Distributed Dynamic Event-Triggered Control for Multi-Agent Systems

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***Abstract*— We propose two distributed dynamic triggering laws to solve the consensus problem for multi-agent systems with event-triggered consensus protocol. Compared with existing triggering laws, the proposed triggering laws involve internal dynamic variables which play an essential role to guarantee that the triggering time sequence does not exhibit Zeno behavior. Some existing triggering laws are special cases of our dynamic triggering laws. Under the condition that the underlying graph is undirected and connected, it is proven that the proposed dynamic triggering laws together with the event-triggered consensus protocol make the state of each agent converges exponentially to the average of the agents' initial states. Numerical simulations illustrate the effectiveness of the theoretical results.**

I. INTRODUCTION

Multi-agent (average) consensus problem, where a group of agents seeks to agree upon certain quantity of interest (e.g., the average of their initial states), has been widely investigated because it has many applications such as mobile robots, autonomous underwater vehicles, unmanned air vehicles, etc. There are many results obtained in this field, such as [1]–[3] and the references therein. In these papers, agents have continuous-time dynamics and actuation. However, in practice, it is in most cases at discrete points in time that agents communicate with their neighbors and take actions. There are also many papers that study the agents with discrete-time dynamics or continuous-time dynamics with discontinuous information transmission, for example see [4]–[6]. In these papers, time-driven sampling is used to determine when agents should establish communication with its neighbors. Time-driven sampling is often implemented by periodic sampling. A significant drawback of periodic sampling is that it requires all agents to exchange their information synchronously, which is not so easy to be realized in real systems, especially when the number of agents is large.

In addition to time-driven sampling, event-driven sampling has been proposed [7], [8]. In event-driven sampling actuation updates and inter-agent communications occur only when some specific events are triggered, for instance, a measure of the state error exceeds a specified threshold. Event-driven sampling is normally implemented by event-triggered or self-triggered control. The event-triggered control is often piecewise constant between the triggering times. The triggering times are determined by the triggering laws. Many researchers studied event-triggered control for multi-agent systems recently [9]–[19]. A key challenge in event-triggered control for multi-agent systems is how to design triggering laws to determine the corresponding triggering times, and to exclude Zeno behavior. For continuous-time multi-agent systems, Zeno behavior means that there are infinite number of triggers in a finite time interval [20].

In [21], by introducing an internal dynamic variable, a new class of event-triggering mechanisms is presented. The idea of using internal dynamic variables in event-triggered and self-triggered control can also be found in [22], [23]. In this paper, we modify the dynamic event triggering mechanism in [21] and extend it to multi-agent systems in a distributed manner.

We have the following main contributions: we propose two dynamic triggering laws which are distributed in the sense that they do not require any a priori knowledge of global network parameters; we prove that the proposed dynamic triggering laws yield consensus exponentially fast; and we show that they are free from Zeno behavior. We show also that the triggering laws in [9]–[11] are special cases of the control laws considered in this paper.

The rest of this paper is organized as follows. Section II introduces the preliminaries. The main results are stated in Section III. Simulations are given in Section IV. Finally, the paper is concluded in Section V.

Notations: $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices. $\mathbf{1}_n$ denotes the column one vector with dimension n. I_n is the n dimension identity matrix. $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues. Given two symmetric matrices M, N, $M \geq N$ means $M - N$ is positive semi-definite. $|S|$ is the cardinality of set S.

II. PRELIMINARIES

In this section, we present some definitions from algebraic graph theory [24] and the considered multi-agent system.

A. Algebraic Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ denote a weighted undirected graph with the set of agents (vertices or nodes) $\mathcal{V} = \{v_1, \dots, v_n\}$, the set of links (edges) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the (weighted) adjacency matrix $A = A^\top = (a_{ij})$ with nonnegative elements a_{ij}. A link of $\mathcal{G}$ is denoted by $(v_i, v_j) \in \mathcal{E}$ if $a_{ij} > 0$, i.e., if agents v_i and v_j can communicate with each other. It is assumed that $a_{ii} = 0$ for all $i \in \mathcal{I}$, where $\mathcal{I} = \{1, \dots, n\}$. Let $\mathcal{N}_i = \{j \in \mathcal{I} \mid a_{ij} > 0\}$

