A non-monotonic approach to periodic event-triggered control with packet loss

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***Abstract*— This paper treats the topic of periodic event-triggered control (PETC) for Networked Control Systems in a setup where the network provides a maximum sampling rate and a maximum number of successive packet losses. The packet loss is tackled with a non-monotonic approach, meaning that the trigger rules are designed such that a Lyapunov function is guaranteed to decrease only between two successful triggering instants. The stability properties of the resulting continuous-time sampled-data system are analyzed in two ways. The first one uses different restricted dynamical systems and the second one applies theory on non-monotonic Lyapunov functions. Based on this theory an example for another trigger rule is given that can be derived using this approach. The theoretical results are demonstrated through a known numerical example.**

I. INTRODUCTION

In Networked Control Systems (NCSs) one faces several problems induced by the network, as described in [1]. One of the interesting challenges is how to sample the measurements before sending them to the controller. An approach to reduce the amount of transmissions is the event-based sampling approach, introduced in [2]. In [3] a Lyapunov based framework for event-triggered control was presented that initiated a huge amount of work in this area. Again when dealing with NCSs it is of special interest to analyze how event-triggered control behaves in combination with other network induced imperfections such as packet loss. This interplay was first studied in a stochastic framework. For instance in [4] the behavior of systems with integrator dynamics is analyzed and problems with the unstructured data traffic of event-triggered control have been identified. To generate a more structured data traffic is one of the main benefits of periodic event-triggered control (PETC). In PETC, introduced in [5], the state of the plant is sampled at periodic time instants and the information is sent to the controller if a trigger condition is violated at one of those instants.

An early work on the design of trigger rules for NCSs is given in [6] in a distributed setup. Another distributed approach is shown in [7], where losses are modeled as additional delays. A stochastic formulation of packet loss due to random loss and jamming attacks is given in [8], where a trigger rule is designed to guarantee almost sure asymptotic stability for a deterministic discrete-time system. A more recent approach in event-triggered control is to allow thresholds to be dynamically generated. This method is known as dynamic event-triggered control and it is used to deal with packet loss in NCSs in [9]. To deal with network-induced imperfections in a PETC framework is the goal of [10] which focuses on communication delays as well as packet loss. The main contribution therein is to come up with a new Lyapunov-Krasovskii functional that significantly improves the behavior under delay. The packet loss is then covered by lowering the generated threshold such that the condition that guarantees $\mathscr{H}_\infty$ stability in the delayed setup is guaranteed to hold at every sampling instant albeit the occurrence of packet loss.

In this paper we will show that one can deal with packet loss in PETC by applying stability results that employ Lyapunov functions that are non-monotonically decreasing along solutions of the dynamical system. Such results are given in [11] for nonlinear time-varying and in [12] for various system classes, including discontinuous dynamical systems. The benefits of non-monotonic Lyapunov functions were used in [13] with the goal to simplify computation of Lyapunov functions, and in [14] and [15] in the context of optimization in NCSs.

In this work we focus on PETC with packet loss. The first goal is to derive a trigger rule with easily verifiable conditions, dependent on the system dynamics and amount of packet loss, to guarantee stability of the closed-loop continuous-time system. We will derive this trigger rule by computing conditions such that the dynamical system, restricted to successful triggering instants, is globally exponentially stable. The second contribution lies in generalizing these results as a non-monotonic approach. We show that stability of the closed-loop system can be proven using a theorem for non-monotonic Lyapunov functions. Furthermore we will show based on this theorem that the approach can be used to synthesize other trigger rules for the given setup than the one used in this paper.

The remainder is structured as follows. In Section II a precise definition of the problem is given. Following this we derive a trigger rule for stabilization in the presence of packet loss based on different discretizations in Section III. In Section IV we generalize our approach using non-monotonic Lyapunov functions and give an example for a trigger rule from literature that can be designed using this approach. The paper closes with a numerical example in Section V and a conclusion in Section VI.

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II. PROBLEM SETUP

In this paper we consider the LTI System

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t), \quad x_c(0^-) = x_{c,0} \tag{1}$$

with $x_c(t) \in \mathbb{R}^n$, $u_c(t) \in \mathbb{R}^p$ for all $t \geq 0^-$ and matrices A_c, B_c with suitable dimensions, where $t = 0^-$ stands for $\lim_{t \to 0, t<0} t$. We assume (A_c, B_c) to be stabilizable. We observe a scenario where the control loop is closed over a communication network that provides a certain service guarantee. This guarantee tuple (h,m) consists of the maximum sampling time h and the maximum number of successive packet losses m. We assume h such that the discretized system with sampling time h is stabilizable as well (see e.g. [16] for sufficient conditions on h, A_c, B_c). To generate the control input, we use a sampled-data controller whose update sequences are determined through a periodic event-triggered scheme. A new element of this sequence is added when a certain trigger rule is violated at the periodic time instants, with period h, it is being evaluated, i.e.,

$$\tilde{\mathscr{T}} = \{t \geq 0 : t = kh \wedge \text{trigger rule is violated at t}\} \tag{2}$$

where $k \in \mathbb{N}$ and $\tilde{\mathscr{T}} =: \{\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2, \dots\}$. Due to the possibility of packet loss the actual resulting sequence of update times

$$\mathscr{T} = \{t \geq 0 : t \in \tilde{\mathscr{T}} \wedge \text{triggering successful}\}, \tag{3}$$

