Time-constrained Event-triggered Model Predictive Control for Nonlinear Continuous-time Systems

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***Abstract*—In this paper we propose a new event-triggered scheme for nonlinear continuous-time systems with additive bounded disturbances. Unlike existing results, the proposed event-triggered strategy is not derived from Lyapunov stability analysis. Instead, it is obtained from the time interval when the state reaches a local region around the origin. By guaranteeing that this time interval becomes smaller as the optimal control problem is solved, we ensure that the state converges to the prescribed set in finite time.**

I. Introduction

Model Predictive Control (MPC) has been one of the most popular control strategies for both linear and nonlinear systems [1]. In this control scheme, the control action is determined by solving an Optimal Control Problem (OCP) online, based on the knowledge about the dynamics and the current state of the plant. Many results haveen obtained over the last decades, such as stability, feasibility and robustness in the face of disturbances, which are all critical to many control applications, see e.g., [2]–[4].

In another direction of research from networked control systems, an aperiodic formulation of executing control tasks in contrast to typical periodic executions has been receiving attention in recent years. This control strategy is known as 'Event-triggered control', and the reader may refer to [5], [6]. The application of event-triggered control is motivated by the fact that by reducing the frequency of executing control tasks, we may achieve the reduction of over-usage of communication resources and the energy consumption of battery powered devices. Event-triggered strategies have been derived from different performance guarantees, such as $\mathcal{L}_2$ and $\mathcal{L}_\infty$ gain stability [7], [8] and Input to State Stability (ISS) [9].

In this paper, we are interested in applying event-triggered strategy to MPC, where the OCP is solved only when certain prescribed performances cannot be guaranteed. Introducing the aperiodic formulation to the MPC framework, not necessarily limited to the scope of networked control systems, has potential advantages over the conventional periodic MPC, since it could alleviate computation load by reducing the number of solving OCPs. The aperiodic framework for MPC has been recently examined in [10]–[12] for the linear case, [13] for the nonlinear case with no disturbances, and [14], [15] for the nonlinear case with additive disturbances. In this paper, we limit the scope to the nonlinear case with additive disturbances. For nonlinear systems, stability has been mainly analysed by checking if the optimal cost, regarded as a Lyapunov function candidate, is decreasing. In the formulation of event-triggered MPC, therefore, the OCP is solved only when the optimal cost is not guaranteed to decrease as proposed in [14], [15].

In contrast to the above event-triggered strategies, we take another approach which does not use the Lyapunov stability analysis. The main motivations of this are as follows:

(i) In case of nonlinear systems, the stability analysis and the corresponding event-triggered rules involve several parameters that are in general not known explicitly, if there are no state constraints. An example of such parameters is the Lipschitz constant of the stage cost, see e.g., [3], [14]. Thus, it is sometimes not suitable to include the parameters in the event-triggered strategies.

(ii) Checking if the optimal cost is decreasing may provide a complex structure of the event-triggered condition, which could yield delays due to the computation time. This may be critical when the triggering rule is 'event-based' case, where the condition needs to be checked continuously by monitoring the current state measurement.

Motivated by the above problems, our approach does not involve the optimal cost as a Lyapunov function candidate; instead, the performance will be evaluated by the *time interval*, from the current time to when the state reaches to a local region around the origin. By guaranteeing that this time interval becomes smaller as the OCP is solved, the state becomes closer to the origin and eventually converges to a prescribed set in finite time. The event-triggered rule is then proposed as one of the general types of event-triggered strategies, in which the OCP is solved only when the error between the actual and predictive state exceeds a certain threshold.

The remainder of this paper is organized as follows: in Section II the optimal control problem is formulated. In Section III, we provide the event-triggering rule and show the corresponding feasibility. In Section IV, we analyze the convergence properties of the proposed scheme. In Section V, we additionally propose a self-triggered strategy. In Section VI, a simulation result is provided. Finally, in Section VII the conclusion is given.

The notations used in the sequel are as follows. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}_{\geq 0}$, $\mathbb{N}_{\geq 1}$ be the real, non-negative real, non-negative integers and positive integers, respectively. We denote $||x||$ as a Euclidean norm of vector x, and $||x||_P$ as a weighted norm of vector x, i.e., $||x||_P = \sqrt{x^\mathsf{T} P x}$. Given a compact set

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$\Phi \subseteq \mathbb{R}^n$, we denote $\partial\Phi$ as the boundary of Φ. The function $\phi(x,u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called Lipschitz continuous with a weighted matrix P and Lipschitz constant L_ϕ in $x \in \Omega$, if $||\phi(x_1,u) - \phi(x_2,u)||_P \leq L_\phi ||x_1 - x_2||_P$ where $x_1, x_2 \in \Omega$.

