High-order Barrier Functions: Robustness, Safety and Performance-Critical Control

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Abstract—In this paper, we propose a notion of high-order (zeroing) barrier functions that generalizes the concept of zeroing barrier functions and guarantees set forward invariance by checking their higher order derivatives. The proposed formulation guarantees asymptotic stability of the forward invariant set, which is highly favorable for robustness with respect to model perturbations. No forward completeness assumption is needed in our setting in contrast to existing high order barrier function methods. For the case of controlled dynamical systems, we relax the requirement of uniform relative degree and propose a singularity-free control scheme that yields a locally Lipschitz control signal and guarantees safety. Furthermore, the proposed formulation accounts for "performance-critical" control: it guarantees that a subset of the forward invariant set will admit any existing, bounded control law, while still ensuring forward invariance of the set. Finally, a non-trivial case study with rigid-body attitude dynamics and interconnected cell regions as the safe region is investigated.

I. INTRODUCTION

Optimizing system performance while satisfying safety guarantees is an important goal for controlling dynamical systems. For a general nonlinear system wherein an analytical solution is difficult to compute, model predictive control (MPC) and barrier function techniques are two relevant tools to guarantee constraint satisfaction i.e., safety. MPC [1]–[3] is a powerful tool that takes all safety constraints into account at every discrete time instant and solves an optimization problem up to a finite horizon with the system performance metric as the objective function. This inevitably brings heavy computational burden for online implementation and the resulting controller provides constraint satisfaction and optimality. Barrier functions, on the other hand, provide a system-level certificate that guarantees the forward invariance of a set, usually referred to as the "safety set", that can be designed in parallel to a performance-optimizing controller [4]. This modular formulation gives designers greater flexibility.

There are several types of "barrier functions" in the literature. One is related to *barrier Lyapunov functions* [5] that were introduced and extensively studied for constrained control problems. Barrier Lyapunov functions are constructed so that they tend to infinity when the system's state approaches the boundary of the safety set. Using backstepping techniques, barrier Lyapunov functions are extendable to high-order control systems. The term "barrier" is taken from

optimization theory [6] wherein barrier/penalty terms are used to avoid exploration of unwanted regions. An extension of this methodology, later coined *reciprocal barrier functions* [7], is presented in [8]. Reciprocal barrier functions also blow up at the safety boundary and guarantee forward invariance of the safe set if a Lyapunov-like condition holds. Another form of barrier functions, also known as barrier certificates, arise from system verification. Those barrier certificates are Lyapunov-like functions that are used to verify safety of nonlinear and stochastic systems [9], [10]. In those methods, the unsafe region is described by the superlevel set of a real-valued function and if the derivative of this function is negative definite, then the system is verified to be safe. The controlled version is also discussed in [11]. A major limitation of reciprocal barrier functions is that a large control signal is required when the system's state is close to the boundary of the safety set, making it sensitive to noises in the system. On the other hand, barrier certificates ensure invariance of every level set, which indicates that the condition imposed is too strong and restrictive.

Recently, [7], [12] proposed *zeroing barrier functions* (ZBFs) that are well-defined both inside and outside the safe set, and only ensure invariance of the safe set. More importantly, ZBFs provide robustness properties with respect to model perturbations. Robustness is addressed by ensuring asymptotic stability of the forward invariant set and an Input-to-State stability property of the safe set is established. *Zeroing control barrier functions* (ZCBFs), the controlled version of zeroing barrier functions, originally addressed relative degree one constraints, and robustness was further studied in [13]. This tool is applicable in a wide range of applications, e.g., in multi-robot coordination, verification and control [14]–[16].

Recently, [17]–[20] have started to investigate conditions on the higher order derivatives of constraint functions to guarantee set invariance. This is motivated by two facts: 1) by examining the conditions on the high-order derivative terms, an alternative method to find barrier functions is provided; 2) many constraints have higher relative degrees with respect to the underlying system, e.g., a position constraint for a mechanical system. Thus a systematic framework for higher order barrier functions is highly relevant for real-world applications. Although many important results have been obtained in [17]–[20], we argue that the formulations therein have certain limitations in the sense discussed below and can be considered as special cases of the results presented here.

