Tracking a mobile target by multi-robot circumnavigation using bearing measurements

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***Abstract*— In this paper, we study a problem of target tracking and circumnavigation with a network of autonomous agents. We propose a distributed algorithm to estimate the position of the target and drive the agents to rotate around it while forming a regular polygon and keeping a desired distance. We formally show that the algorithm attains exponential convergence of the agents to the desired polygon if the target is stationary, and bounded convergence if the target is moving with bounded speed. Numerical simulations corroborate the theoretical results and demonstrate the resilience of the network to addition and removal of agents.**

I. Introduction

The problem of target tracking and circumnavigation finds applications in numerous fields. A new challenging application is the tracking of an underwater target with a network of AUVs, where the vehicles are employed to study, and possibly reproduce, the behavior of small sea animals. The design of a control algorithm that drives an agent to approach a target and to follow a circle trajectory around it has been studied in [1]–[5]. The solutions proposed for the single agent [5], have been extended to multi-agent systems, where great attention has been given to the formation of a regular polygon inscribed in a desired circle, centered at the target position [6]–[9]. This type of formation is optimal to solve triangulation problems and it is a good solution to control agents that cannot easily stop moving, such as UAVs. For example, an application of the circumnavigation to escorting and patrolling missions is analyzed in [10]. In a large number of applications, the position of the target is unknown to the agents, so that a localization procedure is required to achieve the tracking. In [11] a peer-to-peer collaborative localization is studied for a network of sensors, while in [12] a stereo-vision-type estimation is realized by the leading agent, sending its visual measurements of the target to its followers. The problem of using the information obtained from an identification process of unknown characteristics of a system to update a control algorithm is known in the literature as the *dual problem* [13], [14]. In [5], [15], the circumnavigation of a target with unknown position is formally modeled as a dual problem. In particular, in [15] the dual problem is solved using distance measurements, while in [5] it is solved using bearing measurements. In [16] the robot and landmark localization problem is solved using data originated from bearing-only measurements. Similarly to the solution adopted in [5] and [9], we propose a distributed control algorithm based on bearing measurements, and an estimator to localize a mobile target. In this paper, we propose a different control strategy where every agent updates its control signal on the basis of information it received by other agents within a defined communication radius, irrespectively of the distance from the target. Moreover, we let the rotational speed of the agents depend explicitly on the desired distance from the target. Our analysis leads to the following improvements: the angular velocity of the agents about the target does not grow unbounded when the desired distance from the target is small; we formally prove exponential convergence of the agents to a regular polygon, rather than simply asymptotic convergence.

The control algorithm is simulated in ROS [17], where each simulated agent is implemented as a separate ROS node. The simulations also demonstrate the resiliency of the algorithm to addition and removal of some agents, showing that the agents rearrange themselves to form a different polygon when one agent enters or leaves the network. Each agent starts in a monitoring position, and when it receives the first bearing measurement it effectively enters the network.

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II. Preliminaries

The set of the positive integers is denoted as $\mathbb{N}$. All vectors in $\mathbb{R}^n$, with $n \in \mathbb{N}$, are column vectors. Transposition of a vector or of a matrix is denoted as $(\cdot)^\mathsf{T}$. The operator $[\cdot]_{ij}$ denotes the element at the ith row and at the jth column of a matrix. The operator $\|\cdot\|$ denotes the Euclidean norm of a vector and the corresponding induced norm of a matrix. The set of the unit-norm vectors in $\mathbb{R}^2$ is denoted as $\mathcal{S}^1$. The circle of radius d centered at $x \in \mathbb{R}^2$ is denoted as $\mathcal{C}(x, d)$. The n-by-n identity matrix is denoted as I_n. Let $\varphi \in \mathcal{S}^1$; defining $\theta(t)$ as the counterclockwise angle between the vector and the x-axis of a reference frame, φ can be represented as $\varphi = [\cos\theta, \sin\theta]^\mathsf{T}$.

Lemma 1: Let $\varphi \in \mathcal{S}^1$ and let $\bar{\varphi}$ be obtained by rotating φ by $\pi/2$ radians clockwise. Then $I_2 - \varphi\varphi^\mathsf{T} = \bar{\varphi}\bar{\varphi}^\mathsf{T}$.

Proof: Letting $\varphi = [\cos\theta, \sin\theta]^\mathsf{T}$, we have $\bar{\varphi} = [-\sin\theta, \cos\theta]^\mathsf{T}$, and the claim is verified by inspection. ■

Definition 1 (Persistence of excitation [18]): A time-varying vector $\bar{\varphi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ is *persistently exciting (p.e.)*

if there exist $\epsilon_1, \epsilon_2, T > 0$ such that

$$\epsilon_1 \leq \int_{t_0}^{t_0+T} (U^\intercal \bar{\varphi}(t))^2 \, \mathrm{d}t \leq \epsilon_2 \tag{1}$$

for all $t_0 \geq 0$ and all $U \in \mathcal{S}^1$. ■

Persistence of excitation requires that the vector $\bar{\varphi}$ rotates sufficiently in the plane that the integral of the semi-positive definite matrix $\bar{\varphi}\bar{\varphi}^\intercal$ is uniformly definite positive over any interval of some positive length T.

