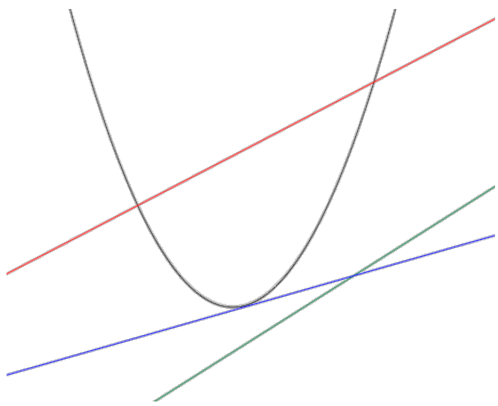
KTH Teknikvetenskap

Topics in Applied Algebraic Geometry
Lecture 4: Introduction to The Chow ring and the intersection product



1 Geometric reformulation

We have defined the Chow groups $A_k(X)$ given as equivalence classes of dimension k subvarieti-es. Two equivalent varieties $Z_1 \sim Z_2$ are said to be *rationally equivalent*, because the equivalence class relation can be reformulated via a rational curve: $\mathbb{P}^1$.

The product of (affine or projective) algebraic variety , $X_1 \times X_2$, is an algebraic variety defined by the *Segre embedding*:

$$s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}, s(x_0 : \ldots : x_n, y_0 : \ldots : y_m) = (\ldots, x_i y_j, \ldots).$$

The image $s(\mathbb{P}^n \times \mathbb{P}^m) = V(z_{ik}z_{jl} - z_{il}z_{jk})$ where $S(\mathbb{P}^{nm+n+m}) = \mathbb{C}[z_{ij}]_{0\leq i\leq n, 0\leq j\leq m}$.

Definition 1.1. Let $X_1 \subset \mathbb{P}^n, X_2 \subset \mathbb{P}^m$ be subvarieties or open subsets. The image $s(X_1 \times X_2)$ is the product variety. Notice that $\mathbb{C}^n \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$.

Facts 1.2. The following is true:

1. The product of irreducible subspaces is irreducible.

2. The graph of a morphism $f : X \to Y$ is a closed subset of $X \times Y$.

   Let $X \subset \mathbb{C}^m, Y \subset \mathbb{C}^n$ be affine varieties, then $f = (p_1, \ldots p_n) : X \to \mathbb{C}^n$. Recall that $\Gamma = \{(x, f(x) = (p_1(x), \ldots, p_m(x))\}$. Consider the product map:

   $$f \times id : X \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$$

   Then $\Gamma = (f \times id)^{-1}(\Delta)$ where Δ is the diagonal, which is the closed subset $V(z_{ij})_{i\leq j}$.

3. The projections $p : X \times Y \to X, q : X \times Y \to Y$ are morphisms.

4. The projection of a closed subvariety is a closed subvariety.

   Consider $p : \mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$, and $Z = V(f_i), f_i \in \mathbb{C}[y_0 : \ldots : y_n, x_1, \ldots, x_m]$(homogeneous in the y_i), $\deg(f_i) = d_i$. Let $x_* \in \mathbb{C}^m$, then $x_* \in p(Z)$ if and only if $\emptyset \neq V(f_i(y, x_*))$. By proj. Nullstellensatz this is equivalent to $(y_0, \ldots, y_n) \not\subset \sqrt{(f_i(y, x_*))}$ i.e. $(y_0, \ldots, y_n)^t \not\subset (f_i(y, x_*))$ for all t. If we denote by $H_t = \{x_* \in \mathbb{C}^m \; s.t. \; (y_0, \ldots, y_n)^t \not\subset (f_i(y, x_*))\}$. Then $p(Z) = \cap H_t$. It is then enough to show that H_t is closed. Let x_* such that $(y_0, \ldots, y_n)^t \subset (f_i(y, x_*))$ and consider the linear map:

   $$\oplus \mathbb{C}[y_0 : \ldots : y_n]_{t-d_i} \to \mathbb{C}[y_0 : \ldots : y_n]_t \; (g_i(y)) \to \sum g_i(y) f_i(y, x_*)$$

   This map is surjective when a maximal minor is not zero. So the complement of $p(Z)$ is an open condintion.

5. The product of two affine (resp. projective) varieties is affine (resp. projective).

Let X be an algebraic variety and let $V \subset X \times \mathbb{P}^1$ be a $(k+1)$-dimensional subvariety such that $p : V \to \mathbb{P}^1$ is dominant. Let $q : V \to X$ then $q(p^{-1}(0)), q(p^{-1}(\infty)) \in Z_k(X)$ are rationally equivalent. We will denote $q(p^{-1}(0)) - q(p^{-1}(\infty)) = \partial V$ and we say that $Z \sim 0$ if there exists $V_1, \ldots, V_k \in A_{k+1}(X \times \mathbb{P}^1)$ with dominant $p_i : V_i \to \mathbb{P}^1$ such that

$$Z = \sum \partial V_i$$

This is the same equivalence relation used in the previous lecture.