In this paper, we propose a novel definition of high-order barrier functions (HOBFs) that generalizes the concept of zeroing barrier functions [7], [12] and the formulations in

[17]–[20]. In our formulation, extended class $\mathcal{K}$ functions are incorporated instead of linear functions [17], [18] or class $\mathcal{K}$ functions [19]. Apart from this definition generalization, the contributions of this paper are stated as follows:

1) In our formulation, the forward completeness assumption in [18], [19] is no longer required. More importantly, the forward invariant set is proven to be asymptotically stable for the first time in an HOBF setting and inherits all the robustness properties of ZBFs as in [12].

2) For the controlled system, we allow the relative degree to vary in the safe region, which relaxes the uniform relative degree assumption in [18], [19]. The high-order control barrier function is constructed by introducing a truncating function to the original constraint. The obtained control law is shown to be Lipschitz continuous and the safe set is guaranteed to be forward invariant.

3) In many applications, a pre-designed nominal control law must be implemented without modification in a desired region to ensure satisfaction of the task. This is coined a *performance-critical task*. Most ZCBF methods aim to be minimally invasive, but do not specify when the nominal control will be implemented *a priori*. Our formulation allows one to design performance-critical regions where the nominal input will be used.

Notation: The Lie derivatives of a function $h(\boldsymbol{x})$ for the system $\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})\boldsymbol{u}$ are denoted by $L_{\mathfrak{f}}h = \frac{\partial h}{\partial \boldsymbol{x}}\mathfrak{f}(\boldsymbol{x})$ and $L_{\mathfrak{g}}h = \frac{\partial h}{\partial \boldsymbol{x}}\mathfrak{g}(\boldsymbol{x})$, respectively. The notations $\prec, \preceq$ and $\succ, \succeq$ are used to denote element-wise vector inequalities. The interior and boundary of a set $\mathscr{A}$ are denoted $\text{Int}(\mathscr{A})$ and $\partial \mathscr{A}$, respectively. The distance from a point $\boldsymbol{x}$ to a set $\mathscr{A} \subset \mathbb{R}^n$ is given by $\|\boldsymbol{x}\|_{\mathscr{A}} := \inf_{\boldsymbol{w} \in \mathscr{A}}\|\boldsymbol{x} - \boldsymbol{w}\|$. The tangent cone to the set $\mathscr{A}$ at the point $\boldsymbol{x}$ is defined as $\mathcal{T}_{\mathscr{A}}(\boldsymbol{x}) := \{\boldsymbol{z} : \liminf_{\tau \to 0}\|\boldsymbol{x} + \tau\boldsymbol{z}\|_{\mathscr{A}}/\tau = 0\}$. Denote $\mathbb{R}^n_+ := \{\boldsymbol{a} \in \mathbb{R}^n : a_i \geq 0\}$, where a_i corresponds to the ith component of $\boldsymbol{a}$. We note that $\boldsymbol{x} \in \partial\mathbb{R}^n_+$ if $\boldsymbol{a}^\top\boldsymbol{x} = 0$ for some nonzero vector $\boldsymbol{a} \in \mathbb{R}^n_+$ and $\boldsymbol{x} \in Int(\mathbb{R}^n_+)$ if $\boldsymbol{a}^\top\boldsymbol{x} > 0$ for all nonzero vectors $\boldsymbol{a} \in \mathbb{R}^n_+$.

II. High-order barrier functions

In this section, we propose a novel HOBF definition, which generalizes the zeroing barrier functions from [7], [12]. The proposed HOBF formulation is more general than previous constructions [18]–[20], and is robust to perturbations.

Consider a nonlinear system on $\mathbb{R}^n$,

$$\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) \tag{1}$$

with $\mathfrak{f}$ locally Lipschitz continuous. Denote by $\boldsymbol{x}(t, \boldsymbol{x}_0)$ the solution of (1) starting from $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. A set $\mathscr{A} \subset \mathbb{R}^n$ is called *forward invariant*, if for any initial condition $\boldsymbol{x}_0 \in \mathscr{A}$, $\boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathscr{A}$ for all $t \in I(\boldsymbol{x}_0)$. Here $I(\boldsymbol{x}_0)$ denotes the maximal time interval of existence of $\boldsymbol{x}(t, \boldsymbol{x}_0)$.

Let $h(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. We define the associated sets as $\mathscr{C}_h = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) \geq 0\}$, $\mathscr{C}_{h,\delta} = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) \geq \delta\}$.

High-order barrier functions are dependent on extended class $\mathcal{K}$ functions, which are defined as follows:

Definition 1 (Extended class $\mathcal{K}$ function [7]). *A continuous function $\alpha : (-b, a) \to (-\infty, \infty)$ for $a, b \in \mathbb{R}_{>0}$ is an extended class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$.*

Note for clarity, the extended class $\mathcal{K}$ functions addressed here will be defined for $a, b = \infty$.