with $\mathscr{T} =: \{\tau_0 = \tilde{\tau}_0, \tau_1, \tau_2, \dots\}$, is a subset of $\tilde{\mathscr{T}}$ where at most m successive elements of $\tilde{\mathscr{T}}$ are not contained in $\mathscr{T}$. Without loss of generality, we assume $\tau_0 = \tilde{\tau}_0$, i.e., the first instant when triggering is necessary is assumed to be successful. Furthermore we define indices that represent the same time instants in $\mathscr{T}$ and $\tilde{\mathscr{T}}$. Starting with a nonnegative integer k_τ that specifies due to (3) the ${k_\tau}^{\text{th}}$ instant when triggering is successful, there exists $k_\tau|_{\tilde{\tau}} \in \mathbb{N}_0$ such that the ${k_\tau|_{\tilde{\tau}}}^{\text{th}}$ instant when triggering is necessary happens at the same time as the ${k_\tau}^{\text{th}}$ successful instant, i.e., $\tau_{k_\tau} = \tilde{\tau}_{k_\tau|_{\tilde{\tau}}}$. Using $\mathscr{T}$ we can define the last successfully transmitted state $\hat{x}_c(t)$ that is initialized with $\hat{x}_{c,0}$ and updated at all successful triggering instants, i.e., $\hat{x}_c(t) = \hat{x}_{c,0}$ for $t \in [0^-, \tau_0)$ and for all $k_\tau \in \mathbb{N}_0$

$$\hat{x}_c(t) = \begin{cases} x_c(t), & t \in \mathscr{T} \\ \hat{x}_c(\tau_{k_\tau}), & t \in (\tau_{k_\tau}, \tau_{k_\tau+1}). \end{cases} \tag{4}$$

Using (4) the resulting periodic event-triggered controller is then given as

$$u_c(t) = K\hat{x}_c(t). \tag{5}$$

Note that for $x_c(t)$ it holds, due to continuity of the solution, that $x_c(0) = x_c(0^-) = x_{c,0}$.

To design the controller and trigger rules we will use the discretized system with sampling time h, i.e.,

$$x_{k+1} = Ax_k + Bu_k, \; k \in \mathbb{N}_0 \tag{6}$$

with $A = \mathrm{e}^{A_c h}$, $B = \int_0^h \mathrm{e}^{A_c s} B_c ds$, $u_k = u_c(t = kh := t_k)$ and $x_0 = x_{c,0}$, thus $x_k = x_c(t = t_k)$. According to the definition of k_τ and $k_\tau|_{\tilde{\tau}}$ above we define $k_\tau|_h := \frac{\tau_{k_\tau}}{h}$. For analysis purposes we define another discretization of the sampled-data system (1) that is a restriction to the successful trigger times, i.e., it is defined on $\mathscr{T}$ as

$$\tilde{x}_{k_\tau+1} = f(\tilde{x}_{k_\tau}, k_\tau), \; k_\tau \in \mathbb{N}_0 \tag{7}$$

with $f(\cdot,\cdot)$ such that $\tilde{x}_{k_\tau} = x_{k_\tau|_h} = x_c(t = \tau_{k_\tau})$ for all $k_\tau \in \mathbb{N}_0$.

For the case $\mathscr{T} = \tilde{\mathscr{T}}$, i.e. the case of no packet loss, [5] states suitable trigger rules such that global exponential stability of the continuous-time sampled-data system is guaranteed. In this work we want to derive trigger rules and easily verifiable conditions, dependent on the system dynamics, service guarantees and the chosen controller, to check if there exist parameters for them such that global exponential stability can still be guaranteed under the given tuple (h,m) provided by the network. To derive these rules we will at first use an approach using standard Lyapunov techniques for the different discretizations introduced above. Later in Section IV we will show, that this approach can be generalized using non-monotonic Lyapunov functions as described in [12].

III. APPROACH BASED ON DIFFERENT DISCRETIZATIONS

The idea is to derive a controller and triggering setup for system (6). This setup will be analyzed to compute bounds on the parameters of the trigger rule using system (7), to guarantee exponential stability of this restriction despite packet loss. The last step is to draw conclusions from stability properties of (7) for system (1).

A. Design controller and state trigger rule

First of all we define trigger errors $e_c(t) = \hat{x}_c(t) - x(t)$ and their discrete counterpart $e_k = e_c(t_k) = \hat{x}_c(t_k) - x_c(t_k) = \hat{x}_k - x_k$. Using this trigger error e_k we can compute a closed-loop representation of the discretized system (6) between two successful triggering instants, i.e.,

$$\begin{aligned} x_{k+1} &= \underbrace{(A+BK)}_{A_1} x_k + \underbrace{BK}_{B_1} e_k \\ e_{k+1} &= \underbrace{(I-BK)}_{A_2} e_k + \underbrace{(I-(A+BK))}_{B_2} x_k \end{aligned} \tag{8}$$

for all $k \in [k_\tau|_h, k_\tau+1|_h)$ with $k_\tau \in \mathbb{N}_0$, where $k_\tau|_h$ and $k_\tau+1|_h$ represent the discrete-time indices of the ${k_\tau}^{\text{th}}$ and $(k_\tau+1)^{\text{th}}$ successful triggering instant as defined in Section II. We use this representation now to derive an explicit formula for the evolution of x_k and e_k.

