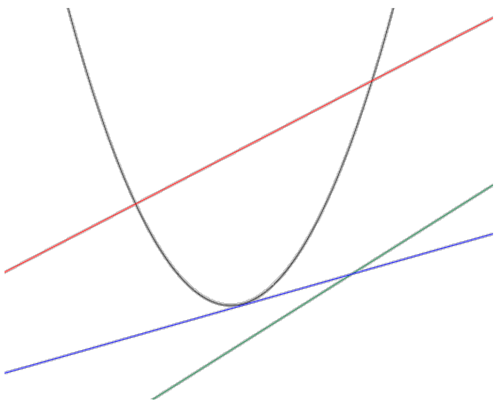
KTH Teknikvetenskap

Topics in Applied Algebraic Geometry
Lecture 3: Introduction to The Chow group of a variety

We will introduce the basics of intersection theory and the Chow group of an algebraic variety. We start by stating the basic notations and facts.



1 Notation

We refer to [Ha, chap I] for more details. $X \subseteq \mathbb{C}^n$ denotes an algebraic variety (reduced and irreducible scheme), i.e. the variety defined by a prime ideal $I(X) \subset \mathbb{C}[x_1, \ldots, x_n]$. The quotient ring $A(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X)$ is the *coordinate ring* A projective variety $Y \subseteq \mathbb{P}^n$ is defined by a homogeneous prime ideal of the graded ring $S = \mathbb{C}[x_0, \ldots, x_n]$ and its coordinate ring is $S(Y) = \mathbb{C}[x_0, \ldots, x_n]/I(Y)$. Recall that a projective variety Y is covered by affine patches $U = \cup Y_i$ for $i = 0, \ldots n$, and $A(Y_i) =\cong S(Y)_{(x_i)}$.

A function $f : Y \to \mathbb{C}$ is *regular* at a point $p \in Y$ if there exist an open U containing p and two polynomials $g, h \in S$ of the same degree such that $f = g/h$ on U. A function is regular on Y if it is regular at every point. We will denote by $\mathcal{O}(Y)$ the ring of regular functions on Y.

Let $p \in Y$. A pair (U, f) where f is a regular function at p and U is the corresponding open set is called a *germ* of f near p. Two germs $(U, f) = (V, g)$ are equal if $f = g$ on $U \cap V$ (eq. relation). The local ring $\mathcal{O}_p$ is the the ring of equivalence classes of germs, its maximal ideal m_p is the ideal of germs vanishing at p. More generally, given $(U, f), (V, g)$ where f (resp g) is regular on U (resp V) one defines the equivalence relation $(U, f) = (V, g)$ if $f = g$ on $U \cap V$ and the ring of eq. classes $K(Y)$ is called the *function field* of Y and its elements are *rational functions*.

Recall the following facts [Ha, Thm 3.2, Thm 3.4]

- If $X \subseteq \mathbb{C}^n$ then $A(X) = \mathcal{O}(X), \mathcal{O}_p = A(X)_{(m_p)}$ and $K(X) = Q(A(X))$.
- $Y \subseteq \mathbb{P}^n$ then $\mathcal{O}(Y) = \mathbb{C}, \mathcal{O}_p = S(X)_{(m_p)}$ and $K(X) = S(X)_{((0))}$

Example 1.1.

- $K(\mathbb{P}^n) = \{f/g, f, g \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of the same degree $\}$.
- ....

Recall that a variety $X \subseteq \mathbb{C}^n$ defined by $I(X) = (f_1, \ldots, f_k)$ is *smooth* (non singular) at $p \in X$ if the Jacobian matrix $[\frac{\partial f_i}{\partial x_j}(p)]$ has maximal rank. Equivalently if the local ring $\mathcal{O}_p$ is regular, i.e. $\dim(m_p/m_p^2) = \dim(X)$.

Let $Y \subseteq \mathbb{P}^n$, a *subvariety* $Z \subseteq Y$ is a variety defined by a prime ideal of the ring $S(Y)$. The *local ring* of Z on Y is the ring of equivalence classes of (U, f) where f is reg U and $U \cap Z \neq \emptyset$ and it is denoted by $\mathcal{O}_{Z,Y}$. If Z is a point p it coincides with $\mathcal{O}_p$. Note also that if X is affine then the defining ideal $I_{Z,Y} = \{f \in \mathcal{O}_{Z,X} s.t. f \in I(X)\}$.

Given an affine algebraic variety $X \subseteq \mathbb{C}^n$ its *dimension* is defined as

- The maximal length och chains of subvarities:

$$\emptyset = V_0 \subset V_1 \subset \ldots \subset V_k = X$$

- Equivalentily $\dim(X) = \dim(A(X))$.