Lemma 2 (in [18]): Let $\bar{\varphi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ be piecewise continuous; if $\bar{\varphi}$ is p.e., then the system

$$\dot{w}(t) = -k\bar{\varphi}(t)\bar{\varphi}^\intercal(t)w(t), \tag{2}$$

with $k > 0$, is globally exponentially stable. ■

Lemma 3 (in [5]): If $A(t) \in \mathbb{R}^{n\times n}$ is continuous for all $t > 0$, and positive constants r, b exist such that for every solution of the homogeneous differential equation $\dot{p}(t) = A(t)p(t)$ it holds that $\|p(t)\| \leq b\|p(t_0)\|e^{-r(t-t_0)}$ for $0 \leq t_0 < t < \infty$, then, for each $f(t)$ bounded and continuous on $[0, \infty)$, every solution of the non-homogeneous equation $\dot{p}(t) = A(t)p(t) + f(t)$, with $p(t_0) = 0$, is also bounded for $t \in [0, \infty)$. In particular, if $\|f(t)\| \leq K_f < \infty$, then the solution of the perturbed system satisfies

$$\|p(t)\| \leq b\|p(t_0)\|e^{-r(t-t_0)} + \frac{K_f}{r}(1 - e^{-r(t-t_0)}).$$

■

For the purposes of this paper, a graph is a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \ldots, N\}$, $N \in \mathbb{N}$, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, with the constraint $(v, v) \notin \mathcal{E}$ for all $v \in \mathcal{V}$. The elements of $\mathcal{V}$ are called the *vertexes* of the graph, and the elements of $\mathcal{E}$ are called the *edges* of the graph. Given two vertexes v, u, a path from v to u is a sequence of vertexes $(v = v_1, v_2, \ldots, v_P = u)$ such that $(v_k, v_{k+1}) \in \mathcal{E}$ for all $k \in \{1, \ldots, P-1\}$. A graph is *connected* if there exists a vertex with a path to all the other vertexes. The Laplacian matrix of the graph is the matrix $L \in \mathbb{R}^{N\times N}$ such that $[L]_{ij} = -1$ if $(j, i) \in \mathcal{E}$, $[L]_{ij} = 0$ if $i \neq j$ and $(j, i) \notin \mathcal{E}$, and $[L]_{ii} = -\sum_{j\in\mathcal{V}\setminus\{i\}}[L]_{ij}$. It is a well-known result in graph theory [19] that if L is the Laplacian matrix of a connected graph, then the system $\dot{x}(t) = -Lx(t)$ converges asymptotically to the consensus subspace (i.e., $|x_i(t) - x_j(t)| \to 0$) from every initial condition.

III. System Model

In this paper, we consider a set of N planar autonomous agents indexed as $1, \ldots, N$, with kinematics

$$\dot{y}_i(t) = u_i(t), \tag{3}$$

where $y(t) \in \mathbb{R}^2$ is the position of the ith agent, and $u_i(t)$ is a control input. The agents are required to locate and circumnavigate a target, whose position is denoted as $x(t)$, while forming a regular polygon inscribed in the *desired circle* $\mathcal{C}(x(t), D^*)$, where $D^* > 0$ is a desired distance from the target. To achieve this objective, each agent can measure its bearing with respect to the target; the bearing

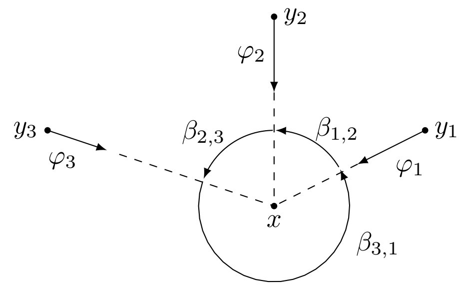

Fig. 1: Counterclockwise angles in a network of three agents. We have $\nu_1 = 2$, $\nu_2 = 3$ and $\nu_3 = 1$.

is represented as the unit vector $\varphi_i(t)$ in the direction $x(t) - y_i(t)$:

$$\varphi_i(t) = \frac{x(t) - y_i(t)}{\|x(t) - y_i(t)\|}. \tag{4}$$

Moreover, the agents have a communication radius $\rho > 0$, and can exchange the bearing measurements with other agents within that radius. We denote as $\mathcal{N}_i(t)$ the set of the agents laying within the communication radius of agent i, namely $\mathcal{N}_i(t) = \{j \in \{1, \ldots, N\} : \|y_j(t) - y_i(t)\| \leq \rho\}$. Let $\beta_{ij}(t) \geq 0$ be the positive counterclockwise angle from $\varphi_i(t)$ to $\varphi_j(t)$. The agent $j \in \mathcal{N}_i(t)$ that attains the minimum $\beta_{ij}(t)$ is called the *counterclockwise neighbor* of agent i at time t, and it is denoted as $\nu_i(t)$, see Figure 1. Moreover, we let $\beta_i(t) = \beta_{i,\nu_i(t)}(t)$ if $\mathcal{N}_i(t)$ is nonempty, $\beta_i(t) = 0$ if $\mathcal{N}_i(t)$ is empty. Then, the control objective is formally written as

$$\lim_{t\to\infty} D_i(t) = D^*, \tag{5}$$

$$\lim_{t\to\infty} \beta_i(t) = \frac{2\pi}{N}, \tag{6}$$

for all $i \in \{1, \ldots, N\}$, where

$$D_i(t) = \|x(t) - y_i(t)\|. \tag{7}$$

Each agent maintains an estimate $\hat{x}_i(t)$ of the position of the target, which is updated using the bearing measurements. Following [5], [9], we set