Lemma 1: Observe the closed-loop representation from equation (8). Then for natural numbers $j,k \in [k_\tau|_h, k_\tau+1|_h)$, $k_\tau \in \mathbb{N}_0$ and $j > k$, it holds that

$$\begin{aligned} x_j &= A_{1|j-k} x_k + B_{1|j-k} e_k \\ e_j &= A_{2|j-k} e_k + B_{2|j-k} x_k \end{aligned} \tag{9}$$

with $A_{1|1} = A_1, B_{1|1} = B_1, A_{2|1} = A_2, B_{2|1} = B_2$ and for $i \geq 1$

$$\begin{aligned} A_{1|i+1} &= A_1 A_{1|i} + B_1 B_{2|i}, \quad B_{1|i+1} = A_1 B_{1|i} + B_1 A_{2|i}, \\ A_{2|i+1} &= A_2 A_{2|i} + B_2 B_{1|i}, \quad B_{2|i+1} = A_2 B_{2|i} + B_2 A_{1|i}. \end{aligned}$$

Proof: We prove Lemma 1 via induction. Using the initial matrices we see that (9) equals (8) for $j-k=1$. Now we assume (9) holds for some $j > k$. Thus we have to show that under this assumption (9) holds for $j+1$ as well. Combining (8) and (9) we compute $x_{j+1} = A_1 x_j + B_1 e_j = A_1 A_{1|j-k} x_k + A_1 B_{1|j-k} e_k + B_1 A_{2|j-k} e_k + B_1 B_{2|j-k} x_k$ and $e_{j+1} = A_2 e_j + B_2 x_j = A_2 A_{2|j-k} e_k + A_2 B_{2|j-k} x_k + B_2 A_{1|j-k} x_k + B_2 B_{1|j-k} e_k$. Thus, one observes that (9) holds for all natural numbers $j,k \in [k_\tau|_h, k_\tau+1|_h)$, $j > k$, $k_\tau \in \mathbb{N}_0$. ∎

As mentioned in the beginning we assume stabilizability of (A_c, B_c) and h to be such that (A,B) is stabilizable. Thus, we can compute a controller K and accordingly $P > 0$ such that the condition

$$V(x_{k+1}) - V(x_k) \le -\theta^2 \|x_k\|^2 + \|e_k\|^2 \tag{10}$$

where $V(x_k) = x_k^\top P x_k$ and $\theta > 0$ holds for all $k \in \mathbb{N}_0$. Thus it also holds that

$$\lambda_{\min}(P)\|x_k\|^2 \le V(x_k) \le \lambda_{\max}(P)\|x_k\|^2. \tag{11}$$

We start our derivations with a trigger rule where triggering is necessary when

$$\|e_k\|^2 > \sigma^2 \|x_k\|^2. \tag{12}$$

In case of no packet loss the condition $\sigma < \theta$ is known to be sufficient for global exponential stability, cf. [5], [17]. We will compute a new bound on σ in the following for the case of m successive packet losses. Our derivations will lead to a trigger rule that does not guarantee that (12) with $\sigma < \theta$ holds despite packet loss. We will also add another condition later that will be only active after packets have been lost and will not change the behavior in the nominal case.

B. Compute bounds on inter-event intervals

It is clear that in a PETC setup the inter-event times are lower bounded by the sampling time h. In the next result we will state and prove that the inter-event times are also uniformly upper bounded, since this will be needed in the stability proof.

Lemma 2: Assume system (1) is controlled with the periodic event-triggered controller (5) according to trigger rule (12) where $\sigma < \theta$. Then the interval between two successful triggering instants in case of at most m successive packet losses is uniformly upper bounded, i.e.,

$$\tau_{k_\tau+1} - \tau_{k_\tau} \le \eta \; \forall k_\tau \in \mathbb{N}_0. \tag{13}$$

Proof: The proof consists of three steps. In the first step an upper bound on the time between a successful triggering instant and the next necessary triggering instant will be given, i.e., η_1 will be computed such that $\eta_1 \ge \tilde{\tau}_{k_\tau|_{\tilde{\tau}}+1} - \tau_{k_\tau} = \tilde{\tau}_{k_\tau|_{\tilde{\tau}}+1} - \tilde{\tau}_{k_\tau|_{\tilde{\tau}}}$ for all $k_\tau \in \mathbb{N}_0$. In the second step it is shown that an upper bound on the time between a necessary but not successful instant and the next necessary instant can be computed, i.e., η_2 such that $\eta_2 \ge \tilde{\tau}_{k_{\tilde{\tau}}+1} - \tilde{\tau}_{k_{\tilde{\tau}}}$ for all $k_{\tilde{\tau}}$ that satisfy $\tilde{\tau}_{k_{\tilde{\tau}}} \notin \mathscr{T}$. The proof is concluded by the observation that the overall bound is a finite sum of these bounds since there are at most m unsuccessful triggering attempts, i.e., (13) holds with $\eta = \eta_1 + m\eta_2$.

