranges over all degree d monomials of degree $d : x^d = \sum_0^n x_i^{d_i}, \sum d_i = d$. This map is an embedding (closed immersion). consider the n-dimensional subvariety $v_d(\mathbb{P}^n) \in A_n(\mathbb{P}^N)$, where $N = \binom{n+d}{d} - 1$. Then $v_d(\mathbb{P}^n) = \deg(v_d(\mathbb{P}^n))\mathcal{H}^{N-n}$, where $\mathcal{H}$ is a hyperplane in $\mathbb{P}^N$. Then $\deg(v_d(\mathbb{P}^n)) = v_d(\mathbb{P}^n)\mathcal{H}^n$ and because the map is one-to-one $\deg(v_d(\mathbb{P}^n)) = (v_d^{-1}\mathcal{H})^n$.

Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ general hyperplanes in $\mathbb{P}^{\binom{n+d}{d}-1}$ the $v_d^*(\mathcal{H}_i) = dH_i$ generic hypersurfaces of degree d and thus

$$\deg(v_d(\mathbb{P}^n)) = d^n.$$

For example $v_2(\mathbb{P}^1)$ is the conic $x_0x_2 = x_1^2$.

Example 3.5. There is always a map $\deg : A_0(X) \to \mathbb{Z}$ defined as $\deg(\sum n_i P_i) = \sum n_i$. For a surface $X \subset \mathbb{P}^n$ the map $A_1(X) \times A_1(X) \to A_0(X) \to \mathbb{Z}$ induces a bilinear form which gives $A_1(X)$ the structure of a lattice. For $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $[L_1] = [\{p\} \times \mathbb{P}^1], [L_2] = [\mathbb{P}^1 \times \{q\}]$ the intersection products are given by:

$$L_1 \cdot L_1 = L_1^2 = x_1^2 = 0, L_2 \cdot L_2 = L_2^2 = x_2^2 = 0, L_1 \cdot L_2 = [(p, q)] = 1.$$

The last example we introduce is the *dual hypersurface*. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Let $F(x_0, \dots, x_n)$ be the defining equations. The the equation

$$\sum_0^n \frac{\partial F}{\partial x_i}(p)x_i$$

defines a hyperplane $\mathbb{P}^{n-1} = T_p(X) \subset \mathbb{P}^n$.

Consider the dual projective space $\mathbb{P}^{n*}$ i.e.

$$(a_0x_0 + a_1x_1 + \ldots + a_nx_n = 0) \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n \Leftrightarrow [(a_0 : \ldots : a_n)] \in \mathbb{P}^{n*}$$

and consider the map $\gamma : X \to \mathbb{P}^{n*}$ defined by $\gamma(p) = T_p(X)$. The image $\gamma(X) = X^*$ is called the dual variety of the hypersurface X. It is a classical (not easy) fact that $(X^*)^* = X$ (biduality).

Proposition 3.6. *If $\deg(F) > 1$ then X^* is a hypersurface of degree $d(d-1)^{n-1}$.*

Bevis. The map γ is one-to-one (biduality) and thus

$$\deg(X^*) = [X^*] \cdot \Pi_1^{n-1}[H_0] = [X]\Pi_1^{n-1}[\gamma^{-1}(H_i)].$$

Let H_i be the coordinate hyperplane $x_i = 0$, then $\gamma^{-1}(H) = \{p \mid \frac{\partial F}{\partial x_i}(p) = 0\}$. i.e. a hypersurface of degree $d - 1$. It follows that $\deg(X^*) = d(d-1)^{n-1}$. □

Consider the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ then the dual is again a surface of degree 2 and in fact $X \cong X^*$. (Homework).

Litteraturförteckning

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