II. Problem Formulation

We consider applying model predictive control to the following nonlinear system:

$$\dot{x}(t) = \phi(x(t), u(t)) + w(t) \tag{1}$$

subject to

$$u(t) \in \mathcal{U} \subseteq \mathbb{R}^m, \quad w(t) \in \mathcal{W} \subseteq \mathbb{R}^n, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state of the plant, $u(t) \in \mathbb{R}^m$ is the control input, and $w(t) \in \mathbb{R}^n$ is the additive bounded disturbance. The constraint sets for the control input and the disturbance $\mathcal{U}$, $\mathcal{W}$, are assumed to be compact containing the origin in their interiors. We let t_k, $k \in \mathbb{N}_{\geq 0}$ be sampling instants when the OCP is solved, and Δ_k be the sampling intervals, i.e., $\Delta_k = t_{k+1} - t_k$. At t_k, the controller solves the OCP involving the predictive state trajectory denoted as $\hat{x}(\xi)$ and the control input $u(\xi)$ for $\xi \in [t_k, t_k + T_p]$, based on the current state $x(t_k)$ where T_p is the prediction horizon. We consider the following quadratic cost function to be minimized:

$$J(x(t_k), u(\cdot)) = \int_{t_k}^{t_k+T_p} ||\hat{x}(\xi)||_Q^2 + ||u(\xi)||_R^2 \mathrm{d}\xi,$$

where $Q = Q^\mathsf{T} \succ 0$, $R = R^\mathsf{T} \succ 0$.

Before stating the OCP, we define a set Φ given by

$$\Phi = \{x \in \mathbb{R}^n : V_f(x) \leq \varepsilon^2\},$$

where $V_f(x) = x^\mathsf{T} P x$ and $P = P^\mathsf{T} \succ 0$. Regarding the parameter ε and the matrix P, we assume the following :

Assumption 1. *There exists a local stabilizing controller $\kappa(x) = Kx \in \mathcal{U}$ satisfying*

$$\frac{\mathrm{d}V_f}{\mathrm{d}x}\,\phi(x, Kx) \leq -x^\mathsf{T}(Q + K^\mathsf{T} R K)x \tag{3}$$

for all $x \in \Phi$.

Many methods have been proposed to numerically calculate ε and P satisfying (3), see e.g., [2]. Assumption 3 has been useful to show stability of MPC for both linear and nonlinear systems. Although this paper does not use the Lyapunov analysis to guarantee stability, this assumption will be useful to guarantee feasibility provided in the next section.

Regarding the nonlinear dynamics, we assume the following:

Assumption 2. *The function $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^2, $\phi(0,0) = 0$, and Lipschitz continuous with a weighted matrix P and Lipschitz constant L_ϕ in $x \in \mathbb{R}^n$.*

We further define a set $\mathcal{X}_f$ given by

$$\mathcal{X}_f = \{x \in \mathbb{R}^n : V_f(x) \leq \varepsilon_f^2\},$$

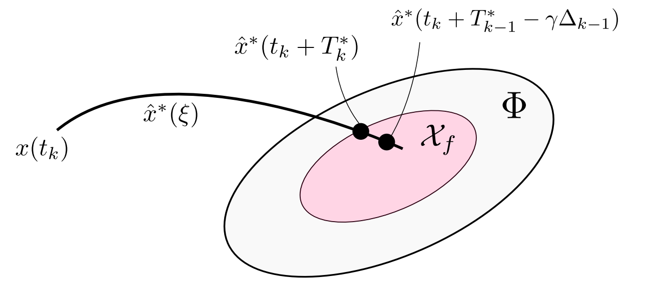

Fig. 1. Graphical representation of the two regions Φ, $\mathcal{X}_f$ and the optimal state trajectory $\hat{x}^*$. T^*_k denotes the time interval to reach $\mathcal{X}_f$. Due to the constraint (6), T^*_k satisfies $T^*_k \leq T^*_{k-1} - \gamma\Delta_{k-1} < T^*_{k-1}$.

where $0 < \varepsilon_f < \varepsilon$. Thus the set $\mathcal{X}_f$ is smaller than Φ, i.e., $\mathcal{X}_f \subset \Phi$. The illustration of the regions $\mathcal{X}_f$ and Φ are depicted in Fig. 1.

