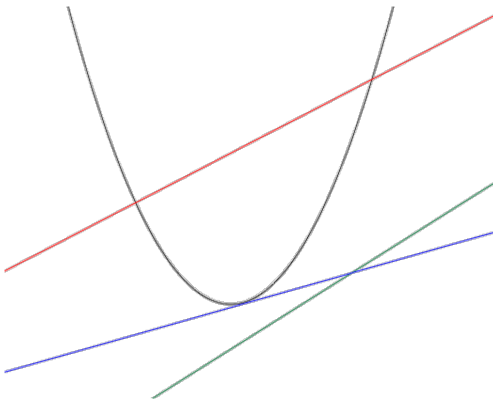
KTH Teknikvetenskap

Topics in Applied Algebraic Geometry
Lecture 2: Introduction to intersection theory

We will introduce the basics of intersection theory and the Chow ring of an algebraic variety. We start by describing the subvarieties of the Grassmanian and their intersection.



1 The Grassmanian variety

Let V be a $\mathbb{C}$-vector space of dimension n. We denote by $G(k,V)$ the pace of all linear subspaces of V of dimension k. We will also use the symbol $G(k,n)$ when the role of a given space V is not important.

Note that by projectivization the same symbol is used to denote:

$$\mathbb{G}(k-1,n-1) = \{H \cong \mathbb{P}^{k-1}, H \subset \mathbb{P}^{n-1}\}.$$

Observe that $G(1,n) = \mathbb{P}^{n-1}$.

Consider the following map (the Plücker embedding)

$$pl : G(k,n) \to \mathbb{P}^{\binom{n}{k}-1} \text{ defined by } V \to \bigwedge^{k} V.$$

If $\{v_1,\ldots,v_k\}$ is a basis for V, the the image is given by $pl([v_1,\ldots,v_k]) = (\ldots, M_{i_1\ldots i_k}, \ldots)$ where

$$M_{i_1\ldots i_k} = \det \left[C_{i_1} - - - C_{i_k}\right]$$

for $i_1 < \ldots < i_k$, where

$$M = \begin{bmatrix} v_1 \\ -- \\ v_k \end{bmatrix} = \left[C_1 - - - C_n\right].$$

The map does not depend on the choice of the basis. This map is an embedding. The Grassmanian $G(k,n)$ is then isomorphic to its image which is a closed subvariety of $\mathbb{P}^{\binom{n}{k}}$, in other words it is a *projective algebraic variety.* For a k-dimensional subset $V \in \mathbb{C}^n$ we call $p_l(V)$ its *Plücker coordinates*.

Definition 1.1. A collection $\mathcal{F} = (H_1 \subset H_2 \subset \ldots H_n = V)$ where H_i are subspaces of dimension i is called a *complete flag* of V. Correspondingly we have the projective flag : $\mathcal{F} = (\mathbb{P}(H_1) \subset \mathbb{P}(H_2) \subset \ldots \mathbb{P}(H_n) = \mathbb{P}(V))$

A subspace $V \subset \mathbb{C}^n$ of dimension k is said to be $\mathcal{F}$-generic if it intersects all the components of $\mathcal{F}$ in the expected dimension, i.e. $V \cap H_i = \{0\}$ for all $i < n-k = \mathrm{codim(V)}$ and $\dim(V \cap H_i) = k+i-n$ otherwise.

Lemma 1.2. *$G(k,n)$ is a projective algebraic variety of dimension $k(n-k)$.*

For every complete flag $\mathcal{F}$, let $G_{\mathcal{F}} = \{V \in G(k,n), \mathcal{F}\text{-generic}\}$. Note that every element $V \in G(k,n)$ is $\mathcal{F}$-generig for some complete flag $\mathcal{F}$, and thus $G(k,n) \cong \bigcup_{\mathcal{F}} G_{\mathcal{F}}$. Given a complete flag $\mathcal{F}$ fix a basis $\{e_1,\ldots,e_n\}$ of $H_n = \mathbb{C}^n$, so that $\{e_{n-i+1},\ldots,e_n\}$ is a basis of H_i. Then each V $\mathcal{F}$-generic has a basis of the form $\{u_1,\ldots,u_k\}$ where $u_i = e_i + m_{i1}e_{k+1} + \ldots + m_{i(n-k)}e_n$. To prove this notice that $H_{n-k+1} \cap V =< e_k, e_{k+1}, \ldots, e_n > \cap V$ is a line and that $H_{n-k} \cap V =< e_{k+1}, \ldots, e_n > \cap V = \{0\}$. This implies that there is a vector $v = ae_k + \sum_{k+1}^n a_i e_j$ spanning $H_{n-k+1} \cap V$ with $a \neq 0$. Continue then considering all the intersections. Notice that a vector $u_k = e_k + m_{i1}e_{k+1} + \ldots + m_{i(n-k)}e_n$ is uniquw, other wise bu substraction we would violate the required intersections. The coefficients (m_{ij}) ar the entries of a $k \times (n-k)$ matrix. $\square$