The Chow group of X is $A_*(X) = \sum_k A_k(X)$.

2 Push forward and pull back.

Let us now see when we can define a *pull back* and *push forward* operation on cycles. Given an open $U \subset V$ it is not always the case that we can extend subvarieties (push forward), e.g. $\mathbb{C}^n \subset \mathbb{P}^n$ or $\{pt\} \in \mathbb{C}^n$.

2.1 Push forward.

The category of maps $f : X \to Y$ for which we can define $f_* : A_k(X) \to A_k(Y)$ are the *proper map* (universally closed).

Definition 2.1. A map $f : X \to Y$ is proper if $p^{-1}(Z)$ is compact for all compact $Z \subset Y$. equivalently if for any $g : Z \to Y$ the map f' :

$$\begin{array}{cccccc} & X \times_Y Z & \to & X & \\ f' & \downarrow & & \downarrow & f \\ & Z & \to & Y & \end{array}$$

maps closed subsets to closed subsets.

Facts 2.2. The following is true.

1. Let $f : X \to Y$ be a dominant morphism of varieties of the same dimension. Then $K(X)$ is a finite dimensional vector space over $K(Y)$ whose dimension is called *degree* of f and is equal to the number of points in a general fiber.

2. if f is proper then every fiber consists of the same number of points (counted with multiplicity).

Let $f : X \to Y$ be a proper morphism then we can define $f_* : Z_k(X) \to Z_k(Y)$ as

$$f_*([Z]) = [K(Z) : K(f(Z))][f(Z)] \text{ if } \dim(f(Z)) = \dim(Z) \text{ and 0 otherwise .}$$

If $r \in K(X)^*$ then $f_*([r]_X) = [N(r)]_Y$ if $\dim(Y) = \dim(X)$ and 0 otherwise. The norm $N(r)$ is defined as follows. The rational function $N(r) \in K(Y)$ is the determinant of the endomorphism on the $K(Y)$-vector space $K(X)$ given by the multiplication by r. It follows that:

Proposition 2.3 (Hmw)**.** *Let $f : X \to Y$ be a proper morphism, then there are well defined push-forward maps: $f_* : A_k(X) \to A_k(Y)$.*

Consider $r \in K(V)^*$ for a subvariety of X. The rational function $r : V \to \mathbb{P}^1$ is dominant over an open $U \subset V$. Let $W = \Gamma_r \subset V \times \mathbb{P}^1$ and $[r] = \partial\Gamma$. Viceversa consider $W \subset X \times \mathbb{P}^1$ such that $\partial W \sim 0$ then $p : W \to \mathbb{P}^1$ gives rise to a rational function $r \in K(W)^*$ and $\partial W = q_*[r]$.

Definition 2.4. Let $f : X \to Y$ be a morphism. For any point $p \in X$ the local ring $\mathcal{O}_y$ is an $\mathcal{O}_{f(y)}$-module. The morphism f is *flat* if $\mathcal{O}_y$ is flat for each point.

All fibers of a flat morphism have the same dimension, called the *relative dimension* of f. If f is a flat morphism of relative dimension r then we can define a pull back map:

$$f^* : Z_k(Y) \to Z_{k+r}(X)$$

as $f^*([Z]) = [f^{-1}(Z)]$. The map is well defined on A_*. Let $Z = \partial W$ for $W \subset Y \times \mathbb{P}^1$ mapping dominantly $g : W \to \mathbb{P}^1$. Let $V = (f \times id)^{-1}(Z) \subset X \times \mathbb{P}^1$, and let $h : V \to \mathbb{P}^1$. Denote $p : X \times \mathbb{P}^1 \to X, q : Y \times \mathbb{P}^1 \to Y$. then

$$f^*Z = f^*q_*([g^{-1}(0)]-[g^{-1}(\infty)]) = p_*(f\times id)^*([g^{-1}(0)]-[g^{-1}(\infty)]) = p_*([h^{-1}(0)]-[h^{-1}(\infty)]) = \partial V$$

Litteraturförteckning

[FU] W. Fulton, Intersection Theory, Springer 1984

[FUintro] Introduction to Intersection Theory in Algebraic Geometry, AMS regional conference series in mathematics, no 54, 1984.

[Ha] R. Hartshorne, Algebraic Geometry, Springer 1977.