By minimizing $J(x(t_k), u(\cdot))$, we aim at finding an optimal control trajectory $u^*(\xi)$ and the corresponding state $\hat{x}^*(\xi)$ for $\xi \in [t_k, t_k + T_p]$, subject to several constraints we will define in the following. One of the constraints imposed in this paper is that the optimal predictive state $\hat{x}^*(\xi)$ converges to $\mathcal{X}_f$. The illustration of $\hat{x}^*(\xi)$ is also shown Fig. 1. Here, we further denote T^*_k, $k \in \mathbb{N}_{\geq 0}$ as the time interval when the optimal state reaches $\mathcal{X}_f$, i.e., $\hat{x}^*(t_k + T^*_k) \in \partial\mathcal{X}_f$. Based on above notations, we define the following OCP:

(Problem 1) : For t_k, $k \in \mathbb{N}_{\geq 1}$, given $x(t_k)$ and T^*_{k-1}, find the optimal control and predictive state trajectory $u^*(\xi)$, $\hat{x}^*(\xi)$ for all $\xi \in [t_k, t_k + T_p]$ that minimizes $J(x(t_k), u(\cdot))$, subject to

$$\begin{cases} \dot{\hat{x}}(\xi) = \phi(\hat{x}(\xi), u(\xi)),\ \xi \in [t_k, t_k + T_p] & (4) \\ u(\xi) \in \mathcal{U} & (5) \\ \hat{x}(t_k + T^*_{k-1} - \gamma\Delta_{k-1}) \in \mathcal{X}_f & (6) \end{cases}$$

where $0 < \gamma < 1$ and $\Delta_{k-1} = t_k - t_{k-1}$. For the initial time t_0, $k = 0$, find $u^*(\xi)$, $\hat{x}^*(\xi)$ for all $\xi \in [t_0, t_0 + T_p]$ that minimizes $J(x(t_0), u(\cdot))$ subject to (4)*,* (5) *and $\hat{x}(t_0 + T_p) \in \mathcal{X}_f$.*

Unlike the standard MPC set up presented in [2]–[4], the constraint (6) is no longer used as a terminal constraint for $\hat{x}(t_k + T_p)$. Instead, it guarantees that the predictive state is inside $\mathcal{X}_f$ at $t_k + T^*_{k-1} - \gamma\Delta_{k-1}$. Since T^*_k denotes the time interval to reach $\mathcal{X}_f$, T^*_k satisfies $T^*_k \leq T^*_{k-1} - \gamma\Delta_{k-1} < T^*_{k-1}$. This means that the time interval T^*_k becomes smaller than the previous time T^*_{k-1}, as long as Problem 1 becomes feasible. We will make use of this property in convergence analysis that this time interval is guaranteed to decrease and become small enough to achieve $x(t) \in \Phi$ within finite time steps.

In the proposed event-triggered strategy, the obtained optimal control input trajectory will be applied until the next OCP will be solved to update the control input. This is formulated as

$$u(t) = u^*(t), \quad t \in [t_k, t_{k+1}) \tag{7}$$

Once the state reaches Φ, we apply the local controller $\kappa(x)$

to stabilize the system such that the state stays in Φ for all the future time. This control strategy is referred to as 'Dual mode MPC', as adopted in some papers, see e.g., [4].

Remark 1. In Problem 1, we can have the state constraint if it is needed, e.g., $x(\xi) \in \mathcal{X}$. As shown in [3], to guarantee the feasibility in the presence of disturbances, the constraint for $x(\xi)$ must be restricted to a smaller region characterized by L_ϕ, in the formulation of the OCP. For more specific definitions of this region, see [4].

III. Main algorithm and Feasibility

A. Main algorithm

We will in the following propose a general type of event-triggered strategy, in which the OCP is solved when the error between the predictive state and the actual state exceeds a certain threshold:

***(Event-triggered strategy)*:** Consider the nonlinear system (1), subject to (2) and the OCP in Problem 1. Suppose that the current time is t_k when the OCP is solved. The event-triggered condition, which determines the next time t_{k+1} when the OCP is again solved, is then given by

$$||x(t) - \hat{x}^*(t)||_P < (\varepsilon - \varepsilon_f) e^{-L_\phi T_k^*}, \quad t_k < t \leq t_k + T_k^* \quad (8)$$

The OCP is solved only when (8) is violated. If (8) is satisfied for all $t_k < t \leq t_k + T_k^*$, then we set $t_{k+1} = t_k + T_k^*$.