A. High-order barrier functions

The class of high-order barrier functions considered in this paper is defined as follows. Given a rth-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, and sufficiently smooth extended class $\mathcal{K}$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \cdots, \alpha_r(\cdot)$, we define a series of functions as

$$\psi_0(\boldsymbol{x}) = h(\boldsymbol{x}), \psi_k(\boldsymbol{x}) = (\tfrac{d}{dt} + \alpha_k)\psi_{k-1}, \ 1 \leq k \leq r, \tag{2}$$

with the corresponding sets: $\mathscr{C}_{\psi_{k-1}} = \{x : \psi_{k-1}(\boldsymbol{x}) \geq 0\}$.

Definition 2 (High-order (zeroing) barrier function). *A rth-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a **high-order (zeroing) barrier function** of degree r for system (1) if there exist differentiable extended class $\mathcal{K}$ functions $\alpha_k, k = 1, 2, \cdots, r$ and an open set $\mathscr{D}$ with $\mathscr{C} := \bigcap_{k=1}^r \mathscr{C}_{\psi_{k-1}} \subset \mathscr{D} \subset \mathbb{R}^n$ such that*

$$\psi_r(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathscr{D}, \tag{3}$$

with $\psi_k(\boldsymbol{x})$ defined in (2).

Proposition 1. *Consider an autonomous system in (1) and a rth order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. If h is an HOBF defined on the open set $\mathscr{D}$ with $\mathscr{C} := \bigcap_{k=1}^r \mathscr{C}_{\psi_{k-1}} \subset \mathscr{D} \subset \mathbb{R}^n$, then $\mathscr{C}$ is forward invariant.*

Proof. For all $\boldsymbol{x} \in \mathscr{C} \subset \mathscr{D}$, $\psi_r(\boldsymbol{x}) \geq 0$, we obtain

$$\frac{\partial\psi_{k-1}}{\partial\boldsymbol{x}}\mathfrak{f}(\boldsymbol{x}) = \frac{d}{dt}\psi_{k-1}(\boldsymbol{x}) = -\alpha_k(\psi_{k-1}(\boldsymbol{x})) + \psi_k(\boldsymbol{x})$$
$$\geq -\alpha_k(\psi_{k-1}(\boldsymbol{x})), \quad 1 \leq k \leq r, \forall \boldsymbol{x} \in \mathscr{C}.$$

We thus have

$$\frac{\partial\psi_{k-1}}{\partial\boldsymbol{x}}\mathfrak{f}(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \partial\mathscr{C}_{\psi_{k-1}} \cap \mathscr{C} \subset \mathscr{C},$$

Thus, by definition of the tangent cone,

$$\mathfrak{f}(\boldsymbol{x}) \in \mathcal{T}_{\mathscr{C}_{\psi_{k-1}}}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \partial\mathscr{C}_{\psi_{k-1}} \cap \mathscr{C} \subset \mathscr{C}.$$

Let $\text{Act}(\boldsymbol{x})$ denote the set of active constraints, i.e., $\text{Act}(\boldsymbol{x}) = \{k : \psi_{k-1}(\boldsymbol{x}) = 0\}$. Thus, for $\boldsymbol{x} \in \partial\mathscr{C}$, the following holds

$$\mathfrak{f}(\boldsymbol{x}) \in \mathcal{T}_{\mathscr{C}_{\psi_{k-1}}}(\boldsymbol{x}), \quad k \in \text{Act}(\boldsymbol{x})$$

This implies that $\mathfrak{f}(\boldsymbol{x}) \in \mathcal{T}_{\mathscr{C}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathscr{C}$. Since $\mathfrak{f}$ is locally Lipschitz, the application of Brezis's Theorem [21, Theorem 4] ensures that the set $\mathscr{C}$ is forward invariant. $\square$

Remark 1. *Nagumo's Theorem [22, Theorem 4.7] has been applied in the barrier function community to guarantee forward invariance. However, we need to point out that Nagumo's theorem cannot be applied in the previous proof because, to guarantee forward invariance, it requires forward completeness of the system (1), which is not assumed in our case. Instead, Brezis's theorem dictates that with a locally Lipschitz continuous vector field $\mathfrak{f}$ and a closed set $\mathscr{A}$, $\mathfrak{f}(\boldsymbol{x}) \in \mathcal{T}_{\mathscr{A}}$ for all*

$\boldsymbol{x} \in \mathscr{A}$ implies that $\mathscr{A}$ is forward invariant up to the maximal time interval. If we further assume the set $\mathscr{A}$ is compact, then the solution remains in $\mathscr{A}$ for all $t \geq t_0$.