$$\dot{\hat{x}}_i(t) = -k_e(I_2 - \varphi_i(t)\varphi_i^\intercal(t))(\hat{x}_i(t) - y_i(t)), \tag{8}$$

where $k_e > 0$ is an estimation gain. The estimate error associated to $\hat{x}_i(t)$ is denoted $\tilde{x}_i(t) = \hat{x}_i(t) - x(t)$. Considering (8) and Lemma 1, the dynamics of $\tilde{x}_i(t)$ can be written as $\dot{\tilde{x}}_i(t) = k_e\bar{\varphi}_i(t)\bar{\varphi}_i^\intercal(t)(\tilde{x}_i(t) + D_i(t)\varphi_i(t)) - \dot{x}(t)$, which, exploiting the orthogonality between $\varphi_i(t)$ and $\bar{\varphi}_i(t)$, becomes

$$\dot{\tilde{x}}_i(t) = -k_e\bar{\varphi}_i(t)\bar{\varphi}_i^\intercal(t)\tilde{x}_i(t) - \dot{x}(t). \tag{9}$$

The estimate $\hat{x}_i(t)$ is used to compute $u_i(t)$ as

$$u_i(t) = k_d(\hat{D}_i(t) - D^*)\varphi_i(t) + k_\varphi D^*(\alpha + \beta_i(t))\bar{\varphi}_i(t), \tag{10}$$

where $k_d, k_\varphi, \alpha > 0$. Note that the control signal $u_i(t)$ is made up of two contributions: a *radial term* $k_d(\hat{D}_i(t) - D^*)\varphi_i(t)$ drives the agent towards the desired circle, and a *tangential term* $k_\varphi D^*(\alpha + \beta_i(t))\bar{\varphi}(t)$ makes the agent circumnavigate the target while attaining the desired formation with the other agents. Differently from [5], [9], we let the tangential term depend on the desired distance from the target, D^*, in order to avoid high angular velocities when the desired distance from the target is small. Another important property of control law (10) is that $u_i(t)$ is always nonzero. In fact, since $\varphi_i(t)$ and $\bar{\varphi}_i(t)$ are orthogonal, and since $\alpha + \beta_i(t) > 0$, we have that $u_i(t) = 0$ would require $\hat{D}_i(t) - D^* = 0$ and $D^* = 0$, which is not possible since $D^* > 0$. This property also implies that the closed-loop system has no equilibria.

In order to apply estimate law (8) and control law (10), we need that the bearing vector $\varphi_i(t)$ is well defined for any $t \geq 0$, a condition that, by (4), corresponds to $D_i(t) = \|x(t) - y_i(t)\| > 0$ for all $t \geq 0$. Therefore, our results section will begin by showing that $D_i(t) > 0$ is guaranteed under appropriate initial conditions.

Differentiating (4) and (7) with respect to time, and applying (3), (8), and (10), we derive the differential equations that govern the evolution of $D_i(t)$ and $\varphi_i(t)$ for $D_i(t) > 0$:

$$\dot{D}_i(t) = k_d(\hat{D}_i(t) - D^*) + \varphi_i^\intercal(t)\dot{x}(t), \tag{11}$$

$$\dot{\varphi}_i(t) = \frac{-k_\varphi(\alpha + \beta_i(t))D^* + \bar{\varphi}_i^\intercal(t)\dot{x}(t)}{D_i(t)}\bar{\varphi}_i(t). \tag{12}$$

We introduce the error $\Delta_i(t) = D_i(t) - D^*$ between the distance of agent i from the target and the desired distance. The dynamics of $\Delta_i(t)$ is the same as the dynamics (11) of $D_i(t)$, and can be written as

$$\dot{\Delta}_i(t) = -k_d(\Delta_i(t) - \tilde{D}_i(t)) + \varphi_i^\intercal(t)\dot{x}(t), \tag{13}$$

where we have denoted $\tilde{D}_i(t) = D_i(t) - \hat{D}_i(t)$.

IV. Tracking a Stationary Target

In this section, we consider the case of a stationary target ($\dot{x}(t) = 0_2$), and we show that the algorithm composed of the estimate law (8), and the control law (10) reaches the control objectives (5) and (6).

We begin by showing that the distance $D_i(t)$ between each agent and the target is always positive, which guarantees that the bearing measurements $\varphi_i(t)$ are always defined. Defining the initial error of estimate as $\|\tilde{x}_i(0)\| = \|\hat{x}_i(0) - x\|$, we are going to need the following assumption on the initial conditions.