□

We would guess from (8) that, while T_k^* is large the condition (8) will be violated before arriving $t = t_k + T_k^*$, that is, $t_{k+1} < t_k + T_k^*$. As the OCP is solved and T_k^* becomes smaller, then longer time will be expected to satisfy (8) and we will obtain $t_{k+1} = t_k + T_k^*$ when (8) is satisfied for all $t_k < t \leq t_k + T_k^*$.

Based on the above event-triggered strategy, the main algorithm is provided below.

Algorithm 1:

(i) *Initialization :* At initial time t_0, if $x(t_0) \in \Phi$, then apply the local controller $\kappa(x)$ as a dual mode strategy. Otherwise, solve Problem 1 for t_0 to obtain $u^*(\xi)$, $\hat{x}^*(\xi)$ for all $[t_0, t_0 + T_p]$. Calculate T_0^* as the time interval to reach $\mathcal{X}_f$, i.e., $\hat{x}^*(t_0 + T_0^*) \in \partial\mathcal{X}_f$. The controller $u^*(\xi)$ is then applied for $[t_0, t_1)$, where t_1 is the next time to solve Problem 1, which is obtained from the event-triggered strategy.

(ii) For any t_k, $k \in \mathbb{N}_{\geq 1}$, solve Problem 1 to obtain $u^*(\xi)$, $\hat{x}^*(\xi)$ for all $\xi \in [t_k, t_k + T_p]$. Calculate T_k^* as the time interval when the state reaches $\mathcal{X}_f$, i.e., $\hat{x}^*(t_k + T_k^*) \in \partial\mathcal{X}_f$.

(iii) Apply $u^*(\xi)$ for $[t_k, t_{k+1})$, where t_{k+1} is the next time step to solve Problem 1, which is obtained from the event-triggered strategy. Once we obtain $x(t) \in \Phi$ at a certain time $t \in [t_k, t_{k+1})$, then switch to the local controller $\kappa(x)$ as a dual mode strategy.

(iv) $k \leftarrow k + 1$ and go back to the step (ii).

Remark 2. Note that the proposed method does not involve the optimal cost as a Lyapunov function to derive the event-triggered strategy. This has some advantages over the existing event-triggered MPC strategies, since several assumptions are not required to obtain the event-triggered condition. For instance, one of the fairly standard assumptions is the Lipschitz continuity of the stage and terminal cost, see [3], [4], [14], [15], [18]. However, the corresponding Lipschitz constant values depend on the supremum of the state, i.e., $\sup_{t\in[0,\infty)} \{||x(t)||\}$ (see *Lemma 3* in [18] or *Lemma 3.2* in [17]). This value is, however, difficult to be obtained since the supremum of $||x(t)||$ is in general not known when there are no state constraints. Thus, it may not be appropriate to include these parameters in the event-triggered condition such as in (8).

Remark 3. From (8), if we set the parameter ε_f as small as possible, we may obtain longer inter-event time Δ_k. However, as ε_f becomes smaller, then the constraint of (6) becomes more restrictive, and thus the prediction horizon T_p may need to be longer to guarantee the feasibility at the initial time t_0. Therefore, regarding how to select the value of ε_f, we need to take the problem of feasibility into account.

B. Feasibility

Under the event-triggered strategy provided in the previous subsection, we next show the feasibility of Problem 1. To achieve this, we first let $\bar{u}(\xi)$, $\xi \in [t_{k+1}, t_{k+1} + T_p]$ be the control trajectory applied from t_{k+1} given by

$$\bar{u}(\xi) = \begin{cases} u^*(\xi), & \xi \in [t_{k+1}, t_k + T_k^*] \\ \kappa(\bar{x}(\xi)), & \xi \in (t_k + T_k^*, t_{k+1} + T_p], \end{cases} \quad (9)$$

where $\bar{x}(\xi)$ is the predictive state obtained by applying $\bar{u}(\xi)$, that is, $\dot{\bar{x}}(\xi) = \phi(\bar{x}(\xi), \bar{u}(\xi))$ starting from $\bar{x}(t_{k+1}) = x(t_{k+1})$.

The following theorem guarantees that (9) is a feasible controller for Problem 1 at t_{k+1}, provided that the feasibility at t_k is guaranteed and the disturbances are upper bounded accordingly:

Theorem 1. *Suppose that the current time is t_k, and the feasibility of Problem 1 is guaranteed at t_k. Then, Problem 1 under the proposed event-triggered strategy is guaranteed to be feasible at t_{k+1} with a feasible controller* (9)*, if the additive bounded disturbance w satisfies*

$$||w||_P \leq \frac{\lambda_{\min}(\hat{Q}_P)}{2e^{L_\phi T_0^*}}(1-\gamma)\varepsilon_f, \quad (10)$$

where $\hat{Q}_P = P^{-1/2}(Q + K^\mathsf{T} R K)P^{-1/2}$.

