KTH Teknikvetenskap

Topics in Applied Algebraic Geometry
Lecture 1: Algebraic Kinematics

We will introduce the basic algebraic terminology within the theory of serial link-mechanisms.

1 Rigid Body Motion Space

A rigid body motion in 3-space is a composition of a rotation and a translation, the former is an element of the special orthogonal group $SO_3(\mathbb{R})$ and the latter a vector in $\mathbb{R}^3$. In other world it is an element of $SE_3(\mathbb{R})$, the semi-direct product of $\mathbb{R}^3$ and $SO_3(\mathbb{R})$. More precisely $SE_3(\mathbb{R})$ is a 6-dimensional Lie-group, the set $SO_3(\mathbb{R}) \times \mathbb{R}^3$ with the group operation:

$$(R_2, t_2) \circ (R_1, t_1) = (R_2 R_1, R_2 t_1 + t_2).$$

It is convenient to embed $\mathbb{R}^3$ in the quaternions. Consider $\mathbb{R}^4$ as the division algebra of the quaternions. Let 1 be the unit element and $\{i, j, k\}$ be the quaternions units i.e. $i^2 = j^2 = k^2 = ijk = -1$. Then $\{1, i, j, k\}$ is a basis for $\mathbb{R}^4$. Moreover for

$$a = (a_1, \ldots, a_4) = (a_0, \mathbf{a}) \in \mathbb{R}^4 \text{ let } a^* = (-a_1, \ldots, -a_4), |a| = \sqrt{aa^*}.$$

The multiplication is always intended as moltiplication of quaternions:

$$(a_0, \mathbf{a})(b_0, \mathbf{b}) = (a_0 b_0 - \mathbf{ab}, a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

We identify $\mathbb{R}^3 \cong \{a \in \mathbb{R}^4 \text{ s.t. } a_0 = 0\}$

Consider now $\mathbb{P}^3_{\mathbb{R}}$ (as the quotient of $\mathbb{R}^4$) and the map:

$$\phi : \mathbb{P}^3_{\mathbb{R}} \to SO_3(\mathbb{R})$$

defined by $\phi(a)x = axa^*$ for $a \in \mathbb{P}^3_{\mathbb{R}}, x \in \mathbb{R}^3$. This is an isomophism.

Consider now the following quadric hypersurface:

$$\mathcal{Q} = \{(p, q) = (p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3) \in \mathbb{P}^7_{\mathbb{R}} \text{ s.t. } \sum p_i q_i = 0\} \subset \mathbb{P}^7_{\mathbb{R}}$$

It is called the **Study quadric** and it is the natural ambient space for all the rigid body motions. Consider the subset:

$$\mathcal{Q}' = \mathcal{Q} \setminus \{(p, q) | p_0 = p_1 = p_2 = p_3 = 0\}.$$

Notice that if $(p, q) \in \mathcal{Q}'$ then by viewing p, q as quaternians

$$pq^* = (-p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3, ...) = (0, ..) \in \mathbb{R}^3 \subset \mathbb{R}^4$$

Consider the map:

$$\phi : \mathcal{Q}' \to SO_3(\mathbb{R}) \times \mathbb{R}^3, (p, q) \mapsto (\phi(q), \frac{pq^*}{qq^*}).$$

This map is an isomorphism (homework!) and thus

$$SE_3(\mathbb{R}) \cong \mathcal{Q}' \subset \mathcal{Q} \subset \mathbb{P}^3_{\mathbb{R}}$$

$\mathcal{Q}'$ has and induced group structure:

$$(p_1, q_1)(p_2, q_2) = (p_1 p_2, p_1 q_2 + p_2 q_1).$$

In solving polynomial systems it is convenient to work with algebraically closed fields, so from now on we will consider:

$$\mathcal{Q} = \{(p_0, \ldots, p_3, q_1 \ldots, q_3) \,:\, p_0q_0 + \ldots + p_3q_3 = 0\} \subset \mathbb{P}^7_{\mathbb{C}}$$

and $\mathcal{Q}' = \mathcal{Q} \setminus \{p_0^2 + p_1^2 + p_1^2 + p_3^2 = 0\}$ so that $\mathcal{Q}_{\mathbb{R}} \subset \mathcal{Q}$ and $\mathcal{Q}'_{\mathbb{R}} = \mathcal{Q}' \cap \mathcal{Q}_{\mathbb{R}}$.