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and $\deg_i = \sum_{j=1}^{n} a_{ij}$ denotes the neighbors' index set and weighted degree of agent v_i, respectively. The degree matrix of graph $\mathcal{G}$ is $\mathrm{Deg} = \mathrm{diag}([\deg_1, \cdots, \deg_n])$. The Laplacian matrix is $L = (L_{ij}) = \mathrm{Deg} - A$. A path of length k between agent v_i and agent v_j is a subgraph with distinct agents $v_{i_0} = v_i, \ldots, v_{i_k} = v_j \in \mathcal{V}$ and edges $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E},\ j = 0, \ldots, k-1$. An undirected graph is connected if there exists at least one path between any two agents. For a connected graph we have the following well known results.

Lemma 1: Let $K_n = I_n - \frac{1}{n}\mathbf{1_n}\mathbf{1_n}^\top$ and assume graph $\mathcal{G}$ is connected, then its Laplacian matrix L is positive semi-definite. Moreover, we have

$$0 \le \rho_2(L) K_n \le L. \tag{1}$$

Proof: For the proof of (1), please see Lemma 2.1 in [18].

B. System Model

We consider a set of n agents that are modelled as single integrators

$$\dot{x}_i(t) = u_i(t),\ i \in \mathcal{I}, t \ge 0, \tag{2}$$

where $x_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$ is the control input.

In the literature, the distributed consensus protocol, $u_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t)$, is often considered, e.g., [1], [2]. To implement the this consensus protocol, continuous-time state information from neighbors is needed. However, it is often impractical to require continuous communication in physical applications.

Inspired by the idea of event-triggered control for multi-agent systems [9], we use the following event-triggered consensus protocol instead

$$u_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t^j_{k_j(t)}), \tag{3}$$

where $k_j(t) = \mathrm{argmax}_k \{t^j_k \le t\}$ with the increasing $\{t^j_k\}_{k=1}^\infty,\ j \in \mathcal{I}$ to be determined later. We assume $t^j_1 = 0, j \in \mathcal{I}$. Note that the control protocol (3) only updates at the triggering times and is constant between two consecutive triggering times.

For simplicity, let $x(t) = [x_1(t), \ldots, x_n(t)]^\top$, $\hat{x}_i(t) = x_i(t^i_{k_i(t)})$, $\hat{x}(t) = [\hat{x}_1(t), \ldots, \hat{x}_n(t)]^\top$, $e_i(t) = \hat{x}_i(t) - x_i(t)$, and $e(t) = [e_1(t), \cdots, e_n(t)]^\top = \hat{x}(t) - x(t)$. Then we can rewrite the multi-agent system (2)–(3) in the stack vector form $\dot{x}(t) = -L\hat{x}(t) = -L(x(t) + e(t))$.

III. DYNAMIC EVENT-TRIGGERED CONTROL

In this section, we propose the dynamic triggering laws to determine the triggering time sequence and we prove that they lead to consensus for the multi-agent system (2)–(3).

A. Continuous Approach

We first give the following lemma.

Lemma 2: Consider the multi-agent system (2)–(3). Suppose that $\mathcal{G}$ is undirected. The average of all agents' states $\bar{x}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t)$ is a constant, i.e., $\bar{x}(t) = \bar{x}(0), \forall t \ge 0$.

Proof: This is straightforward since $\dot{\bar{x}}(t) = 0$.

Consider a Lyapunov candidate:

$$V(t) = \frac{1}{2} x^\top(t) K_n x(t) = \frac{1}{2}\sum_{i=1}^{n} [x_i(t) - \bar{x}(0)]^2. \tag{4}$$

Then the derivative of $V(t)$ along the trajectories of (2)–(3) satisfies

$$\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{n} [x_i(t) - \bar{x}(0)] \dot{x}_i(t) \\
=& -\sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} L_{ij}(x_j(t) + e_j(t)) \\
\overset{*}{=}& -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n} \sum_{j=1, j\ne i}^{n} e_i(t) L_{ij}(x_j(t) - x_i(t)) \\
\le& -\sum_{i=1}^{n} q_i(t) - \sum_{i=1}^{n} \sum_{j=1, j\ne i}^{n} L_{ij} e_i^2(t) \\
& -\sum_{i=1}^{n} \sum_{j=1, j\ne i}^{n} L_{ij} \frac{1}{4}(x_j(t) - x_i(t))^2 \\
\overset{*}{=}& -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t),
\end{aligned} \tag{5}$$