Assumption 1: For every agent of the network it holds that $D_i(0) > 0$ and $\|\tilde{x}_i(0)\| < D^*$. ■

Lemma 4: Under Assumption 1, estimate law (8), and control law (10), the distance $D_i(t)$ between each agent and the target is bounded as

$$D_i(t) \geq \min\{D_i(0), D^* - \|\tilde{x}_i(0)\|\}, \tag{14}$$

$$D_i(t) \leq \max\{D_i(0), D^* + \|\tilde{x}_i(0)\|\}. \tag{15}$$

Proof: Integrating (13) with $\dot{x}(t) = 0_2$, adding D^* on both sides, and recalling that $D_i(t) = \Delta_i(t) + D^*$, yields

$$D_i(t) = D^* + \Delta_i(0)e^{-k_d t} + k_d \int_0^t e^{-k_d(t-\tau)}\tilde{D}_i(\tau)\,\mathrm{d}\tau\,. \tag{16}$$

Now observe that, by the triangular inequality, we have $\|x - y_i(t)\| \leq \|y_i(t) - \hat{x}_i(t)\| + \|\hat{x}_i(t) - x\|$, which, since $D_i(t) = \|x - y_i(t)\|$, $\hat{D}_i(t) = \|\hat{x}_i(t) - y_i(t)\|$, and $\tilde{D}_i(t) = D_i(t) - \hat{D}_i(t)$, can be rewritten as

$$\tilde{D}_i(t) \leq \|\tilde{x}_i(t)\|. \tag{17}$$

Using (17), and observing from (9) that with $\dot{x} = 0_2$ the norm of the estimate error is nonincreasing, we have

$$\tilde{D}_i(t) \leq \|\tilde{x}_i(0)\| \tag{18}$$

for all $t \geq 0$. Using (18) in (16), and computing the integral explicitly, we have

$$D_i(t) \geq D^* + \Delta_i(0)e^{-k_d t} - \|\tilde{x}_i(0)\|(1 - e^{-k_d t}), \tag{19}$$

$$D_i(t) \leq D^* + \Delta_i(0)e^{-k_d t} + \|\tilde{x}_i(0)\|(1 - e^{-k_d t}). \tag{20}$$

Adding and subtracting $D^* e^{-k_d t}$ from the right-hand side of (19) and (20), we have

$$D_i(t) \geq D_i(0)e^{-k_d t} + (D^* - \|\tilde{x}_i(0)\|)(1 - e^{-k_d t}), \tag{21}$$

$$D_i(t) \leq D_i(0)e^{-k_d t} + (D^* + \|\tilde{x}_i(0)\|)(1 - e^{-k_d t}). \tag{22}$$

Finally, using (21) under Assumption 1 and (22), we have (14) and (15). ■

Our next step is to prove that the estimates $\hat{x}_i(t)$ converge to the real position of the target x.

Lemma 5: Under Assumption 1, estimate law (8), and control law (10), the norm of the estimate error $\|\tilde{x}_i(t)\|$ exponentially converges to zero for every agent in the network.

Proof: Since the dynamics of the estimate error (9) is a system in the form of (2), it is sufficient to prove that $\bar{\varphi}_i(t)$ is p.e. in order to prove the exponential convergence of the estimate. Therefore, we need to prove that condition (1) is satisfied for the vector $\bar{\varphi}_i(t)$. Since $\bar{\varphi}_i(t) \in \mathcal{S}^1$, we have $(U^\intercal\bar{\varphi}_i(t))^2 \leq 1$ for all $U \in \mathcal{S}^1$, and therefore $\int_{t_0}^{t_0+T}(U^\intercal\bar{\varphi}_i(t))^2\,\mathrm{d}t \leq T$ for all $t_0 \geq 0$ and all $U \in \mathcal{S}^1$. Therefore, an upper bound ϵ_2 satisfying condition (1) exists and it is equal to T; we shall use Lemma 1 in [5] to prove that a lower bound $\epsilon_1 > 0$ also exists. Defining $\gamma_{ui}(t)$ as the angle between $U \in \mathcal{S}^1$ and the bearing vector $\bar{\varphi}_i(t)$, we write the dynamics of its derivative as $\dot{\gamma}_{ui}(t) = k_\varphi D^*(\alpha + \beta_i(t))/D_i(t)$. Using (15) and recalling that $\beta_i(t) > 0$, we have $\dot{\gamma}_{ui}(t) \geq k_\varphi D^*\alpha/D_{i,\max}$, where $D_{i,\max} = \max\{D_i(0), D^* + \|\tilde{x}_i(0)\|\}$. Therefore, the dynamics of $\gamma_{ui}(t)$ is lower bounded as $\gamma_{ui}(t) \geq \gamma_{ui}(t_0) + k_\varphi D^*\alpha t/D_{i,\max}$, and we can always find a $\epsilon_1 > 0$ and a $T > 0$ such that (1) is satisfied. The previous proof is valid for each agent independently; therefore, we can state that $\bar{\varphi}_i(t)$ is p.e. for every agent. Hence, by Lemma 2, the estimate error norms of all the agents of the network globally exponentially converge to zero. ■

Next, we prove that the desired circle $\mathcal{C}(x, D^*)$ is an attractive limit cycle for the networked agents.

Lemma 6: Under Assumption 1, estimate law (8), and control law (10), the circle $C(x, D^*)$ is an attractive limit cycle for the trajectories of (3).