Example 1.3. $G(1, n) = \mathbb{P}^{n-1}$ is covered by $\mathbb{C}^{n-1} = \{L \in \mathbb{C}^n : L \not\subset H_{n-1}\}$. Given a line $L \in G_{\mathcal{F}}$ the coordinates are $(1, a_2, \ldots a_n)$ giving the affine coordinates in the corresponding chart: $\mathbb{C}^{n-1}$.

Example 1.4. $G(2, 4)$ is a 4 dimensional variety in $\mathbb{P}^5$. It is embedded as a quadric hypersurface defined by $M_{12}M_{34} - M_{13}M_{24} - M_{14}M_{23} = 0$.

2 Schubert cells

The Grassmanian has a decomposition in disjoint union of cells, called *Schubert cells*, see [A03] for a complete treatment. Representing $V \in G(k, n)$ by a $(k \times n)$ matrix and using Gauss elimination then (up to isomorphism) the matrix representation will have a pivot 1 at position (i, λ_i), for $i = 1, \ldots, k$ and $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k \leq n$, and zeroes above, below and to the right. If we call Σ_λ, with $\lambda = (\lambda_1, \ldots, \lambda_k)$ the set of such matrices than

$$G(k, n) = \bigcup_\lambda \Sigma_\lambda.$$

The dimension of each Σ_λ is given by the free terms to the left of (i, λ_i) and thus $\dim(\Sigma_\lambda) = \sum_i (\lambda_i - i)$. Moreover given any flag $\mathcal{F}$, these cells can be interpreted as:

$$\Sigma_\lambda = \Sigma_\lambda(\mathcal{F}) = \{V \in G(k, n) : \dim(V \cap H_{\lambda_i}) = i\}.$$

Notice that one can show that this representation does not depend on the choice of the Flag(notice that they are determined by the vanishing of certain coordinates).

Example 2.1. Consider $G(2, 4)$, the Grassmanian of projective lines in $\mathbb{P}^3$ and a flag: $p \in l \subset H \subset \mathbb{P}^3$. The Schubert cells are:

$$\Sigma_{1,2} = \begin{bmatrix} 1000 \\ 0100 \end{bmatrix}, \dim(\Sigma_{(1,2)}) = 0, \{L = l\} = [l]$$

$$\Sigma_{1,3} = \begin{bmatrix} 1000 \\ 0x10 \end{bmatrix}, \dim(\Sigma_{(1,3)}) = 1, \{L, p \in L \subset H\} = \mathbb{C}$$

$$\Sigma_{1,4} = \begin{bmatrix} 1000 \\ 0xy1 \end{bmatrix}, \dim(\Sigma_{(1,4)}) = 2, \{L, p \in L\} = \mathbb{C}^2$$

$$\Sigma_{2,3} = \begin{bmatrix} x100 \\ y010 \end{bmatrix}, \dim(\Sigma_{(2,3)}) = 2, \{L, L \subset H\} = \mathbb{C}^2$$

$$\Sigma_{2,4} = \begin{bmatrix} x100 \\ y0z1 \end{bmatrix}, \dim(\Sigma_{(2,4)}) = 3, \{L, l \cap L = pt\} = \mathbb{C}^3$$

$$\Sigma_{3,4} = \begin{bmatrix} xy10 \\ zw01 \end{bmatrix}, \dim(\Sigma_{(3,4)}) = 4, \mathbb{C}^4$$

Definition 2.2. The *Schubert varieties* are defined as the closures (in the Zariski topology) of the Schubert cells:

$$\sigma_\lambda = \sigma_\lambda(\mathcal{F}) = \overline{\Sigma_\lambda(\mathcal{F})} = \{V \in G(k,n) \,:\, \dim(V \cap H_{\lambda_i}) \geq i\}.$$

Note that it is also common to use different indexing in the literature $\lambda_j \to \overline{\lambda_j} = n - k - \lambda_j + j$ The advantage is that the latter gives a partition of $k(n-k)$ to which one can associate a *Young Tableau*, see [F97].