Definition 2 and Proposition 1 are generalizations of similar concepts proposed in [18] and [19]. In [18], each α_k is restricted to the class of linear functions, i.e., $\alpha_k(v) = a_k v, a_k > 0, 1 \leq k \leq r$, whereas our results hold for any extended class-$\mathcal{K}$ function. In [19], the HOBFs are not well-defined outside of their safe sets due to the restriction to class-$\mathcal{K}$ functions. Here we let each α_k be an extended class $\mathcal{K}$ function, which is well-defined even if $\psi_{k-1}(\boldsymbol{x}) < 0, 1 \leq k \leq r$. This is important to address robustness as will be shown in the following section.

B. Asymptotic stability of the set $\mathscr{C}$

Here, we assume the system (1) is forward complete to comply with the conditions for asymptotic stability to a set. Before addressing asymptotic stability, we first recall a generalized comparison lemma from [23]. The vector inequalities used here are to be interpreted component-wise.

Definition 3. *A function $\boldsymbol{p} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is called quasimonotone nondecreasing if, for $1 \leq i \leq n$ and all $1 \leq j \leq n, j \neq i$, $x_i = y_i, x_j \leq y_j$ implies that*

$$p_i(t, \boldsymbol{x}) \leq p_i(t, \boldsymbol{y}) \tag{4}$$

for the ith component of $\boldsymbol{p}(t, \cdot)$ and for each t.

To understand this definition, we present a simple example. Suppose $\boldsymbol{p}(t, \boldsymbol{v}) = A\boldsymbol{v}$. If $\boldsymbol{p}$ is quasimonotone nondecreasing, then all the off-diagonal elements in A must be nonnegative. Also one can verify that, if $\boldsymbol{p}$ is quasimonotone nondecreasing, then $\boldsymbol{y} - \boldsymbol{x} \succeq \boldsymbol{0}$, $\boldsymbol{a}^\top(\boldsymbol{y} - \boldsymbol{x}) = 0$ for some nonzero vector $\boldsymbol{a} \in \mathbb{R}_+^n$ implies that $\boldsymbol{a}^\top(\boldsymbol{p}(t, \boldsymbol{y}) - \boldsymbol{p}(t, \boldsymbol{x})) \geq 0$.

Lemma 1. *[23, Modified from Theorem 1.5.4] Consider the vectorial differential system*

$$\frac{d}{dt}\boldsymbol{v} = \boldsymbol{p}(t, \boldsymbol{v}), \boldsymbol{v}(t_0) = \boldsymbol{v}_0 \tag{5}$$

where $\boldsymbol{p} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is quasimonotone nondecreasing and let $\boldsymbol{r}(t)$ be the maximal solution existing on $[t_0, \infty)$. Suppose that a continuous function $\boldsymbol{m} \in C[\mathbb{R}_+, \mathbb{R}^n]$ satisfies, for some fixed Dini derivative[1],

$$D\boldsymbol{m}(t) \preceq \boldsymbol{p}(t, \boldsymbol{m}(t)), \quad t \in [t_0, \infty). \tag{6}$$

Then, $\boldsymbol{m}(t_0) \preceq \boldsymbol{v}_0$ implies

$$\boldsymbol{m}(t) \preceq \boldsymbol{r}(t), \quad t \in [t_0, \infty). \tag{7}$$

The difference between this Lemma and Theorem 1.5.4 of [23] is that we do not need the domain of $\boldsymbol{p}$ to be $\mathbb{R}_+ \times \mathbb{R}_+^n$, nor do we require $\boldsymbol{m}(t_0), \boldsymbol{v}_0$ to be in $\mathbb{R}_+^n$. The proof is almost identical and presented here for completeness.