where

$$q_i(t) = -\frac{1}{2}\sum_{j=1}^{n} L_{ij}(x_j(t) - x_i(t))^2 \ge 0, \tag{6}$$

and the equalities denoted by $\overset{*}{=}$ hold since

$$\sum_{i=1}^{n} q_i(t) = -\sum_{i=1}^{n} \frac{1}{2}\sum_{j=1}^{n} L_{ij}(x_j(t) - x_i(t))^2 = x^\top(t) L x(t),$$

and the inequality holds since $ab \le a^2 + \frac{1}{4}b^2$.

Similar to [9] and [17], if we use the following law to determine the triggering time sequence:

$$\begin{aligned}
t^i_1 =& 0 \\
t^i_{k+1} =& \max_{r \ge t^i_k} \Big\{ r : e_i^2(t) \le \frac{\sigma_i}{2L_{ii}} q_i(t), \forall t \in [t^i_k, r] \Big\},
\end{aligned} \tag{7}$$

with $\sigma_i \in (0, 1)$, then, from (5) and (7), we have

$$\begin{aligned}
\dot{V}(t) \le& -\sum_{i=1}^{n} \frac{1}{2} q_i(t) + \sum_{i=1}^{n} L_{ii} e_i^2(t) \\
\le& -\frac{1}{2}(1 - \sigma_{\max}) \sum_{i=1}^{n} q_i(t) \\
=& -\frac{1}{2}(1 - \sigma_{\max}) x^\top(t) L x(t) \\
\le& -(1 - \sigma_{\max}) \rho_2(L) V(t),
\end{aligned} \tag{8}$$

where $\sigma_{\max} = \max\{\sigma_1, \ldots, \sigma_n\} < 1$ and the last inequality holds due to (1). Then $V(t) \le V(0) e^{-(1-\sigma_{\max})\rho_2(L)t}$. This implies that system (2)–(3) under triggering law (7) reaches consensus exponentially.

Remark 1: We refer to (7) as a static triggering law since it does not involve any extra dynamic variables except

$x_i(t), \hat{x}_i(t)$ and $x_j(t), j \in \mathcal{N}_i$. The static triggering law (7) is distributed since each agent's control action only depends on its own state information and its neighbors' state information, without any a prior knowledge of any global parameters, such as the eigenvalues of the Laplacian matrix.

Remark 2: If we consider the same graph that considered in [9], i.e., $a_{ij} = 1$ if $(i,j) \in \mathcal{E}$, then $L_{ii} = |\mathcal{N}_i|$. From the facts $a(1 - a|\mathcal{N}_i|) \leq \frac{1}{4|\mathcal{N}_i|}$ and $(\sum_{j=1}^n (x_j(t) - x_i(t)))^2 \leq 2|\mathcal{N}_i| \sum_{j=1}^n (x_j(t) - x_i(t))^2$, we have $\frac{\sigma_i a(1-a|\mathcal{N}_i|)}{|\mathcal{N}_i|}(\sum_{j=1}^n (x_j(t) - x_i(t)))^2 \leq \frac{\sigma_i}{2|\mathcal{N}_i|} q_i(t)$. In other words, the distributed triggering law (10) in [9] is a special case of the static triggering law (7).

The main purpose of using the event-triggered control is to reduce the overall need of actuation updates and communication between agents, so it is essential to exclude Zeno behavior. However, we do not know whether Zeno behavior can be excluded or not in the above triggering law. In order to explicitly exclude Zeno behavior, in the following we propose a dynamic triggering law.

Inspired by [21], we propose the following internal dynamic variable η_i to agent v_i:

$$\dot{\eta}_i(t) = -\beta_i \eta_i(t) + \xi_i(\frac{\sigma_i}{2} q_i(t) - L_{ii} e_i^2(t)), i \in \mathcal{I}, \quad (9)$$

with $\eta_i(0) > 0$, $\beta_i > 0$, $\xi_i \in [0, 1]$, and $\sigma_i \in [0, 1)$. These dynamic variables are correlated in the triggering law, as defined in our first main result.