To prove that (9) is a feasible controller of Problem 1, we will show that the followings are satisfied;

(i) By applying $\bar{u}(\xi)$, $\xi \in [t_{k+1}, t_k + T_k^*]$, the predictive state is in Φ by the time $t_k + T_k^*$. That is, $\bar{x}(t_k + T_k^*) \in \Phi$. This ensures that $\bar{u}(\xi)$ is admissible for all $\xi \in (t_k + T_k^*, t_{k+1} + T_p]$, since the local controller $\kappa(\bar{x})$ can be applied.

(ii) $T_k^* - \gamma\Delta_k > 0$. This ensures that the time interval to reach $\mathcal{X}_f$ in the constraint (6) is guaranteed to be positive at t_{k+1}.

(iii) By applying $\bar{u}(\xi)$, $\xi \in (t_k + T_k^*, t_{k+1} + T_k^* - \gamma\Delta_k]$, the predictive state $\bar{x}$ converges to $\mathcal{X}_f$ by the time $t_{k+1} + T_k^* - \gamma\Delta_k$. That is,

$$\bar{x}(t_{k+1} + T_k^* - \gamma\Delta_k) \in \mathcal{X}_f$$

Proof: To prove (i), we first use the fact that the difference between $\bar{x}$ and $\hat{x}^*$ is upper bounded by

$$\begin{aligned} ||\bar{x}(\xi) - \hat{x}^*(\xi)||_P &\leq ||x(t_{k+1}) - \hat{x}^*(t_{k+1})||_P e^{L_\phi(\xi - t_{k+1})} \\ &\leq e^{-L_\phi T_k^*}(\varepsilon - \varepsilon_f) \cdot e^{L_\phi(\xi - t_{k+1})} \end{aligned}$$

for $\xi \in [t_{k+1}, t_k + T_k^*]$. The second inequality follows from the event-triggered strategy (8). Letting $\xi = t_k + T_k^*$ and from the triangular inequality, we obtain

$$\begin{aligned} &||\bar{x}(t_k + T_k^*)||_P \\ &\quad \leq ||\hat{x}^*(t_k + T_k^*)||_P + (\varepsilon - \varepsilon_f) e^{-L_\phi(t_{k+1} - t_k)} \\ &\quad \leq \varepsilon_f + \varepsilon - \varepsilon_f \\ &\quad = \varepsilon \end{aligned}$$

Thus, $\bar{x}(t_k + T_k^*) \in \Phi$ and the proof of (i) is completed.

The proof of (ii) is obtained from the fact that we have $\Delta_k \leq T_k^*$ from the event-triggered strategy, and thus $T_k^* - \gamma\Delta_k \geq (1-\gamma)T_k^* > 0$.

By using $\bar{x}(t_k + T_k^*) \in \Phi$ and from (3), we obtain $\dot{V}_f(\bar{x}(\xi)) \leq -\bar{x}^\mathsf{T}(\xi)(Q + K^\mathsf{T}RK)\bar{x}(\xi) \leq -\lambda_{\min}(\hat{Q}_P)V_f(\bar{x}(\xi))$ for $\xi \in (t_k + T_k^*, t_{k+1} + T_k^* - \gamma\Delta_k]$. Furthermore, from the Gronwall-Bellman inequality and the feasibility condition (10), we obtain

$$\begin{aligned} &||\bar{x}(t_k + T_k^*)||_P \\ &\quad \leq ||\hat{x}^*(t_k + T_k^*)||_P + \frac{w_{\max}}{L_\phi} e^{L_\phi T_k^*}(1 - e^{-L_\phi \Delta_k}) \\ &\quad \leq \varepsilon_f + \frac{(1-\gamma)}{2L_\phi}\varepsilon_f \lambda_{\min}(\hat{Q}_P)(1 - e^{-L_\phi\Delta_k}) \end{aligned}$$

Denoting $\eta = \frac{(1-\gamma)}{2L_\phi}\lambda_{\min}(\hat{Q}_P)$, and by using comparison lemma, we obtain

$$\begin{aligned} &V_f(\bar{x}(t_{k+1} + T_k^* - \gamma\Delta_k)) \\ &\quad \leq V_f(\bar{x}(t_k + T_k^*)) e^{-\lambda_{\min}(\hat{Q}_P)(1-\gamma)\Delta_k} \\ &\quad \leq \varepsilon_f^2 \left(1 + \eta(1 - e^{-L_\phi\Delta_k})\right)^2 e^{-2L_\phi\eta\Delta_k} \\ &\quad \leq \varepsilon_f^2 \end{aligned}$$