Proof: When the target is stationary, the dynamics of the distance error $\Delta_i(t)$ is

$$\dot{\Delta}_i(t) = -k_d\Delta_i(t) + k_d\tilde{D}_i(t). \tag{23}$$

From Lemma 5 (resp., (23)), we know that $\tilde{D}_i(t)$ (resp., $\Delta_i(t)$) converges to zero exponentially. Hence, the desired circle $\mathcal{C}(x, D^*)$ is an attractive invariant region. Since the closed-loop system has no equilibria and is planar, we can conclude that the desired circle is an attractive limit cycle. ■

Last, we need to prove that the agents asymptotically form a regular polygon on the desired circle, or in other words, that $\beta_i(t) \to 2\pi/N$ for all $i \in \{1, \ldots, N\}$. The crucial step to this result is to write down the dynamics of $\beta_i(t)$. Recalling that $\beta_i(t) = \beta_{i,\nu_i(t)}(t)$ whenever $\nu_i(t)$ is defined, let us consider the generic counterclockwise angle $\beta_{ij}(t)$ from $\varphi_i(t)$ to $\varphi_j(t)$. Define the auxiliary variable $c_{ij}(t) = \cos\beta_{ij}(t) = \varphi_i^\intercal(t)\varphi_j(t)$, and take its derivative with respect to time, obtaining

$$\dot{c}_{ij}(t) = \dot{\varphi}_i^\intercal(t)\varphi_j(t) + \varphi_i^\intercal(t)\dot{\varphi}_j(t). \tag{24}$$

Using the dynamics (12) of the bearing vectors with $\dot{x}(t) = 0_2$, we rewrite (24) as

$$\begin{aligned}\dot{c}_{ij}(t) = &- k_\varphi(\alpha + \beta_i(t))\frac{D^*}{D_i(t)}\bar{\varphi}_i^\intercal(t)\varphi_j(t) \\ &- k_\varphi(\alpha + \beta_j(t))\frac{D^*}{D_j(t)}\varphi_i^\intercal(t)\bar{\varphi}_j(t).\end{aligned} \tag{25}$$

Simple trigonometrical properties show that $\bar{\varphi}_i^\intercal(t)\varphi_j(t) = -\sin\beta_{ij}(t)$ and $\varphi_i^\intercal(t)\bar{\varphi}_j(t) = \sin\beta_{ij}(t)$, while from the chain rule we have $\dot{c}_{ij}(t) = -\sin\beta_{ij}(t) \cdot \dot{\beta}_{ij}(t)$. Therefore, we can rewrite (25) as

$$\begin{aligned}-\sin\beta_{ij}(t) \cdot \dot{\beta}_{ij}(t) = &k_\varphi(\alpha + \beta_i(t))\frac{D^*}{D_i(t)}\sin\beta_{ij}(t) \\ &- k_\varphi(\alpha + \beta_j(t))\frac{D^*}{D_j(t)}\sin\beta_{ij}(t).\end{aligned} \tag{26}$$

When $\sin\beta_{ij} \neq 0$, we can rewrite (26) as

$$\dot{\beta}_{ij}(t) = k_\varphi(\alpha + \beta_j(t))\frac{D^*}{D_j(t)} - k_\varphi(\alpha + \beta_i(t))\frac{D^*}{D_i(t)}. \tag{27}$$

The case $\cos\beta_{ij} = 0$ can be handled similarly by considering the auxiliary variable $s_{ij}(t) = \sin\beta_{ij}(t)$, which leads again to (27). Adding and subtracting $k_\varphi(\beta_j(t) - \beta_i(t))$ from the right-hand side of (27) leads to

$$\begin{aligned}\dot{\beta}_{ij}(t) = &- k_\varphi(\beta_i(t) - \beta_j(t)) - k_\varphi\alpha D^*\left(\frac{1}{D_i} - \frac{1}{D_j}\right) \\ &+ k_\varphi\beta_i(t)\frac{\Delta_i(t)}{D_i(t)} - k_\varphi\beta_j(t)\frac{\Delta_j(t)}{D_j(t)}.\end{aligned} \tag{28}$$

Now recall that, for each agent in the network, $\alpha + \beta_i(t)$ is upper-bounded, $D_i(t)$ is lower-bounded, and $\Delta_i(t) = D_i(t) - D^*$ vanishes exponentially. Therefore, the last three addends in the right-hand side of (28) vanishes exponentially, and we shall denote it as $\delta_{ij}(t)$ for brevity. Hence, (28) becomes $\dot{\beta}_{ij}(t) = k_\varphi(\beta_j(t) - \beta_i(t)) + \delta_{ij}(t)$. In particular, we are interested in the case $j = \nu_i(t)$, which leads to

$$\dot{\beta}_i(t) = k_\varphi(\beta_{\nu_i(t)}(t) - \beta_i(t)) + \delta_i(t), \tag{29}$$

where we have denoted $\delta_i(t) = \delta_{i\nu_i(t)}(t)$. We are now in a suitable position to state the final convergence result of this section.

Assumption 2: The communication radius of the agents satisfies $\rho > 2D^*$. ■

Lemma 7: Under Assumptions 1 and 2, estimate law (8) and control law (10), it holds that $\beta_i(t) \to 2\pi/N$ for all the agents in the network.