Theorem 1: Consider the multi-agent system (2)–(3). Suppose that $\mathcal{G}$ is undirected and connected. Given $\theta_i > \frac{1-\xi_i}{\beta_i}$ and the first triggering time $t_1^i = 0$, agent v_i determines the triggering time sequence $\{t_k^i\}_{k=2}^\infty$ by

$$t_{k+1}^i = \max_{r \geq t_k^i} \Big\{ r : \theta_i \Big(L_{ii} e_i^2(t) - \frac{\sigma_i}{2} q_i(t) \Big) \leq \eta_i(t), \quad \forall t \in [t_k^i, r] \Big\}, \quad (10)$$

with $q_i(t)$ defined in (6) and $\eta_i(t)$ defined in (9). Then the consensus is achieved exponentially and there is no Zeno behavior.

Proof: (i) From equation (9) and condition (10), we have $\dot{\eta}_i(t) \geq -\beta_i \eta_i(t) - \frac{\xi_i}{\theta_i} \eta_i(t)$. Thus

$$\eta_i(t) \geq \eta_i(0) e^{-(\beta_i + \frac{\xi_i}{\theta_i})t} > 0. \quad (11)$$

Consider a Lyapunov candidate: $W(t) = V(t) + \sum_{i=1}^n \eta_i(t)$. Then the derivative of $W(t)$ along the trajectories of (2)–(3) and (9) satisfies

$$\begin{aligned}
\dot{W}(t) &= \dot{V}(t) + \sum_{i=1}^n \dot{\eta}_i(t) \\
&\leq - \sum_{i=1}^n \frac{1}{2} q_i(t) + \sum_{i=1}^n L_{ii} e_i^2(t) - \sum_{i=1}^n \beta_i \eta_i(t) \\
&\quad + \sum_{i=1}^n \xi_i (\frac{\sigma_i}{2} q_i(t) - L_{ii} e_i^2(t)) \\
&\leq - \sum_{i=1}^n \frac{1}{2}(1 - \sigma_i) q_i(t) - \sum_{i=1}^n \beta_i \eta_i(t) + \sum_{i=1}^n \frac{1 - \xi_i}{\theta_i} \eta_i(t) \\
&\leq -(1 - \sigma_{\max}) \rho_2(L) V(t) - k_d \sum_{i=1}^n \eta_i(t) \leq -k_W W(t),
\end{aligned}$$

where $k_d = \min_i \{\beta_i - \frac{1-\xi_i}{\theta_i}\} > 0$ and $k_W = \min\{(1 - \sigma_{\max})\rho_2(L), k_d\} > 0$. Then

$$V(t) < W(t) \leq W(0) e^{-k_W t}. \quad (12)$$

This implies that system (2)–(3) reaches consensus exponentially.

(ii) Next, we prove that there is no Zeno behavior by contradiction. Suppose there exists Zeno behavior. Then there exists an agent v_i, such that $\lim_{k \to +\infty} t_k^i = T_0$ where T_0 is a positive constant.

From (12), we know that there exists a positive constant $M_0 > 0$ such that $|x_i(t)| \leq M_0$ for all $t \geq 0$ and $i = 1, \ldots, n$. Then, we have $|u_i(t)| \leq 2M_0 L_{ii}$.

Let $\varepsilon_0 = \frac{\sqrt{\eta_i(0)}}{4\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})T_0} > 0$. Then from the property of limits, there exists a positive integer $N(\varepsilon_0)$ such that

$$t_k^i \in [T_0 - \varepsilon_0, T_0], \ \forall k \geq N(\varepsilon_0). \quad (13)$$

Noting $q_i(t) \geq 0$ and (11), we can conclude that one sufficient condition to guarantee the inequality in condition (10) is

$$|\hat{x}_i(t) - x_i(t)| \leq \sqrt{\frac{\eta_i(0)}{\theta_i L_{ii}}} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})t}. \quad (14)$$

Again noting $|\dot{x}_i(t)| = |u_i(t)| \leq 2M_0 L_{ii}$ and $|\hat{x}_i(t_k^i) - x_i(t_k^i)| = 0$ for any triggering time t_k^i, we can conclude that one sufficient condition to the above inequality is

$$(t - t_k^i) 2M_0 L_{ii} \leq \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})t}. \quad (15)$$