The 3rd inequality is obtained by the fact that the function $f_\varepsilon(\Delta_k) = (1 + \eta(1 - e^{-L_\phi\Delta_k}))e^{-L_\phi\eta\Delta_k}$ is shown to be a decreasing function of Δ_k with $f_\varepsilon(0) = 1$. Thus we obtain $V_f(\bar{x}(t_{k+1} + T_k^* - \gamma\Delta_k)) \leq \varepsilon_f^2$, and the proof of (iii) is completed.

Based on above, Problem 1 is shown to be feasible at t_{k+1} with the feasible controller given by (9). This completes the proof of Theorem 1. ■

IV. Convergence

Based on the event-triggered strategy and the feasibility theorem presented in the previous section, we provide the following theorem stating that the the state is guaranteed to converge to Φ in finite time.

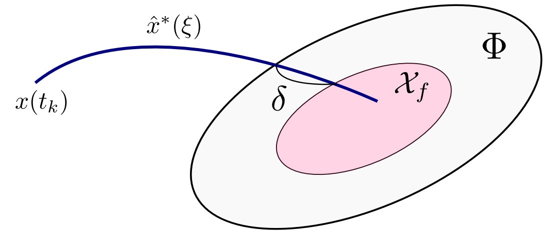

Fig. 2. Denoting t_Φ, $t_{\mathcal{X}_f}$ as the time when the state $\hat{x}^*$ enters Φ, $\mathcal{X}_f$ respectively, $\delta_{\min}$ is the minimum time interval of $t_{\mathcal{X}_f} - t_\Phi$.

Theorem 2. *Consider the nonlinear system* (1)*, subject to* (2) *with additive disturbances satisfying* (10) *and Problem 1 under the proposed event-triggered strategy. Then, the state trajectory enters Φ in finite time.*

To prove convergence, let the time interval $\delta_{\min}$ be defined by

$$\begin{aligned} \delta_{\min} = \inf\{\delta \in \mathbb{R}_{\geq 0} \mid \delta = t_{\mathcal{X}_f} - t_\Phi,\ \hat{x}^*(t_\Phi) \in \partial\Phi, \\ \hat{x}^*(t_{\mathcal{X}_f}) \in \partial\mathcal{X}_f\} \end{aligned} \tag{11}$$

That is, $\delta_{\min}$ is the minimum time interval from when the state is on the boundary of Φ to the time when it reaches the boundary of $\mathcal{X}_f$ (see Fig. 2). Note that we have $\delta_{\min} > 0$ since the predictive state $\hat{x}^*$ is a continuous function. If $x(t_k)$ is outside of Φ, then we have $T_k^* > \delta_{\min}$, i.e.,

$$x(t_k) \notin \Phi \Rightarrow T_k^* > \delta_{\min}. \tag{12}$$

By taking contraposition we obtain

$$T_k^* \leq \delta_{\min} \Rightarrow x(t_k) \in \Phi. \tag{13}$$

This means that if T_k^* becomes small enough to fulfill $T_k^* \leq \delta_{\min}$, then we obtain $x(t_k) \in \Phi$. The following proof illustrates that the time to achieve $T_k^* \leq \delta_{\min}$ is shown to be upper bounded by finite time steps.

Proof: As the first step to prove the convergence, we derive the inter-event time Δ_k of our proposed event-triggered strategy (8). By using the upper bound $||x(t) - \hat{x}^*(t)|| \leq \frac{w_{\max}}{L_\phi}(e^{L_\phi(t-t_k)} - 1)$ and the feasibility condition (10), a sufficient condition to satisfy (8) is then given by

$$\frac{\lambda_{\min}(\hat{Q}_P)(1-\gamma)\varepsilon_f}{2L_\phi e^{L_\phi T_0^*}}(e^{L_\phi(t-t_k)} - 1) < (\varepsilon - \varepsilon_f)e^{-L_\phi T_k^*}. \tag{14}$$

The time $\hat{t}_{k+1}$ when the condition (14) is violated is explicitly given by

$$\begin{aligned} \hat{t}_{k+1} &= t_k + \frac{1}{L_\phi}\ln(1 + \rho(T_k^*)) \\ &\geq t_k + \frac{1}{L_\phi}\ln(1 + \rho_{\min}), \end{aligned} \tag{15}$$

where $\rho(T_k^*)$ and $\rho_{\min}$ are given by

$$\rho(T_k^*) = \frac{2L_\phi(\varepsilon - \varepsilon_f)e^{L_\phi(T_0^* - T_k^*)}}{\lambda_{\min}(\hat{Q}_P)(1-\gamma)\varepsilon_f} \tag{16}$$