It will be convenient to embed $\mathcal{Q}$ in a much higher dimensional projective space vi 4×4 matrices. For $t = (x, y, z, w) \in \mathbb{C}^4$ consider the matrix $M(t) = xId + yI + zJ + wK$ where:

$$Id = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The map $M : \mathcal{Q} \to \mathbb{P}^{63}_{\mathbb{C}}$ defines as

$$M(p, q) = \begin{bmatrix} M(p) & 0 \\ M(q) & M(p) \end{bmatrix}$$

up to multiplication by scalar. this map is injective. Observe that

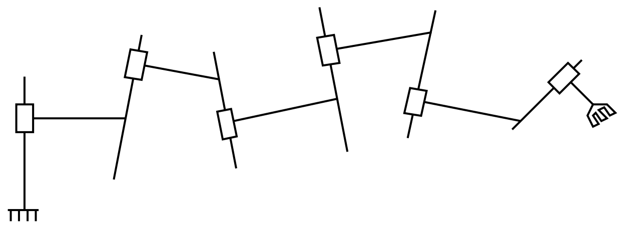
$$M(p, q) = \begin{bmatrix} p_0 & - & - & - & - & - & - & - \\ p_1 & - & - & - & - & - & - & - \\ p_2 & - & - & - & - & - & - & - \\ p_3 & - & - & - & - & - & - & - \\ q_0 & - & - & - & - & - & - & - \\ q_1 & - & - & - & - & - & - & - \\ q_2 & - & - & - & - & - & - & - \\ q_3 & - & - & - & - & - & - & - \end{bmatrix}$$

The inverse M^{-1} is then defined by the first column.

Remark 1.1. (Homework) Consider the group of non-degenerate linear maps on $\mathbb{C}^8$ preserving the bi-linear form Q up to scalar: $\mathbb{C}^* \times SO_8(\mathcal{Q})$. The restriction $M|_{\mathcal{Q}*}$ identifies $\mathcal{Q}'$ with $SO_8(\mathcal{Q})$.

2 The 3R-chain and the 6R-chain

An m-revolute serial chain linkage (mR-chain) consists of $(m + 1)$ rigid links connected by m revolute joints. Here is a 6-R chain.



One end is attached to the ground and at the other end there is *hand* intended to move around in space. After fixing an initial pose we can measure the rotation angles at the m-joints (with respect to the initial). The initial pose fixes a coordinate frame at the ground and at the hand. The transformation from hand-coordinates to ground-coordinates is a function of the rotation angles. The angles are elements of the unit circle identified with $\mathbb{P}^1_{\mathbb{R}}$. This transformation can be then described as:

$$\Phi_{\mathbb{R}} : \underbrace{\mathbb{P}^1_{\mathbb{R}} \times \ldots \times \mathbb{P}^1_{\mathbb{R}}}_{m} \to SE_3(\mathbb{R}) \cong \mathcal{Q}' \subset \mathcal{Q}.$$

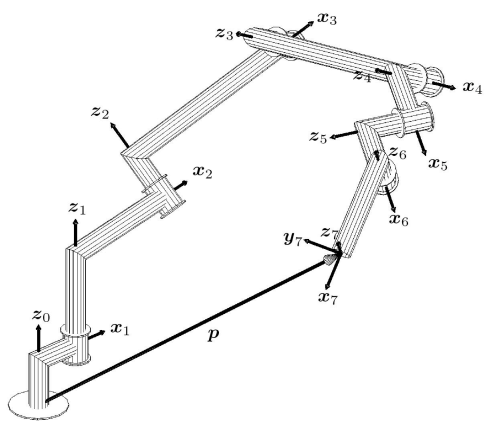
This map maps an m-ple of angles to the corresponding transformation from hand to ground.

Forward and inverse kinematics are two basis tasks of mechanism analysis. Usually the direct kinematics problem is relatively easy for serial manipulators but often difficult for parallel manipulators. Conversely, the inverse kinematics problem is usually simple for parallel manipulators and often complicated for parallel manipulators.