Then

$$\begin{aligned}
t_{N(\varepsilon_0)+1}^i - t_{N(\varepsilon_0)}^i &\geq \frac{\sqrt{\eta_i(0)}}{2\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i}) t_{N(\varepsilon_0)+1}^i} \\
&\geq \frac{\sqrt{\eta_i(0)}}{2\sqrt{\theta_i L_{ii}^3} M_0} e^{-\frac{1}{2}(\beta_i + \frac{\xi_i}{\theta_i})T_0} = 2\varepsilon_0, \quad (16)
\end{aligned}$$

which contradicts to (13). Therefore, Zeno behavior is excluded.

Remark 3: We refer to (10) as a dynamic triggering law since it involves the extra dynamic variables $\eta_i(t)$. Similar to the static triggering law (7), it is also distributed. The static triggering law (7) can be seen as a limit case of the dynamic triggering law (10) when θ_i grows large. Thus, from the analysis in Remark 2, we can conclude that the distributed triggering law (10) in [9] is a special case of the dynamic triggering law (10).

Remark 4: If we choose $\xi_i = 0$ in (9) and $\sigma_i = 0$ in (10), then $\eta_i(t) = \eta_i(0) e^{-\beta_i t}$ and now the inequality in (10) is $|e_i(t)| \leq \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}} e^{-\frac{\beta_i}{2} t}$. This is the triggering function (7) in [11] with $c_0 = 0, c_1 = \frac{\sqrt{\eta_i(0)}}{\sqrt{\theta_i L_{ii}}}, \alpha = \frac{\beta_i}{2}$. However, we do not need the constraint $\alpha < \rho_2(L)$ which is necessary in [11].

B. Discontinuous Approach

In the above static and dynamic triggering laws, in order to check the inequalities in (7) and (10), each agent still needs to continuously monitor its neighbors's states, which means continuous communication is still needed. In the following, we will modify the above results to avoid this.

To do so, we first upper-bound the derivative of $V(t)$ along the trajectories of (2)–(3) by a different way. Similar to the derivation process to get (5), we have

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} -L_{ij}\hat{x}_j(t) \\ &= -\sum_{i=1}^{n}(\hat{x}_i(t) - e_i(t)) \sum_{j=1}^{n} L_{ij}\hat{x}_j(t) \\ &\overset{**}{=} -\sum_{i=1}^{n}\hat{q}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} e_i(t)L_{ij}\hat{x}_j(t) \\ &\le -\sum_{i=1}^{n}\hat{q}_i(t) - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} L_{ij}e_i^2(t) \\ &\quad -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} L_{ij}\frac{1}{4}(\hat{x}_j(t)-\hat{x}_i(t))^2 \\ &\overset{**}{=} -\sum_{i=1}^{n}\frac{1}{2}\hat{q}_i(t) + \sum_{i=1}^{n} L_{ii}e_i^2(t), \end{aligned} \tag{17}$$

where

$$\hat{q}_i(t) = -\frac{1}{2}\sum_{j=1}^{n} L_{ij}(\hat{x}_j(t) - \hat{x}_i(t))^2 \ge 0, \tag{18}$$

and the equalities denoted by $\overset{**}{=}$ hold since

$$\sum_{i=1}^{n}\hat{q}_i(t) = -\sum_{i=1}^{n}\frac{1}{2}\sum_{j=1}^{n} L_{ij}(\hat{x}_j(t)-\hat{x}_i(t))^2 = \hat{x}^\top(t)L\hat{x}(t).$$

Similar to [10] and [17], if we use the following law to determine the triggering time sequence:

$$\begin{aligned} t_1^i &= 0 \\ t_{k+1}^i &= \max_{r\ge t_k^i}\Big\{r : e_i^2(t) \le \frac{\sigma_i}{2L_{ii}}\hat{q}_i(t), \forall t\in[t_k^i, r]\Big\}, \end{aligned} \tag{19}$$

with $\sigma_i \in (0,1)$, then, from (17) and (19), we have

$$\begin{aligned}\dot{V}(t) &\le -\sum_{i=1}^{n}\frac{1}{2}\hat{q}_i(t) + \sum_{i=1}^{n} L_{ii}e_i^2(t) \\ &\le -\frac{1}{2}(1-\sigma_{\max})\sum_{i=1}^{n}\hat{q}_i(t) \\ &= -\frac{1}{2}(1-\sigma_{\max})\hat{x}^\top(t)L\hat{x}(t). \end{aligned} \tag{20}$$