Proof: Thanks to Assumption 2, and recalling that $D_i(t)$ converges to D^* exponentially, there is a time $T > 0$ such that, for each $t \geq T$, $\mathcal{N}_i(t) = \{1, \ldots, N\} \setminus \{i\}$. In particular, this condition implies that, for each $t \geq T$, each agent i has a counterclockwise neighbor $\nu_i(t)$. In this regime, the dynamics of the angles $\beta_i(t)$ are given by (29). Denoting $\beta(t) = [\beta_1(t), \ldots, \beta_N(t)]$ and similarly for $\delta(t)$, we can rewrite (29) compactly as

$$\dot{\beta}(t) = -L(t)\beta(t) + \delta(t), \tag{30}$$

where $[L(t)]_{ij} = 1$ if $j = i$, $[L(t)]_{ij} = -1$ if $j = \nu_i$ and $[L(t)]_{ij} = 0$ otherwise. Note that $L(t)$ is the Laplacian of a time-varying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}(t))$, with $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E}(t) = \{(\nu_i(t), i) : i \in \mathcal{V}\}$.

First we complete the proof in the case that $\nu_i(t)$ is constant in $[T, \infty)$ for all the agents; then, we show that the proof is easily extended to the case that some agents change their counterclockwise neighbor. If ν_i is constant for all $i \in \{1, \ldots, N\}$, then L is constant, and (30) reduces to a consensus equation over a strongly connected graph with bounded and vanishing disturbances. Under such dynamics, the vector $\beta(t)$ achieves *consensus*, which by definition means that $\beta_i(t) - \beta_j(t) \to 0$ for all pairs $(i, j) \in \{1, \ldots, N\}$. Since the angles $\beta_i(t)$ sum to 2π, consensus is equivalent to $\beta_i(t) \to 2\pi/N$ for all $i \in \{1, \ldots, N\}$, which concludes this part of the proof.

Now consider the case that some agent i changes its counterclockwise neighbor $\nu_i(t)$ at some time $\tau \geq T$ (cfr. Figure 2). Without loss of generality, let $j = \nu_i(\tau^-)$ and $k = \nu_i(\tau^+)$. Since $\mathcal{N}_i(t) = \{1, \ldots, N\} \setminus \{i\}$ for all $t \geq T$, this change cannot be caused by some agent entering or exiting $\mathcal{N}_i(t)$, but must be due $\beta_{ik}(t)$ becoming as small as $\beta_{ij}(t)$ for $t = \tau$, which also implies that $k = \nu_j(\tau^-)$, $j = \nu_k(\tau^+)$ and $\nu_k(\tau^-) = \nu_j(\tau^+)$. Therefore, $L(\tau^+)$ is obtained by $L(\tau^-)$ by simply permuting the jth row (resp., column) with the kth row (resp., column). Moreover, since $\beta_{ij}(\tau) = \beta_{ik}(\tau)$, $\beta_i(t)$ is continuous at $t = \tau$. Conversely, $\beta_j(t)$ and $\beta_k(t)$ switch their values at τ, in fact: $\lim_{t\to\tau^-}\beta_j(t) = \lim_{t\to\tau^-}\beta_{jk}(t) = 0$, $\lim_{t\to\tau^+}\beta_j(t) = \lim_{t\to\tau^-}\beta_k(t)$, and $\lim_{t\to\tau^+}\beta_k(t) = \lim_{t\to\tau^+}\beta_{kj}(t) = 0$. Therefore, $\beta(\tau^+)$ is obtained by $\beta(\tau^-)$ by simply permuting the jth element with the kth element. Hence, the dynamics of $\beta(t)$ are not affected (up to a permutation of two indexes) if some agents change their counterclockwise neighbor, and we can conclude that $\beta_i(t) \to 2\pi/N$. ■

Lemmas 4–7 amount to our main result, which is formalized in the following theorem.

Theorem 1: Consider a network of N autonomous agents under estimate law (8) and control law (10). If Assumptions 1 and 2 hold, the agents converge to the desired circle $\mathcal{C}(x, D^*)$

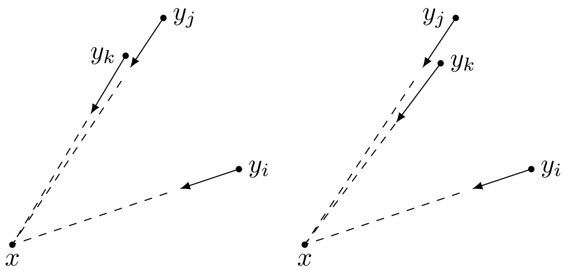

Fig. 2: Example of an agent i changing its counterclockwise neighbor. Left: $t = \tau^-$ and $\nu_i(\tau^-) = j$. Right: $t = \tau^+$ and $\nu_i(\tau^+) = k$.

while forming a regular polygon; i.e., they achieve the control objective (5) and (6). ■

V. Tracking a Mobile Target

In this section, we extend our results to the scenario where the target moves under the constraint $\|\dot{x}(t)\| \leq \epsilon_T$, with $\epsilon_T > 0$. The first result is to prove that the estimate error converges to a neighborhood of zero for all the agents. To this aim, we need the following technical assumption.

Assumption 3: The desired distance from the target satisfies $k_\varphi \alpha D^* - \epsilon_T \geq \omega > 0$. ■

Lemma 8: Under Assumption 3, estimate law (8) and control law (10), the estimate error $\tilde{x}_i(t)$ satisfies

$$\|\tilde{x}_i(t)\| \leq b\|\tilde{x}_i(0)\|e^{-rt} + \frac{\epsilon_T}{r}(1 - e^{-rt}) \tag{31}$$

for all $t \geq 0$, and some $r, b > 0$.