The *codimension* of a subvariety $Z \subseteq Y$ is:

- The maximal length och chains of subvarities:

$$Z = V_0 \subset V_1 \subset \ldots \subset V_k = X$$

- Equivalentily $codim_X(Z) = \dim_{\mathbb{C}}(\mathcal{O}_{Z,Y})$.

2 Weil divisors

Let $Z \subset Y$ be a subvariety of codimension 1 then locally $\dim_{\mathbb{C}}(\mathcal{O}_{Z,Y_i})$ is one and thus $I_{Z,Y_i} = (f_i)$, where $f_i \in Q(\mathcal{O}_{Z,Y_i})$. Moreover $Q(\mathcal{O}_{Z,Y_i}) = K(Y_i)$.

Let us recall some important properties of local one dimensional rings. Let A be a 1-dimensional Noetherian local ring ($\dim(m/m^2) = 1$) and $0 \neq a \in A$. Then the ring A/aA has a finite length (maximal length of a decreasing chain of ideals).

Definition 2.1. We define the order of a as $ord(a) = ord(A/aA) = length(A/aA)$. The *order finction* is

$$ord : Q(A)^* \to \mathbb{Z}, ord(a/b) = ord(a) - ord(b).$$

Given $X \subseteq \mathbb{C}^n$ and $Z \subset X$ of codimension one we have a well defined order funtion:

$$\begin{aligned} ord_Z : Q(\mathcal{O}_{Z,X})^* \to \mathbb{Z},\ ord(f/g) = ord(f) - ord(g) \\ = \text{length}(\mathcal{O}_{Z,X}/f\mathcal{O}_{Z,X}) - \text{length}(\mathcal{O}_{Z,X}/g\mathcal{O}_{Z,X}). \end{aligned}$$

In fact of any rational function $r \in K(X)$ we can define the cycle of r :

$$[r] = \sum_{\text{cod 1 subv}} ord_Z(r)Z.$$

Example 2.2. Consider a plane curve $C \in \mathbb{C}^2$ and two other curved defined by the polynomials F, G. the cycle of the rational:

$$[F/G] = \sum_{p \in C} ord_p(F/G)p = \sum_p (mult_p(C, F) - mult_p(C, G))$$

where $mult_p(C, F) = \text{length}(\mathcal{O}_p/F\mathcal{O}_p)$.

3 Chow groups

Let X be an algebraic variety. We denote by $Z_k(X)$ the *group of k-cycles*, i.e. the abelian group freely generated by the subvarieties of dimenion k.

For any $(k+1)$-dimensional $W \subset X$ and any $r \in K(W)$ we can define the cycle

$$[r] = \sum_{V \in Z_k(X)} ord(r)V$$

A cycle $\alpha \in Z_k(X)$ is s said to be equivalent to 0, $\alpha \sim 0$, if there are finitely many $(k+1)$-dimensional subvarieties $W_1, \dots, W_s$ and rational function r_{W_i} such that

$$\alpha = \sum_1^s [r_{W_i}]$$

The cycle equivalent to 0 for a subgroup denoted by $B_k(X)$. The k-th *Chow group* of X is defined as:

$$A_k(X) = Z_k(X)/B_k(X).$$

Example 3.1. 1. Any codimension one subvariety $Y \subset \mathbb{C}^n$ is the zero of a polynomial. It follows that $A_{n-1}(\mathbb{C}^n) = 0$. Similarly for any point $p \in \mathbb{C}^n$ and considering $p \in L$ where L is a line through p on sees that $p \sim 0$ so that $A_0(\mathbb{C}^n) = 0$. Clearly $A_n(\mathbb{C}^n) = [\mathbb{C}^n] = \mathbb{Z}$. ($[\mathbb{C}^n]$ must be a basis element and $t \to t[\mathbb{C}^n]$ is an isomorphism. This is in fact the case for any variety.