The *forward kinematic problem* (FKP) is about constructing such a map. If the rotation angles of the individual joints are known, computing the end effector frame is just a matter of multiplying consecutive transformation matrices. it is not obvious at all how to choose the joint angles such that the end effector attains a certain specified pose (inverse kinematics). The *inverse kinematic problem*(IKP) is about computing the fiber $\Phi^{-1}_{\mathbb{R}}(x)$ for any $x \in \mathcal{Q}'_{\mathbb{R}}$.

We will see that the IKP for a general 6-R chain has a finite number of solutions:16. See [S96, SW07] for more details on the mechanism. In fact for $m = 6$ the map $\Phi_{\mathbb{R}}$ is onto a generically $16 : 1$.

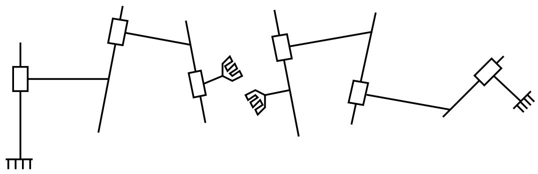
We can be a bit more specific about the map Φ. Consider a the case of a 6R-chain. We have $L_0, \ldots, L_6$ links connected by 6 joints. Let the joint i connect L_{i-1} to L_i. Fix (x, y, z) standard coordinates in $\mathbb{R}^3$. for any coordinate $v \in \{x, y, z\}$ denote by $R_v(\theta)$ the rotation of an angle θ about the v-axis and by $T_\theta(d)$ the translation of distance d in the direction of v. As the mechanism is in the initial pose place coinciding coordinate frames at the links: A_{i-1} at the $(i-1)$-th link and B_i at the i-th link, with origin on the rotation axis of joint i and z-direction parallel to the rotation axis. B_0 denotes a frame at the ground link and A_6 at the hand. We denote by $T_i \in SE_3(\mathbb{R})$ the linear transformation from A_i coordinates to B_i coordinates. Since these coordinates are along the i-th link the transformations only depend on the properties of the mechanism.



In fact the map Φ is given by:

$$(\theta_1, \ldots, \theta_6) \to T_0 \circ R_z(\theta_1) \circ T_1 \circ R_z(\theta_2) \circ \ldots \circ T_5 \circ R_z(\theta_6) \circ T_6.$$

A classical technique to solve the IKP for a 6R-chain is to split the problem into two 3R-chains.



We introduce a joint on the link L_3 which becomes the ending point of a left 3R and a right 3R. Consider $S_1 \in SO_3(\mathbb{R})$ and $S_2 = T_3^{-1} \circ S_1$, The transformation S_1 is the last T_i on the left and S_2 is the hand point of a $3R$ on the right, $S_1 \circ S_2^{-1} = T_3$. Given a hand position H, then $(\theta_1, \ldots, \theta_6)$ is a solution of the IKP for the 6R if and only if it is composed by common solutions of the two FKP of two 3R with the following transformation links:

$$\Phi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{Q},\ T_0, T_1, T_2, S_1$$

$$\Phi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{Q},\ H \circ T_6^{-1}, T_5^{-1}, T_4^{-1}, S_2$$

In other words: $\Phi(\theta_1, \ldots, \theta_6) = H \Leftrightarrow \Phi_1(\theta_1, \theta_2, \theta_3) = \phi_2(\theta_4, \theta_5, \theta_6)$

Let us consider the maps Φ_i in the complex projective space. Notice that up to projective equivalence the maps Φ_1, Φ_2 are the Segre embeddings : $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$ and thus the solutions of the IKS are given by:

$$\{(\theta, \alpha) \in (\mathbb{P}^1_{\mathbb{C}})^3 \times (\mathbb{P}^1_{\mathbb{C}})^3 \ : \ \Phi_1(\theta) = \Phi_2(\alpha)\}.$$

Geometrically this is the intersection of two 3-dimensional subspaces of Q. We will see how to compute it using intersection theory.

Litteraturförteckning

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[S96] J.M. Selig. Geometric Fundamentals of Robotics. Monograph in Computer Science, Springer. Second edition.