Noting

$$\begin{aligned} x^\top(t)Lx(t) &= (\hat{x}(t)+e(t))^\top L(\hat{x}(t)+e(t)) \\ &\le 2\hat{x}^\top(t)L\hat{x}(t) + 2e^\top(t)Le(t) \\ &\le 2\hat{x}^\top(t)L\hat{x}(t) + 2\|L\|\|e(t)\|^2 \\ &\le 2\hat{x}^\top(t)L\hat{x}(t) + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\sum_{i=1}^{n}\hat{q}_i(t) \\ &= \Big(2 + \frac{\|L\|\sigma_{\max}}{\min_i L_{ii}}\Big)\hat{x}^\top(t)L\hat{x}(t), \end{aligned} \tag{21}$$

where the first inequality holds since L is positive semi-definite and $a^\top Lb \le 2a^\top La + 2b^\top Lb, \forall a,b\in\mathbb{R}^n$, the second inequality holds since $a^\top La \le \|L\|\|a\|^2, \forall a\in\mathbb{R}^n$, and the last inequality holds due to (19), we then have

$$\begin{aligned}\dot{V}(t) &\le -\frac{(1-\sigma_{\max})\min_i L_{ii}}{4\min_i L_{ii} + 2\|L\|\sigma_{\max}} x^\top(t)Lx(t) \\ &\le -\frac{(1-\sigma_{\max})\min_i L_{ii}}{2\min_i L_{ii} + \|L\|\sigma_{\max}}\rho_2(L)V(t). \end{aligned}$$

Then $V(t) \le V(0)e^{-\frac{(1-\sigma_{\max})\min_i L_{ii}}{2\min_i L_{ii}+\|L\|\sigma_{\max}}\rho_2(L)t}$. This implies that system (2)–(3) under the triggering law (19) reaches consensus exponentially.

Remark 5: Similar to the analysis in Remark 1, (19) is a static triggering law and it is also distributed. Moreover, similar to the analysis in Remark 2, we can conclude that the distributed triggering law (6) in [10] is a special case of the static triggering law (19).

We also do not know whether Zeno behavior can be excluded or not in the static triggering law (19). In the following, in order to explicitly exclude Zeno behavior, we will change the static triggering law (19) to the dynamic one.

Similar to (9), we propose an internal dynamic variable χ_i to agent v_i:

$$\dot{\chi}_i(t) = -\beta_i\chi_i(t) + \xi_i\big(\frac{\sigma_i}{2}\hat{q}_i(t) - L_{ii}e_i^2(t)\big), i\in\mathcal{I} \tag{22}$$

with $\chi_i(0) > 0$, $\beta_i > 0$, $\xi_i \in [0,1]$, and $\sigma_i \in [0,1)$. Our second main result is given in the following theorem.

Theorem 2: Consider the multi-agent system (2)–(3). Suppose that $\mathcal{G}$ is undirected and connected. Given $\theta_i > \frac{1-\xi_i}{\beta_i}$ and the first triggering time $t_1^i = 0$, agent v_i determines the triggering time sequence $\{t_k^i\}_{k=2}^{\infty}$ by

$$\begin{aligned} t_{k+1}^i = \max_{r\ge t_k^i}\Big\{r : \theta_i\Big(L_{ii}e_i^2(t) - \frac{\sigma_i}{2}\hat{q}_i(t)\Big) \le \chi_i(t), \\ \forall t\in[t_k^i, r]\Big\}, \end{aligned} \tag{23}$$

with $\hat{q}_i(t)$ defined in (18) and $\chi_i(t)$ defined in (22). Then the consensus is achieved exponentially and there is no Zeno behavior.

Proof: The proof is similar to the proof in Theorem 1. We thus omit the proof here.

Remark 6: Obviously, the triggering law (23) is dynamic and it is also distributed. One can easily check that every agent does not need to continuously access its neighbors' states when implementing the static and dynamic triggering laws (19) and (23).