2. Consider now $\mathbb{P}^n$. We have seen that $A_n(\mathbb{P}^n) = \mathbb{Z}$. Consider a codimension one subvariety $V = V(g)$ for a homogeneous polynomial g of degree d. Then g/x_0^d is a rational function and $[g/x_0^d] = V - dH_0$ where $H_0 = (x_0)$ which implies that $V \sim dH_0$. We can then define a rurjective function $\mathbb{Z} \to A_{n-1}(\mathbb{P}^n)$ assigning m to mH_0. Note that if $mH_0 \sim 0$ that there exist $r_1, \dots, r_s \in K(\mathbb{P}^n)^*$ such that $dH_0 = \sum[r_i] = \sum d_i H_0$. It follows that $\deg(r = r_1 \cdots r_s) = 0 = d$. If we write $r = f_1^{m_1} \cdots f_k^{m_k} / g_1^{h_1} \cdots g_k^{h_l}$ where the f_i, g_j have no common factors, then $[r] = \sum_1^k m_i V(f_i) - \sum_1^l h_j V(g_j) = dH_0$. It is then $h_j = 0$ for all j and $m_i = 0$ for all i but one, m_1. but since $\sum_1^k m_i = \sum_1^l h_j$ it must be $m_1 = 0$ and thus $dH_0 = 0$. The map is also injective which proves that $A_{n-1}(\mathbb{P}^n) = \mathbb{Z} =< H >$ where H is a coordinate hyperplane. Similarly as for (1) one shows that $A_0(\mathbb{P}^n) = \mathbb{Z}$.

Proposition 3.2. *Let Y be a subvariety if X. Then for any $k \geq 0$ there is an exact sequence:*

$$A_k(Y) \to A_k(X) \to A_k(X \setminus Y) \to 0.$$

The morphisms are induced by inclusion and restriction.

Bevis. A k-dimensional subvariety of Y is defined by restricting a k-dimensional subvariety of X to Y. This proves that the first map is well defined, i.e. $B_k(Y)$ is mapped to $B_k(X)$. The second map $j : A_k(X) \to A_k(X \setminus Y)$ is defined by restriction. Assume that V is a k dimensional subvariety of X contained in the $(k+1)$ subvariety W then for any $r \in \mathcal{O}_{X,W}$

$$ord_W(r) = ord_{W \setminus Y}(r|_{W \setminus Y})$$

which implies that $j([r]) = [r|_{W \setminus Y}]$ and thus $j(B_k(X)) \subset B_k(X \setminus Y)$. Surjection is implied by the fact that subvarieties in $X \setminus Y$ can be extended to X. Assume now that $j(\alpha) \sim 0$ which menas that $j(\alpha) = \sum_1^s [r_{W_i}]$. again by extending (taking the closure) we can assume $[r_{W_i}] = j[r^*_{W_i}]$. The cycle $\beta = \alpha - \sum_1^s [r^*_{W_i}]$ is equivalent to α and $j(\beta) = 0$ which means that all components are contained in Y. It follows that α is the image of an element of $A_k(Y)$. □

Example 3.3. Let U be a (non empty) open set of $\mathbb{C}^n$. Then $A_k(U) = 0$ for $k < n$ and $A_n(U) = \mathbb{Z}$. One sees this by induction. We have remarked that this is true for $n = 1$. For $n \geq 2$ consider the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ whose fiber is $\mathbb{C}^1$, i.e. $\mathbb{C}^n$ can be viewed as an affine bundle of rank 1 over $\mathbb{C}^{n-1}$, which implies ([FU, Thm. 1.9]) that there is a surjective map $\pi : A_{k-1}(\mathbb{C}^{n-1}) \to A_k(\mathbb{C}^n)$ for all $1 \leq k \leq n$. If $k < n$ by induction the groups are 0. If $k = n$ then both are $\mathbb{Z}$. To finish the prove we observe that the inclusion $U \subset \mathbb{C}^n$ induces a surjection $A_k(\mathbb{C}^n) \to A_k(U)$.

Example 3.4. $A_k(\mathbb{P}^n) = \mathbb{Z}$ foe all $0 \leq k \leq n$ and it is generated by any k-plane $H_k = \mathbb{P}^k$. We have observed this for $k = n-1, n$. Let $k < n$ and use induction on n. Let $H \subset \mathbb{P}^n$ be a hyperplane, then

$$A_k(H) \to A_k(\mathbb{P}^n) \to A_k(\mathbb{P}^n \setminus H) \to 0$$

By induction $A_k(H) = \mathbb{Z}$ and $A_k(\mathbb{P}^n \setminus H) = 0$ because $\mathbb{P}^n \setminus H \cong \mathbb{C}^n$, which implies that we have a surjective map $\mathbb{Z} =< H_k >\rightarrow A_k(\mathbb{P}^n)$ induced by the inclusion. Let $d\mathbb{Z} =< dH_k >$ be the kernel. This means that $dH_k = \sum [r_i]$ where r_i are rational on $(k+1)$-dim V_i and thus $H_k \subset V = \cup V_i$.

Litteraturförteckning

[FU] W. Fulton, Intersection Theory, Springer 1984

[FUintro] Introduction to Intersection Theory in Algebraic Geometry, AMS regional conference series in mathematics, no 54, 1984.

[Ha] R. Hartshorne, Algebraic Geometry, Springer 1977.