We will start with the computation of η_1. According to trigger rule (12) we know $\|e_j\|^2 \le \sigma^2 \|x_j\|^2$ until the next triggering is necessary. Thus from $\sigma < \theta$, the dissipation inequality (10) and the bound (11) one can conclude, using standard Lyapunov arguments, that an exponential decrease holds until the next triggering is necessary, i.e., there exist $r \ge 1$ and $\lambda > 0$ such that $\|x_j\| \le r\mathrm{e}^{-\lambda h(j-k_\tau|_h)} \|x_{k_\tau|_h}\|$, for all $j \in \left[\frac{\tau_{k_\tau}}{h} = k_\tau|_h, \frac{\tilde{\tau}_{k_\tau|_{\tilde{\tau}}+1}}{h}\right]$. By the definition of the trigger error we know that $\|\hat{x}_j\| = \|x_{k_\tau|_h}\| = \|e_j + x_j\| \le \|e_j\| + \|x_j\|$ until a new successful triggering instant resets the trigger error to zero, thus $\|e_j\| \ge \|x_{k_\tau|_h}\| - \|x_j\| \ge \left(1 - r\mathrm{e}^{-\lambda h(j-k_\tau|_h)}\right) \|x_{k_\tau|_h}\|$, for all $j \in \left[k_\tau|_h, \frac{\tilde{\tau}_{k_\tau|_{\tilde{\tau}}+1}}{h}\right)$. Due to these calculations we notice that $\|x_j\|$ and $\|e_j\|$ are strictly decreasing respectively increasing. Thus, if we can compute $k_{\eta_1} \in \mathbb{N}$ such that $\sigma\|x_j\| \ge \|e_j\|$ for $j = k_\tau|_h + k_{\eta_1}$ we know that for $\eta_1 = (k_{\eta_1}+1)h$ trigger rule (12) is violated in any case. It remains to compute $k_{\eta_1} \in \mathbb{N}$ such that $\sigma r\mathrm{e}^{-\lambda h k_{\eta_1}} \|x_{k_\tau|_h}\| \ge (1 - r\mathrm{e}^{-\lambda h k_{\eta_1}})\|x_{k_\tau|_h}\|$. Thus $k_{\eta_1} = \left\lceil \frac{1}{h\lambda} \ln(r(\sigma+1)) \right\rceil$ and $\eta_1 = h\left(\left\lceil \frac{1}{h\lambda} \ln(r(\sigma+1)) \right\rceil + 1\right)$.

It remains to compute η_2. This bound can be computed essentially using the same method as before but considering the following aspects. Either the case $\tilde{\tau}_{k_{\tilde{\tau}}} + h = \tilde{\tau}_{k_{\tilde{\tau}}+1}$ holds, i.e., the next necessary triggering instant is immediately after one period (then $\eta_2 = h$ holds) or the trigger rule is not violated in the next step, i.e., $\|e_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\| \le \sigma \|x_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\|$. Then we know $\|x_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\| \le \|x_{k_\tau|_h}\| + \|e_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\| \le \|x_{k_\tau|_h}\| + \sigma \|x_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\|$ and therefore $\|x_{\frac{\tilde{\tau}_{k_{\tilde{\tau}}}}{h}+1}\| \le \frac{1}{1-\sigma} \|x_{k_\tau|_h}\|$. Using this inequality we can compute η_2 with the same methods as η_1 and thus the proof is complete. ∎

Remark 3: Using the same calculation as in Lemma 2 together with the fact that the first instant when triggering is necessary is assumed to be successful one can observe $\tau_0 - t_0 < \eta$ holds as well.

C. Stability analysis of the restricted system

To analyze stability of system (7) we use the general derivative of the Lyapunov function along trajectories of (7) as in [18], i.e.,

$$DV := \frac{1}{\tau_{k_\tau+1} - \tau_{k_\tau}} \left[V(\tilde{x}_{k_\tau+1}) - V(\tilde{x}_{k_\tau}) \right] \; \forall k_\tau \in \mathbb{N}_0. \tag{14}$$

As mentioned above to guarantee stability of this system we introduce an additional trigger rule. This trigger rule forces to trigger an event when

$$V(x_k) - V(x_{k_\tau|_h}) > -\theta^2 \|x_{k_\tau|_h}\|^2, \; \forall k \in [k_\tau|_h, k_\tau+1|_h). \tag{15}$$

Notice that in the case of no packet loss nothing changes, since the demanded decrease is guaranteed already after one time instant due to the dissipation inequality (10) and the fact that $e_{k_\tau|_h} = 0$ due to successful triggering. Furthermore notice that the bound derived in Lemma 2 still holds since an additional trigger rule cannot increase an upper bound on the inter-event times in general.

Now we can establish a condition on the parameter σ such that the origin of the restricted system (7) is globally exponentially stable despite m successive packet losses.

Theorem 4: Assume one can find $r > 0$ and $0 < \sigma < \theta$ such that for all $l \in \{1, \dots, m\}$

$$\begin{aligned} &r - \theta^2 + \sum_{j=1}^{l} c_j(0) \\ &-\theta^2 \sum_{j=1}^{l} \max\left\{0, \frac{\lambda_{\min}(P) - r - \sum_{i=j}^{l} c_i(0)}{\lambda_{\max}(P)}\right\} \le 0 \end{aligned} \tag{16}$$

and

$$r-\theta^2-\frac{\theta^2-\sigma^2}{(1+\sigma)^2}+\sum_{j=1}^{l} c_j(\sigma)$$
$$-\theta^2\sum_{j=1}^{l}\max\left\{0,\frac{\lambda_{\min}(P)-r-\sum_{i=j}^{l} c_i(\sigma)}{\lambda_{\max}(P)}\right\}\le 0 \quad (17)$$

are satisfied with

$$c_i(\sigma)=\left(\frac{1}{1-\sigma}\right)^2\left(\|A_{2|i}\|\sigma+\|B_{2|i}\|\right)^2,\ c_i(0)=\|B_{2|i}\|^2.$$

Then the equilibrium at $\tilde{x}=0$ of the restricted discrete time system (7) with periodic event-triggered controller (5), where triggering is attempted when either (12) or (15) with threshold σ is violated, and with at most m successive packet losses is globally exponentially stable.