Example 2.3. In Plücker coordinates, the Schubert varieties of $G(2,4)$ are given by:

$\sigma_{1,2} = \{V \,:\, M_{13} = M_{14} = M_{23} = M_{24} = M_{34} = 0\}, \sigma_{1,3} = \{V \,:\, M_{14} = M_{23} = M_{24} = M_{34} = 0\},$

$\sigma_{1,4} = \{V \,:\, M_{23} = M_{24} = M_{34} = 0\}, \sigma_{2,3} = \{V \,:\, M_{14} = M_{24} = M_{34} = 0\}, \sigma_{2,4} = \{V \,:\, M_{34} = 0\}.$

3 Enumerative geometry via intersection theory

These varieties form a basis for all the subvarieties of $G(n,k)$ of the corresponding dimension (Chow ring). Every subvariety of a given dimension can be written (as cohomology class) as an integral combination of Schubert cycles of the same dimension. This coefficients are known as the *Littlewood-Richardson coefficients*.

Working out an intersection theory och Schubert varieties (called Schubert calculus) gives rise to a theory for intersecting any two subvarieties in $G(k,n)$. Let us look at one simple example.

We will use the simple $\sigma_\lambda(H_i)$ if we want to specify that we are considering a specific flag whose i-th component is H_i.

Question How many lines in $\mathbb{P}^3$ intersect four general lines $L_1, \ldots, L_4$?

Recall that $\sigma_{24}(L_1)$ denotes all the lines intersecting L_1. Therefore the number (if finite) of the desired lines corresponds to the intersection $\sigma_{24}(L_1) \cdot \sigma_{24}(L_2) \cdot \sigma_{24}(L_3) \cdot \sigma_{24}(L_4)$.

In order to compute this intersection we have to understand the intersections of the Schubert varieties. Since this is a low dimension case we can use the geometry. Whenever intersecting two cycles we will choose two different projective flags: $p_1 \in l_1 \subset H_1 \subset \mathbb{P}^3, p_2 \in l_2 \subset H_2 \subset \mathbb{P}^3$.

$$\sigma_{24} \cdot \sigma_{23} = \{l, l \cap l_1 = pt, l \subset H_2\} = \sigma_{13}(H_2)$$

$$\sigma_{14} \cdot \sigma_{24} = \{l, p_1 \in l, l \cap l_2 = pt\} = \sigma_{13}(< pt, l_2 >)$$

$$\sigma_{23} \cdot \sigma_{23} = \{l, l = H_1 \cap H_2\} = [\sigma_{12}] = 1$$

$$\sigma_{14} \cdot \sigma_{14} = \{[p_1 p_2]\} = [\sigma_{12}] = 1$$

$$\sigma_{23} \cdot \sigma_{14} = \{l, l \subset H_1, p_2 \in l\} = 0 \text{ most planes do not contain } p_2$$

$$\sigma_{24} \cdot \sigma_{24} = \{l, l \cap l_i = pt\} = [\sigma_{12}] = \sigma_{23} + \sigma_{14}\text{[1]}$$

It follows that:

$$\sigma_{24}(L_1) \cdot \sigma_{24}(L_2) \cdot \sigma_{24}(L_3) \cdot \sigma_{24}(L_4) = (\sigma_{23} + \sigma_{14})^2 = \sigma_{23}^2 + 2\sigma_{23} \cdot \sigma_{14} + \sigma_{14}^2 = 2$$

Answer There are exactly two lines in $\mathbb{P}^3$ intersecting four general lines $L_1, \ldots, L_4$.

[1] an easy excercise using the Pieri Formula and Young Tableaus

Litteraturförteckning

[F97] W. Fulton, Young tableaux, London Mathematical Society Student Texts 35, Cambridge University Press, 1997, ISBN 0-521-56724-6 (paperback edition).

[KL72] S. Kleiman and D. Laksov, D.: Schubert calculus. The American Mathematical Monthly 79(10), 1061-1082 (1972)

[A03] A. Hatcher. Vector bundles and K-theory.(2003),http://www.math.cornell.edu/hatcher