Remark 7: The static triggering law (19) can be seen as a limit case of the dynamic triggering law (23) when θ_i grows large. Thus, from the analysis in Remark 5, we can conclude that the distributed triggering law (6) in [10] is a special case of the dynamic triggering tlaw (23).

IV. SIMULATIONS

In this section, a numerical example is given to demonstrate the theoretical results. Consider a connected network of four agents with the Laplacian matrix

$$L = \begin{bmatrix} 3.4 & -3.4 & 0 & 0 \\ -3.4 & 9.8 & -2.1 & -4.3 \\ 0 & -2.1 & 3.2 & -1.1 \\ 0 & -4.3 & -1.1 & 5.4 \end{bmatrix}.$$

We choose an arbitrary initial state $x(0) = [6.2945, 8.1158, -7.4603, 8.2675]^\top$, the average initial state is $\bar{x}(0) = 3.8044$. Fig. 1 (a) shows the state evolutions of the multi-agent system (2)–(3) under the static triggering law (7) with $\sigma_i = 0.5$. Fig. 1 (b) shows the corresponding triggering times for each agent. Fig. 2 (a) shows the state evolutions under the dynamic triggering law (10) with $\sigma_i = 0.5$, $\eta_i(0) = 10$, $\beta_i = 1$, $\xi_i = 1$ and $\theta_i = 1$. Fig. 2 (b) shows the corresponding triggering times for each agent. Fig. 3 (a) shows the state evolutions under the static triggering law (19) with $\sigma_i = 0.5$. Fig. 3 (b) shows the corresponding triggering times for each agent. Fig. 4 (a) shows the state evolutions under the dynamic triggering law (23) with $\sigma_i = 0.5$, $\chi_i(0) = 10$, $\beta_i = 1$, $\xi_i = 1$ and $\theta_i = 1$. Fig. 4 (b) shows the corresponding triggering times for each agent. It can be seen that consensus is achieved when performing the four triggering laws proposed in this paper. Moreover, just as Theorem 1 and Theorem 2 point out, from the simulations we can also see that there is no Zeno behavior under the dynamic triggering law (10) and the dynamic triggering law (23). Although there is also no Zeno behavior under the static triggering law (7) and the static triggering law (19) in the simulations, we still do not know how to prove this in theory. Moreover, the numbers of triggering times determined by dynamic triggering laws are less than that determined by static triggering laws.

V. CONCLUSION

In this paper, we presented two dynamic triggering laws for multi-agent systems with event-triggered control. We showed that, some existing triggering laws are special cases of the proposed dynamic triggering laws and if the communication graph is undirected and connected, consensus is achieved exponentially. In addition, Zeno behavior was excluded by proving that the triggering time sequence of each agent is divergent. Future research directions include considering general linear multi-agent systems and dynamic self-triggered control.

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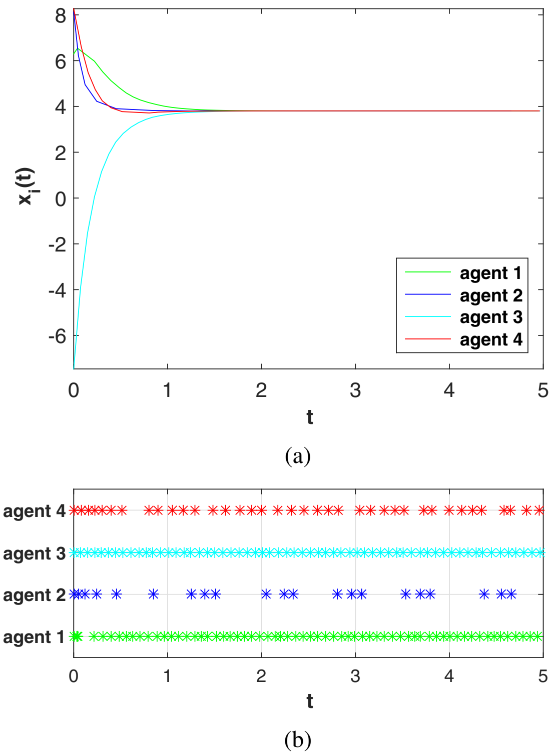


Fig. 1: (a) The state evolutions under the static triggering law (7). (b) The triggering times for each agent.