Proof: To prove the statement we use the Lyapunov function $V(\tilde{x}_{k_\tau})=\tilde{x}_{k_\tau}^\top P\tilde{x}_{k_\tau}$ and analyze the general derivative DV defined in (14). To show global exponential stability of the restricted dynamical system (7) we have to show that there exists a positive constant $\tilde{r}$ such that

$$\frac{1}{\tau_{k_\tau+1}-\tau_{k_\tau}}\left[V(\tilde{x}_{k_\tau+1})-V(\tilde{x}_{k_\tau})\right]\le-\tilde{r}\|\tilde{x}_{k_\tau}\|^2 \quad (18)$$

for all $k_\tau\in\mathbb{N}_0$ since the Lyapunov function can be bounded as in (11).

From Lemma 2 we know $\tau_{k_\tau+1}-\tau_{k_\tau}\le\eta$ for all $k_\tau\in\mathbb{N}_0$. Thus, if one can show existence of $r>0$ such that $V(\tilde{x}_{k_\tau+1})-V(\tilde{x}_{k_\tau})\le-r\|\tilde{x}_{k_\tau}\|^2$, one can immediately conclude that (18) holds with $\tilde{r}=\frac{r}{\eta}$. Therefore we will now focus on the term $V(\tilde{x}_{k_\tau+1})-V(\tilde{x}_{k_\tau})=V(x_{k_{\tau+1}|_h})-V(x_{k_\tau|_h})$. At first we observe which scenarios are possible to happen in between two successful triggering instants. Without loss of generality every successful triggering instant is followed by $q\in\mathbb{N}_0$ time instants where no triggering is necessary. In the notation introduced above this translates to $\tilde{\tau}_{k_\tau|_{\tilde{\tau}}+1}=t_{k_\tau|_h+q+1}$, i.e., the next instant when triggering is necessary is $q+1$ time instants after $t_{k_\tau|_h}$. Since we have to observe the cases when packets are lost these q instants are followed by $l\in\{1,\dots,m\}$ time instants when triggering is necessary but not successful, i.e., $\tilde{\tau}_{k_\tau|_{\tilde{\tau}}+p}=t_{k_\tau|_h+q+p}$, where $p\in\{1,\dots,l\}$ and $l\in\{1,\dots,m\}$. These l time instants are then followed either by an instant when no triggering is necessary or by a successful triggering instant. In the case that no triggering is necessary we know according to trigger rule (15) that $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-\theta^2\|x_{k_\tau|_h}\|^2$ and thus $V(x_{k_{\tau+1}|_h})-V(x_{k_\tau|_h})\le-\theta^2\|x_{k_\tau|_h}\|^2+\left[V(x_{k_{\tau+1}|_h}-V(x_{k_\tau|_h+q+l+1})\right]$ where the term in brackets once again consists of $q\in\mathbb{N}_0$ instants where no triggering is necessary and at most $m-l$ instants where triggering is necessary but not successful. Thus, it suffices to analyze the case where triggering is necessary and successful at the time instant following the l (at most m) time instants when packets are lost, i.e., $\tau_{k_\tau+1}=\tilde{\tau}_{k_\tau|_{\tilde{\tau}}+l+1}=t_{k_\tau|_h+q+l+1}$. Still this scenario provides two different cases that have to be analyzed, depending on the question of q being positive or zero, resulting in condition (17) and (16).

As stated above, to show (18) we have a closer look on $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})$. Due to the fact that triggering is only necessary at $t=t_{k+q+p}$ for $1\le p\le l$, the fact that $e_{k_\tau|_h}=0$ and the dissipation inequality (10) one can compute $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-\theta^2\|x_{k_\tau|_h}\|^2-\sum_{j=k_\tau|_h+1}^{k_\tau|_h+q}(\theta^2-\sigma^2)\|x_j\|^2+\sum_{j=k_\tau|_h+q+1}^{k_\tau|_h+q+l}\|e_j\|^2-\theta^2\|x_j\|^2$. At first we observe the case where $q>0$. We compute a bound on $\|e_j\|$ for $k_\tau|_h+q+1\le j\le k_\tau|_h+q+l$ in this case. Due to Lemma 1 we have

$$\begin{aligned}\|e_j\|&=\|A_{2|j-(k_\tau|_h+q)}e_{k_\tau|_h+q}+B_{2|j-(k_\tau|_h+q)}x_{k_\tau|_h+q}\|\\&\le\|A_{2|j-(k_\tau|_h+q)}\|\|e_{k_\tau|_h+q}\|+\|B_{2|j-(k_\tau|_h+q)}\|\|x_{k_\tau|_h+q}\|\\&\le\left(\|A_{2|j-(k_\tau|_h+q)}\|\sigma+\|B_{2|j-(k_\tau|_h+q)}\|\right)\|x_{k_\tau|_h+q}\|\\&\le\sqrt{c_{j-(k_\tau|_h+q)}(\sigma)}\|x_{k_\tau|_h}\|\end{aligned} \quad (19)$$