Proof: Using Assumption 3, the proof is similar to that of Lemma 3, and it is omitted for brevity. ■

Now we show that the distance between each agent and the target is always positive. Again we need a technical assumption that the desired circumnavigation distance is large enough with respect to the unknown target motion.

Assumption 4: The desired distance D^* satisfies $D^* > b\|\tilde{x}_i(0)\| + \epsilon_T/r + \epsilon_T/k_d$ for all $i \in \{1, \ldots, N\}$, with r and b as defined in Lemma 8. ■

Lemma 9: Under Assumption 4, the estimate law (8) and the control law (10), the distance from the target $D_i(t)$ is always positive.

Proof: Integrating (13) and reasoning as in Lemma 4 gives

$$\begin{aligned} D_i(t) =& D^*(1 - e^{-k_d t}) + D_i(0)e^{-k_d t} \\ &+ \int_0^t e^{-k_d(t-\tau)}[k_d\tilde{D}_i(\tau) + \dot{x}^\intercal(\tau)\varphi_i(\tau)]d\tau. \end{aligned} \tag{32}$$

Using (17) and (31), we have

$$\tilde{D}_i(t) \geq -b\|\tilde{x}_i(0)\| - \frac{\epsilon_T}{r}. \tag{33}$$

Using (33) in (32), and since $|\dot{x}^\intercal(t)\varphi_i(t)| \leq \epsilon_T$, we have

$$\begin{aligned} D_i(t) \geq& D_i(0)e^{-k_d t} \\ &+ \left(D^* - \frac{\epsilon_T}{k_d} - \frac{\epsilon_T}{r} - b\|\tilde{x}_i(0)\|\right)(1 - e^{-k_d t}). \end{aligned} \tag{34}$$

From Assumption 4, it follows from (34) that $D_i(t) \geq 0$ for all $t \geq 0$ and all $i \in \mathcal{V}$. ■

Lemma 10: Consider a network of N agents under (8) and (10) with a mobile target; then, the distance error $\Delta_i(t)$ converges exponentially to a ball centered at zero, of radius $\epsilon_\Delta = \epsilon_T(1/k_d + 1/r)$.

Proof: Integrating (13), taking the absolute value of both sides, and using the triangular inequality, we have

$$|\Delta_i(t)| \leq |\Delta_i(0)|e^{-k_d t} + \int_0^t e^{-k_d(t-\tau)}[k_d\|\tilde{x}_i(\tau)\| + \epsilon_T]d\tau. \tag{35}$$

Using (31), we can rewrite (35) as

$$\begin{aligned} |\Delta_i(t)| \leq& |\Delta_i(0)|e^{-k_d t} \\ &+ \left(-\frac{k_d b\|\tilde{x}_i(0)\|}{r - k_d} + \frac{k_d\epsilon_T}{r(r - k_d)}\right)e^{-(r+k_d)t} \\ &- \left(\frac{\epsilon_T}{r} + \frac{\epsilon_T}{k_d}\right)e^{-k_d t} + \frac{\epsilon_T}{r} + \frac{\epsilon_T}{k_d}. \end{aligned}$$

Letting $t \to \infty$, we have finally

$$\lim_{t\to\infty} |\Delta_i(t)| \leq \epsilon_T\left(\frac{1}{k_d} + \frac{1}{r}\right) \tag{36}$$

for all $i \in \{1, \ldots, N\}$. ■

In order to prove the bounded convergence of the counterclockwise angles, we introduce the error variables $\tilde{\beta}_i(t) = \beta_i(t) - 2\pi/N$; we define the vector $\tilde{\beta}(t) = [\tilde{\beta}_1(t), \ldots, \tilde{\beta}_N(t)]^\intercal$.

Lemma 11: Under Assumptions 3 and 4, the estimate law (8) and the control law (10), the vector $\tilde{\beta}(t)$ converges exponentially to a ball centered at 0.

Proof: Reasoning as in Lemma 7, we write the dynamics of the counterclockwise angle errors as

$$\dot{\tilde{\beta}}(t) = -k_\varphi L\tilde{\beta}(t) + f_\alpha(t) + f_\beta(t) + f_T(t), \tag{37}$$

where L is the Laplacian matrix of the network graph, defined in the proof of Lemma 7, and

$$\begin{aligned} [f_\alpha(t)]_i =& -k_\varphi\alpha D^*\left(\frac{1}{D_i(t)} - \frac{1}{D_{\nu_i(t)}}\right), \\ [f_\beta(t)]_i =& k_\varphi\left(\beta_i(t)\frac{\Delta_i(t)}{D_i(t)} - \beta_{\nu_i(t)}(t)\frac{\Delta_{\nu_i(t)}(t)}{D_{\nu_i(t)}(t)}\right), \\ [f_T(t)]_i =& \dot{x}^\intercal(t)\left(\frac{\bar{\varphi}_i(t)}{D_i(t)} - \frac{\bar{\varphi}_{\nu_i(t)}(t)}{D_{\nu_i(t)}(t)}\right). \end{aligned}$$