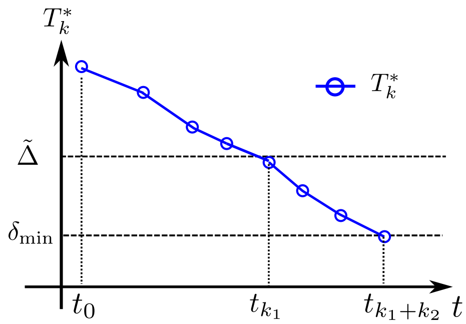

Fig. 3. The illustration of the sequence T_k^*

$$\rho_{\min} = \frac{2L_\phi(\varepsilon - \varepsilon_f)}{\lambda_{\min}(\hat{Q}_P)(1-\gamma)\varepsilon_f} > 0 \tag{17}$$

and we have used $T_0^* \geq T_k^*$. Thus, by denoting $\tilde{\Delta} = \frac{1}{L_\phi}\ln(1+\rho_{\min})$, the condition (8) is satisfied for all $t_k \leq t \leq t_k + \tilde{\Delta}$. On the other hand, if (8) is satisfied for all $t_k \leq t \leq t_k + T_k^*$, then we set $t_{k+1} = t_k + T_k^*$ from the event-triggered strategy. Thus, for the inter-event time $\Delta_k = t_{k+1} - t_k$, we obtain

$$\Delta_k \begin{cases} \geq \tilde{\Delta}, & \text{if } \ T_k^* > \tilde{\Delta} \\ = T_k^*, & \text{if } \ T_k^* \leq \tilde{\Delta} \end{cases} \tag{18}$$

Using (18) and $T_{k+1}^* \leq T_k^* - \gamma\Delta_k$, we obtain the following recursion:

$$T_{k+1}^* \begin{cases} \leq T_k^* - \gamma\tilde{\Delta}, & \text{if } \ T_k^* > \tilde{\Delta} \quad (19) \\ \leq (1-\gamma)T_k^*, & \text{if } \ T_k^* \leq \tilde{\Delta} \quad (20) \end{cases}$$

meaning that T_k^* becomes smaller as the OCP is solved. The example of the sequence of T_k^* is shown in Fig. 3.

Suppose now that at the initial time t_0, we have $T_0^* > \tilde{\Delta}$. Furthermore, we let k_1 be the first (minimum) time step when T_k^* becomes $T_k^* \leq \tilde{\Delta}$, i.e.,

$$k_1 = \inf\{k \in \mathbb{N}_{\geq 0} \mid T_k^* \leq \tilde{\Delta}\}, \tag{21}$$

see Fig. 3. From (19), k_1 satisfies $k_1 \leq K_1$, where

$$K_1 = \frac{1}{\gamma}\left(\frac{T_0^*}{\tilde{\Delta}} - 1\right) \tag{22}$$

Furthermore, we also let k_2 be the first step from k_1, when T_k^* becomes $T_k^* \leq \delta_{\min}$, i.e.,

$$k_2 = \inf\{k \in \mathbb{N}_{\geq 0}, k \geq k_1 \mid T_k^* \leq \delta_{\min}\}. \tag{23}$$

For the case $\delta_{\min} < \tilde{\Delta}$ [1)], and from the recursion of the inequality (20), k_2 satisfies $k_2 \leq K_2$, where

$$K_2 = \frac{\ln(\delta_{\min}/\tilde{\Delta})}{\ln(1-\gamma)}. \tag{24}$$

Therefore, we have shown that there exists a time step k when $T_k^* \leq \delta_{\min}$, where k is upper bounded by $k \leq K_1 + K_2 < \infty$. This means that the state $x(t_k)$ converges to Φ in finite time. This completes the proof of Theorem 2. ■

V. Illustrative Example

As a simulation example we consider the attitude control of single axis satellite modeled as two masses connected by a flexible boom [18]. The state is defined on $\mathbb{R}^4$ and consists of the orientation of the satellite θ_1 and the angle between the star sensor and the instrument package θ_2, and their velocities $\dot{\theta}_1, \dot{\theta}_2$. The state equation is given by $\dot{x} = Ax + Bu + w$, where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\nu_1/J_1 & \nu_1/J_1 & -\nu_2/J_1 & \nu_2/J_1 \\ \nu_1/J_2 & -\nu_1/J_2 & \nu_2/J_2 & -\nu_2/J_2 \end{bmatrix}, \\ B^T &= \begin{bmatrix} 0 & 0 & 0 & L/J_2 \end{bmatrix}. \end{aligned}$$