where we used the fact that triggering is not necessary at $t_{k_\tau|_h+q}$ and the definition of the trigger error for the last inequality. Additionally using that the trigger rule holds at $t_{k_\tau|_h+1}$ combined with (19) leads to $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le\left(-\theta^2-\frac{\theta^2-\sigma^2}{(1+\sigma)^2}+\sum_{j=1}^{l}c_j(\sigma)\right)\|x_{k_\tau|_h}\|^2-\sum_{j=1}^{l}\theta^2\|x_{k_\tau|_h+q+j}\|^2$ where we focus now on the last sum. If there exist $j\in\{1,\dots,l\}$ and $r>0$ such that $V(x_{k_\tau|_h+q+j})-V(x_{k_\tau|_h})\le-r\|x_{k_\tau|_h}\|^2-\sum_{i=k_\tau|_h+q+j}^{k+q+l}\|e_i\|^2$, it holds that $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-r\|x_{k_\tau|_h}\|^2$. If not, we know that for all $j\in\{1,\dots,l\}$ and $r>0$ it holds that $V(x_{k_\tau|_h+q+j})-V(x_{k_\tau|_h})>-r\|x_{k_\tau|_h}\|^2-\sum_{i=k_\tau|_h+q+j}^{k+q+l}\|e_i\|^2=-(r+\sum_{i=j}^{l}c_i(\sigma))\|x_{k_\tau|_h}\|^2$ and thus, due to (11) and nonnegativity of the norm

$$\|x_{k_\tau|_h+q+j}\|^2\ge\max\left\{0,\frac{\lambda_{\min}(P)-r-\sum_{i=j}^{l}c_i(\sigma)}{\lambda_{\max}(P)}\right\}\|x_{k_\tau|_h}\|^2 .$$

Using this together with (17) one observes that $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-r\|x_{k_\tau|_h}\|^2$ holds in this case as well, and thus in general for $q\ge1$. As stated before we know due to Lemma 2 that $\tau_{k_\tau+1}-\tau_{k_\tau}\le\eta,\ \forall k_\tau\in\mathbb{N}_0$ and thus (18) holds with $\tilde{r}=\frac{r}{\eta}$ for $q\ge1$.

It remains to show that the things shown for $q\ge1$ using (17) also hold for $q=0$ using (16). We first recall that in this case the difference of the Lyapunov function simplifies to $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-\theta^2\|x_{k_\tau|_h}\|^2+\sum_{j=k_\tau|_h+1}^{k_\tau|_h+l}\|e_j\|^2-\theta^2\|x_j\|^2$. Due to the fact that $\|e_{k_\tau|_h}\|=0$ and Lemma 1 one observes that $\sum_{j=k_\tau|_h+1}^{k_\tau|_h+l}\|e_j\|^2\le\sum_{j=k_\tau|_h+1}^{k_\tau|_h+l}\|B_{2|j-k_\tau|_h}\|^2\|x_{k_\tau|_h}\|^2=\sum_{i=1}^{l}c_i(0)\|x_{k_\tau|_h}\|^2$ and thus $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le\left(-\theta^2+\sum_{j=1}^{l}c_j(0)\right)\|x_{k_\tau|_h}\|^2-\sum_{j=k_\tau|_h+1}^{k_\tau|_h+l}\|x_j\|^2$. The same arguments that have been used above can be used to show that as in the case $q\ge1$ either $V(x_{k_\tau|_h+q+l+1})-V(x_{k_\tau|_h})\le-r\|x_{k_\tau|_h}\|^2$ holds directly or the bound

$$\|x_{k_\tau|_h+j}\|^2>\max\left\{0,\frac{\lambda_{\min}(P)-r-\sum_{i=j}^{l}c_i(0)}{\lambda_{\max}(P)}\right\}\|x_{k_\tau|_h}\|^2$$

can be derived. Thus with (16), (18) holds for all $k_\tau\in\mathbb{N}_0$ with $\tilde{r}=\frac{r}{\eta}$ and the equilibrium $\tilde{x}=0$ of the discretized system, restricted to successful triggering instants is shown to be globally exponentially stable under the given assumptions. ■

Remark 5: Although conditions (16) and (17) look quite complicated, verification of given σ or finding a suitable σ is not too elaborate. One starts with fixing a value for r that corresponds to a desired convergence rate. Afterwards (16) is an inequality independent of σ that needs to be fulfilled. Regarding (17) one proceeds with a line search to find the maximal value σ such that the inequality is not violated.

D. Consequences for continuous-time sampled-data system

To analyze the continuous-time system, one can use an approach similar to [19]. It uses the uniform upper and lower boundedness of the time intervals between successful triggering from Lemma 2 to compute a uniform upper bound on the increase in between them. Together with exponential stability of the restricted system as shown in Theorem 4 global exponential stability of the sampled-data system can be shown. The details of the proof are omitted here due to space limitations. The result that can be derived is summarized in the following Corollary.

Corollary 6: Assume the continuous-time sampled-data system controlled in the same setup as in Theorem 4. Then there exist $k_c > 1$ and $\lambda_c > 0$ such that $\|x_c(t)\| \leq k_c \mathrm{e}^{-\lambda_c t}\|x_c(0)\|$ for all $x_{c,0}, \hat{x}_{c,0} \in \mathbb{R}^n$ and $t \geq 0$.

IV. STABILITY RESULT USING NON-MONOTONIC LYAPUNOV FUNCTIONS

The approach given in this paper was described to be a non-monotonic approach, since one demands the evolution of V to decrease only between successful triggering instants. Indeed, one can derive the results in a second way using theory on non-monotonic Lyapunov functions as described in [12]. It is shown by an example that this is a general approach to PETC with packet loss.

A. Proof based on non-monotonic Lyapunov functions

The particular result that we use from [12] is Theorem 6.4.7. The Theorem uses a Lyapunov function that is non-monotonically decreasing along solutions of the system to show global exponential stability of the origin of a continuous-time sampled-data system that needs to be formulated as a discontinuous dynamical system (DDS). With Theorem 6.4.7 from [12] we can give a second proof for the main result of the paper.