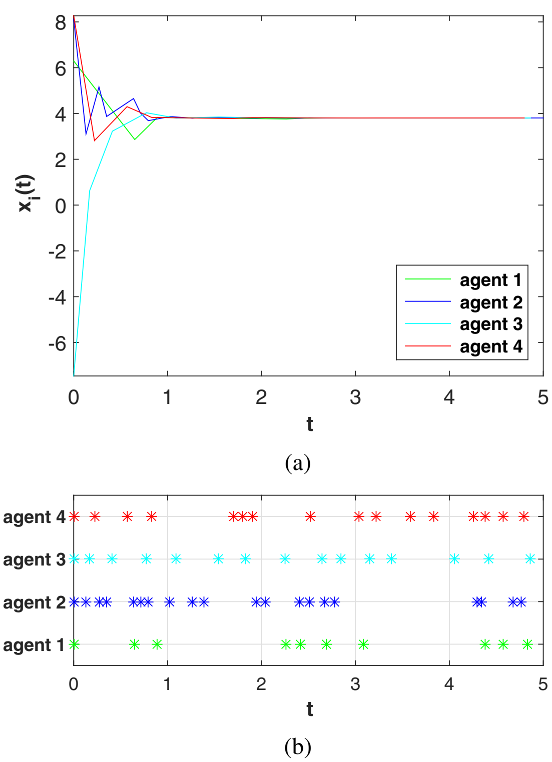
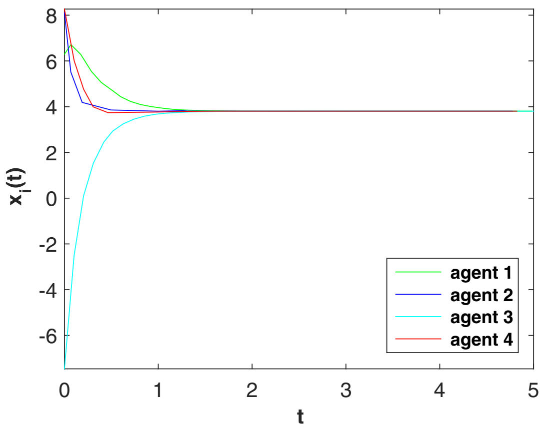
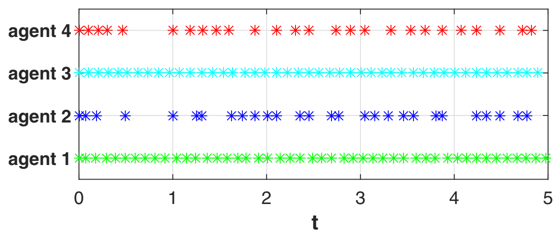


Fig. 2: (a) The state evolutions under the dynamic triggering law (10). (b) The triggering times for each agent.

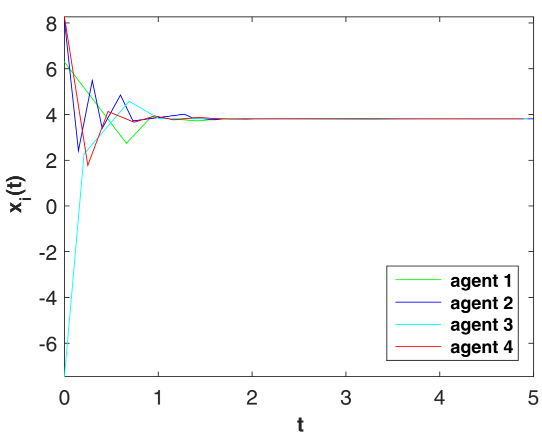


(a)

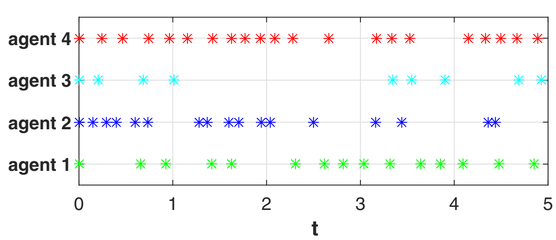


(b)

Fig. 3: (a) The state evolutions under the static triggering law (19). (b) The triggering times for each agent.



(a)



(b)

Fig. 4: (a) The state evolutions under the dynamic triggering law (23). (b) The triggering times for each agent.

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