We set $\nu_1 = 0.09$ as a torque constant, $\nu_2 = 0.08$ as a viscous damping constant, and $L = 1.0$ as a satellite length, and $J_1 = J_2 = 0.1$, as the inertias. The initial state is assumed to be given by $x(t_0) = [\pi \ \ 3\pi/4 \ \ 0 \ \ 0]$. The constraint for the control input is given by $||u|| \leq 10$. The matrices for the stage cost is given by $Q = 0.1I_4, R = 0.1I_1$ and the prediction horizon is $T_p = 6.5$. To characterize the region Φ and the local controller $\kappa(x)$, we follow the steps presented in [2] to obtain $\varepsilon = 1.16$. Letting $\varepsilon_f = 0.7$ and $\gamma = 0.1$, from Theorem 1 the OCP becomes feasible when $||w||_P \leq 0.10$.

Fig. 4 shows the state trajectories under the proposed event-triggered strategy (Blue line) and self-triggered strategy (Red dot line) with disturbances satisfying $||w||_P \leq 0.10$. The local controller $\kappa(x)$ has been applied from $t = 6.84$ (event-triggered) and $t = 6.87$ (self-triggered), when the state enters Φ. From this figure, the state converges to Φ under both event-triggered and self-triggered control strategy. Fig. 5 shows the corresponding triggering instants where the OCP is solved when the value is 1. From this figure, the average time interval of solving the OCP is given by 1.05 for event-triggered case and 0.34 for the self-triggered case. Thus, the self-triggered result requires more frequencies to solve the OCP than the event-triggered case, and this is because it is derived from the sufficient condition of the event-triggered strategy.

From Fig. 4 and Fig. 5, we can conclude that the state converges to Φ by aperiodically solving the OCP.

VI. Conclusion

We have proposed the aperiodic formulation of MPC for the nonlinear systems with additive disturbances. In contrast to the existing ideas, our strategy is not derived from the Lyapunov stability but the time interval to reach to the local region around the origin. We have also shown the feasibility that the OCP is guaranteed, and the convergence that the state converges to Φ in finite time steps. Our proposed framework was also verified through a simulation example.

[1)]For the case $\delta_{\min} \geq \tilde{\Delta}$, we have $T_{k_1}^* \leq \tilde{\Delta} \leq \delta_{\min}$ and the convergence to Φ is already achieved within K_1. Since we want to obtain the upper bound of time steps to achieve the convergence, here we consider $\delta_{\min} < \tilde{\Delta}$.

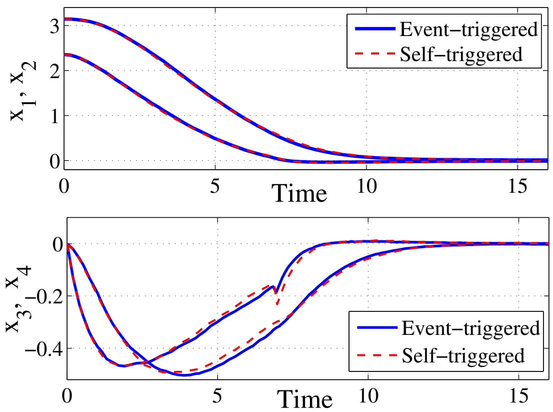

Fig. 4. State trajectories under event-triggered (Blue solid) and self-triggered (Red dot) control strategy.

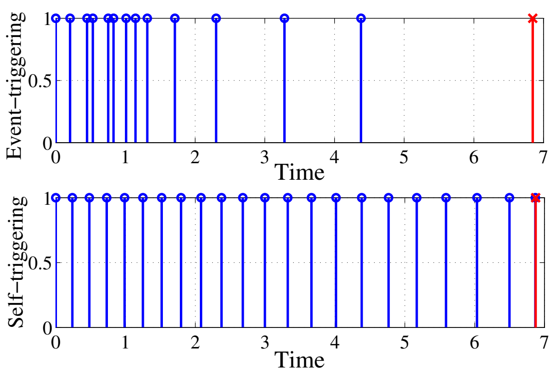

Fig. 5. Triggering instants for event-triggered (upper) and self-triggered (lower) strategy. The red line represents the time when the state enters Φ.

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