Theorem 7: Assume $0 < \sigma < \theta$ fulfills the assumptions of Theorem 4. Then the equilibrium at the origin $\begin{bmatrix} x_c^\top & e_c^\top \end{bmatrix}^\top = \begin{bmatrix} 0^\top & 0^\top \end{bmatrix}^\top$ of the continuous-time sampled-data system (1), (5), where triggering is attempted when either (12) or (15) with threshold σ is violated, with at most m successive packet losses is globally exponentially stable.

Proof: As stated above we need to model the sampled-data system as a discontinuous dynamical system. We use $\xi = \begin{bmatrix} x_c^\top & e_c^\top \end{bmatrix}^\top$ and the sequence of successful triggering instants to rewrite the closed-loop sampled-data system as

$$\dot{\xi}(t) = \underbrace{\begin{bmatrix} A+BK & BK \\ -(A+BK) & -BK \end{bmatrix}}_{A_\xi} \xi(t), \ \tau_{k-1} \leq t \leq \tau_k$$

$$\xi(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \xi(t^-), \qquad t = \tau_k, \ k \in \mathbb{N}_0 \quad (20)$$

with $\xi(0^-) = \begin{bmatrix} x_{c,0}^\top & \hat{x}_{c,0}^\top - x_{c,0} \end{bmatrix}^\top$ and $\tau_{-1} := 0^- = t_0$. As a Lyapunov function we use $v(\xi, t) = \xi^\top \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \xi$ with P as in Section III. We observe that we can now use Theorem 6.4.7 to analyze the stability of (20) where the unbounded discrete subset from [12] with $E = \{\rho_1, \rho_2, \dots\}$ equals the set of successful triggering instants $\mathscr{T} = \{\tau_0, \tau_1, \dots\}$ and $\rho_0 = t_0 = \tau_{-1} = 0^-$. It remains now to show that equations (6.59), (6.60), and (6.62) from [12] hold. At first one can see that (6.59) holds with $r_1 = \min\{\lambda_{\min}(P), 1\}$ and $r_2 = \max\{\lambda_{\max}(P), 1\}$ due to positive definiteness of P (r_i replaces c_i in the notation of [12]).

We will now start with checking (6.62) for ρ_k with $k \in \mathbb{N}$, or equivalently τ_k with $k \in \mathbb{N}_0$, i.e., $\frac{1}{\tau_{k+1}-\tau_k}[v(\phi(\tau_{k+1}, t_0, \xi_0), \tau_{k+1}) - v(\phi(\tau_k, t_0, \xi_0), \tau_k)] \leq -\tilde{r}\|x_c(\tau_k)\|^2 = -\tilde{r}\|\xi(\tau_k)\|^2$, where we used the fact that $e_c(\tau_k) = 0$ for all $k \in \mathbb{N}_0$ and the knowledge from the proof of Theorem 4 that (18) holds for all $k_\tau \in \mathbb{N}_0$ under the assumptions of Theorem 4. Furthermore we have to analyze

$$\frac{1}{\tau_0 - \tau_{-1}} \left[v(\phi(\tau_0, t_0, \xi_0), \tau_0) - v(\phi(\tau_{-1}, t_0, \xi_0), \tau_{-1})\right]. \quad (21)$$

We note that if triggering is necessary immediately after initialization, i.e., $\tau_0 = 0$ we know $\|e_c(0^-)\|^2 > \sigma^2\|x_c(0^-)\|^2$ and $x_c(\tau_0) = x_c(\tau_{-1})$. Thus one can compute (21) $= -\frac{1}{\tau_0-\tau_{-1}}\left[\frac{\sigma^2}{1+\sigma^2}\|e_c(\tau_{-1})\|^2 + \frac{1}{1+\sigma^2}\|e_c(\tau_{-1})\|^2\right] \leq -\frac{1}{\eta}\frac{\sigma^2}{1+\sigma^2}\left[\|e_c(\tau_{-1})\|^2 + \|x(\tau_{-1})\|^2\right] =: -\tilde{r}\|\xi(\tau_{-1})\|^2$. If no triggering is necessary after initialization we know by assumption that $\|e_c(\tau_{-1})\|^2 \leq \sigma^2\|x_c(\tau_{-1})\|^2$, thus (21) $\leq -\frac{1}{\eta}\left[(\theta^2 - \sigma^2)\|x_c(\tau_{-1})\|^2 + \|e_c(\tau_{-1})\|^2\right] \leq -\tilde{r}''\|\xi(\tau_{-1})\|$. Thus we showed that (6.62) holds under the given assumptions with $r_3 = \min\{\tilde{r}, \tilde{r}', \tilde{r}''\}$. It remains to show that (6.60) holds for all ρ_k with $k \in \mathbb{N}_0$. We know that $v(\xi, t) \leq r_2\|\xi(t)\|^2$. Furthermore we know that $\dot{\xi}(t) = A_\xi \xi(t)$ for $t \in (\tau_{k-1}, \tau_k), k \in \mathbb{N}_0$. Thus we can use $\mu(A_\xi) := \lambda_{\max}(\frac{1}{2}(A_\xi^\top + A_\xi))$ to derive $\|\xi(t)\|^2 = \|\mathrm{e}^{A_\xi(t-\tau_k)}\xi(\tau_k)\|^2 \leq \|\mathrm{e}^{\mu(A_\xi)(t-\tau_k)}\|^2\|\xi(\tau_k)\|^2$. Using Lemma 2 and the subsequently given Remark we know $\tau_k - \tau_{k-1} \leq \eta$ for all $k \in \mathbb{N}_0$. Thus, $r_2\|\xi(t)\|^2 \leq r_2\|\mathrm{e}^{\mu(A_\xi)\eta}\|^2\|\xi(\tau_k)\|^2$. Due to the fact that $\|\xi(\tau_k)\|^2 \leq \frac{1}{r_1}v(\xi(\tau_k), \tau_k)$ it follows that $v(\xi(t), t) \leq \frac{r_2}{r_1}\|\mathrm{e}^{\mu(A_\xi)\eta}\|^2 v(\xi(\tau_k), \tau_k)$ for all $t \in (\tau_{k-1}, \tau_k), k \in \mathbb{N}_0$. Thus, (6.60) holds with $f(r) = \frac{r_2}{r_1}\|\mathrm{e}^{\mu(A_\xi)\eta}\|^2 r$. ■