By Theorem 3.17 in [19], the homogeneous form of (37), $\dot{\tilde{\beta}}(t) = -k_\varphi L\tilde{\beta}(t)$, is exponentially stable. According to Lemma 3, in order to prove the bounded convergence of (37), we need to prove the boundedness of $f_\alpha(t) + f_\beta(t) + f_T(t)$. Since $|\beta_i(t)| < 2\pi$ and $|\dot{x}^\intercal(t)\bar{\varphi}_i(t)| \leq \epsilon_T$, using (36), we have, for t sufficiently large, $|f_\alpha(t)| \leq 2\sqrt{N}k_\varphi\alpha D^*/(D^* - \epsilon_\Delta)$, $|f_\beta(t)| \leq 4\pi\sqrt{N}k_\varphi\epsilon_\Delta/(D^* - \epsilon_\Delta)$, and $|f_T(t)| \leq 2\sqrt{N}\epsilon_T/(D^* - \epsilon_\Delta)$. From Lemma 3, we know that there exist $b_\beta > 0$ and $r_\beta = k_\varphi$ such that $\|\tilde{\beta}(t)\| \leq b_\beta\|\tilde{\beta}(0)\|e^{-r_\beta t} + U_\beta/r_\beta(1 - e^{-r_\beta t})$ for all $t \geq 0$, where

$$U_\beta = \frac{\sqrt{N}(2k_\varphi\alpha D^* + 4\pi k_\varphi\epsilon_\Delta + 2\epsilon_T)}{D^* - \epsilon_\Delta}, \tag{38}$$

which, letting $t \to \infty$, yields the desired convergence. ■

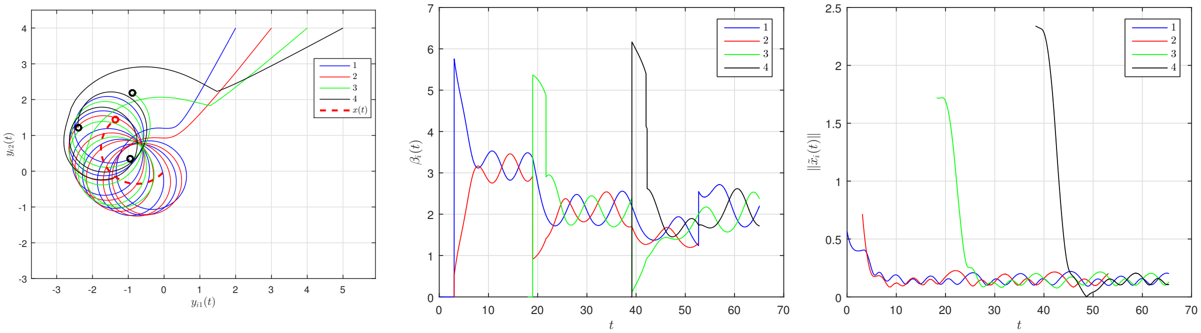

Fig. 3: Results of the simulation. Left: trajectories of the agents. The black circles represent the position of agents 1, 3, 4, and 5 at the end of the simulation (agent 2 is removed at $t = 55$), while the red circle represents the position of the target at the end of the simulation. Center: angles $\beta_i(t)$. Right: estimate errors $\|\tilde{x}_i(t)\|$.

Lemmas 8–11 amount to the main result of this section, which is formalized as the following theorem.

Theorem 2: Consider a network of N autonomous agents under the estimate law (8) and the control law (10), with $\|\dot{x}(t)\| \leq \epsilon_T$. Under Assumption 3 and Assumption 4, the agents converge to an annulus of radii $D^* - \epsilon_\Delta$ and $D^* + \epsilon_\Delta$, containing $\mathcal{C}(x(t), D^*)$, and they are in a formation such that, as $t \to \infty$, $\|\tilde{\beta}(t)\| \leq U_\beta / k_\varphi$, where k_φ is the control gain for the tangential term in (10), and U_β is given by (38). ■

VI. Simulation

In this section, we present a simulation of the proposed algorithm, where we also demonstrate addition and removal of agents. We consider $N = 4$ agents, tracking a target with a slow motion, whose kinematics is $\dot{x}(t) = 0.05[\cos 0.05t,\ \sin 0.05t]^\intercal$, with initial position $[0,\ 0]^\intercal$. The reference distance is $D^* = 1$; notice that, approximating r with $k_e = 0.7$, and recalling that, for the chosen trajectory of the target, $\epsilon_T = 0.07$, D^* satisfies Assumption 4. The agents wait for the detection of $\varphi_i(t)$ in monitoring positions, belonging to the line $y_{i2} = 4$. The gains of the control law are chosen as $k_d = 2, k_\varphi = 0.2, \alpha = 1$, while the estimate gain is chosen as $k_e = 0.7$. The activation times are $a_1 = 2$, $a_2 = 5$, $a_3 = 20$, $a_4 = 40$, $a_5 = 38$, while the removal time for the agent 2 is $r_2 = 55$. The results of the simulation are shown in Figures 3, where we can clearly see the persistent oscillations caused by the target's unknown motion.

VII. Conclusions

We have proposed a distributed algorithm for a problem of target tracking and circumnavigation with a network of planar autonomous agents. The algorithm is amenable to time-varying networks, where agents may come in and out of the network asynchronously. For a stationary target, we have shown that the proposed control algorithm drives the agents to form a regular polygon around the target, while keeping a desired distance from the target. For a mobile target, the agents converge to a region containing the desired configuration, and the size of this region is proportional to the maximum speed of the target. The algorithm is demonstrated in a ROS simulation. Future work includes extending the proposed algorithm to event-triggered communication.

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