B. Remarks on general non-monotonic framework

If this approach is to be used to generate trigger rules that guarantee stability despite packet loss, the main challenges are as follows. One needs to design a trigger rule that guarantees uniform upper and lower boundedness of the inter-event intervals. Additionally the trigger rule needs to guarantee a limited increase of a Lyapunov function in the sense of (6.60) and a decrease between successful triggering instants in the sense of (6.62).

Example 8: In [8] the instants when triggering is necessary are given, according to our notation, as $\tilde{\tau}_{k+1} = \min\{t \in \{\tilde{\tau}_k + h, \tilde{\tau}_k + 2h, \dots\} : t \geq \tilde{\tau}_k + \nu \text{ or } V(Ax(t) + Bu(\tilde{\tau}_k)) > \beta V(x(\tilde{\tau}_k))\}$. Thus the inter-event times are explicitly uniformly upper bounded by $m\nu$ in the case of at most m

TABLE I

Derived bounds on $\sigma > 0$ for $h = 0.02$ and different values of m

m	0	1	2	3	4	5
$\sigma_{\max}$	0.1683	0.1450	0.1060	0.0770	0.0520	0.0300

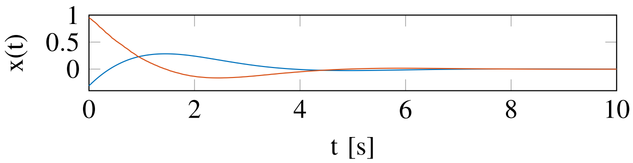

Fig. 1. Evolution of the state x under PETC with packet loss.

successive packet losses. In the setup of that paper control input zero is applied when a packet loss occurs. Thus the conditions $\beta P - (A+BK)^\top P(A+BK) \geq 0$ and $\varphi P - A^\top PA \geq 0$ demand a desired decrease when no triggering is necessary or triggering is successful as well as a maximum increase, that satisfies (6.60) when packets are lost. Thus, if additionally the requirement $\varphi^m \beta < 1$ is fulfilled, the Lyapunov function decreases between successful triggering instants and global exponential stability can be shown using Theorem 6.4.7.

V. Numerical Example

In this section we consider the numerical example used in [5] with $A_c = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $K = \begin{bmatrix} 1 - 4 \end{bmatrix}$. We consider a network that provides a maximum sampling time $h = 0.02s$. According to the rules in Theorem 4 we can now compute bounds on σ for different values of m such that the continuous-time sampled-data system is globally exponentially stable despite the possibility of at most m successive packets being lost. The bounds on $\sigma > 0$ that can be derived are given in Table I.

In the following we assume a network that guarantees a maximal sampling rate of $h = 0.02s$ and at most $m = 3$ successive packet losses. According to Table I we select $\sigma = 0.077$ and simulate the system with initial conditions $x_{c,0} = [-0.309, 0.951]^\top$ and $\hat{x}_{c,0} = [-0.306, 0.942]^\top$. The evolution of the sates is shown in Figure 1, where one can observe that the origin is indeed stabilized. Additionally it is of interest to observe how the trigger error evolves. In Figure 2 the evolution of $\frac{\|e_c(t)\|}{\|x_c(t)\|}$ is shown. Furthermore the value chosen for σ due to Table I as well as the value for σ known to be sufficient in absence of packet losses is shown. The red asterisks mark time instants where the trigger rules are evaluated and the trigger rule with $\sigma = 0.1683$, known to be sufficient for 0 packet loss, is violated. The existence of these instants shows the conceptual difference of our approach to deal with packet loss in PETC. As mentioned before, in other works, e.g. [10], a bound for σ in the case of no packet loss is computed. Afterwards this bound is lowered such that the original trigger rule is never violated despite the fact that packet loss is considered. Thus a behavior as in Fig. 2 implies that we investigate a different approach here that can be used to stabilize the continuous-time system.

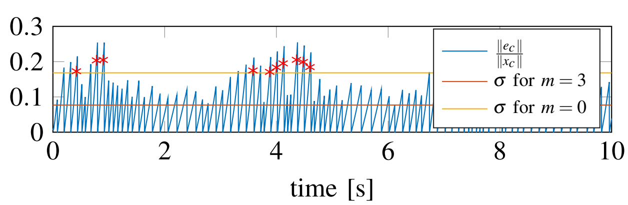

Fig. 2. Evolution of the trigger error and different thresholds.

VI. Conclusions

In this paper we presented a new approach to deal with packet loss in PETC. First a detailed derivation for a specific trigger rule was presented. Building up on this analysis the procedure was generalized as a non-monotonic approach. An example was given to show that other trigger rules can be generated within this framework. This observation gives rise to possible further work including more general network specifications and system dynamics in the setup. Furthermore a constructive development of the framework should focus on including other network-induced imperfections as well.

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