Proof. We first introduce an auxiliary system. From [23, Theorem 1.5.1], we know that for any compact interval $[t_0, T]$,

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[1] For a continuous vectorial function $\boldsymbol{m} : \mathbb{R} \to \mathbb{R}^n$, four forms of Dini derivatives of $\boldsymbol{m}$ at t are defined as follows: $D^+\boldsymbol{m}(t) = \limsup_{h \to 0^+}(\boldsymbol{m}(t + h) - \boldsymbol{m}(t))/h, D^-\boldsymbol{m}(t) = \limsup_{h \to 0^-}(\boldsymbol{m}(t + h) - \boldsymbol{m}(t))/h, D_+\boldsymbol{m}(t) = \liminf_{h \to 0^+}(\boldsymbol{m}(t + h) - \boldsymbol{m}(t))/h, D_-\boldsymbol{m}(t) = \liminf_{h \to 0^-}(\boldsymbol{m}(t + h) - \boldsymbol{m}(t))/h.$

there exists an $\boldsymbol{\epsilon}_0 \succ \boldsymbol{0}$ such that for constant vector $\boldsymbol{\epsilon}, \boldsymbol{0} \prec \boldsymbol{\epsilon} \prec \boldsymbol{\epsilon}_0$, solutions $\boldsymbol{v}(t, \boldsymbol{\epsilon})$ of $\frac{d}{dt}\boldsymbol{v} = \boldsymbol{p}(t, \boldsymbol{v}) + \boldsymbol{\epsilon}, \boldsymbol{v}(t_0) = \boldsymbol{v}_0 + \boldsymbol{\epsilon}$ exist on $[t_0, T]$ and $\lim_{\boldsymbol{\epsilon} \to \boldsymbol{0}} \boldsymbol{v}(t, \boldsymbol{\epsilon}) = \boldsymbol{r}(t)$ uniformly on $[t_0, T]$. From [23, Lemma 1.5.1] and the condition (6), we know that $D_-\boldsymbol{m}(t) \preceq \boldsymbol{p}(t, \boldsymbol{m}(t)), t \geq t_0$, where $D_-\boldsymbol{m}(t) = \liminf_{h \to 0^-}(\boldsymbol{m}(t + h) - \boldsymbol{m}(t))/h$.

It is enough to show that, for arbitrary compact interval $[t_0, T]$ and sufficiently small $\boldsymbol{\epsilon} \succ \boldsymbol{0}$,

$$\boldsymbol{m}(t) \prec \boldsymbol{v}(t, \boldsymbol{\epsilon}), \quad t \in [t_0, T] \tag{8}$$

If (8) is not true for some time instant, since $\boldsymbol{m}(t_0) \preceq \boldsymbol{v}_0 \prec \boldsymbol{v}_0 + \boldsymbol{\epsilon} = \boldsymbol{v}(t_0, \boldsymbol{\epsilon})$ and the continuity of $\boldsymbol{m}(t), \boldsymbol{v}(t, \boldsymbol{\epsilon})$, there exists a $t_1 \in [t_0, T]$ such that, $\boldsymbol{v}(t, \boldsymbol{\epsilon}) - \boldsymbol{m}(t) \succ \boldsymbol{0}$ for all $t \in [t_0, t_1)$ and

$$\boldsymbol{v}(t_1, \boldsymbol{\epsilon}) - \boldsymbol{m}(t_1) \in \partial\mathbb{R}_+^n. \tag{9}$$

(9) means $\boldsymbol{v}(t_1, \boldsymbol{\epsilon}) - \boldsymbol{m}(t_1)$ is at the boundary of $\mathbb{R}_+^n$, hence a nonzero vector $\boldsymbol{a} \in \mathbb{R}_+^n$ exists such that $\boldsymbol{a}^\top(\boldsymbol{v}(t_1, \boldsymbol{\epsilon}) - \boldsymbol{m}(t_1)) = 0$. Employing the quasimonotone nondecreasing property of $\boldsymbol{p}$, it now yields $\boldsymbol{a}^\top(\boldsymbol{p}(t_1, \boldsymbol{v}(t_1, \boldsymbol{\epsilon})) - \boldsymbol{p}(t_1, \boldsymbol{m}(t_1))) \geq 0$. Let $w(t) = \boldsymbol{a}^\top(\boldsymbol{v}(t, \boldsymbol{\epsilon}) - \boldsymbol{m}(t)), t \in [t_0, t_1)$. Since $h < 0, w(t_1 + h) > 0$ (as a result of $\boldsymbol{v}(t, \boldsymbol{\epsilon}) - \boldsymbol{m}(t) \in Int(\mathbb{R}_+^n)$ for $t \in [t_0, t_1)$) and $w(t_1) = 0$, we obtain $D_-w(t_1) = \liminf_{h \to 0^-}(w(t_1 + h) - w(t_1))/h \leq 0$

However, from the quasimonotone nondecreasing property, we get $D_-w(t_1) = \boldsymbol{a}^\top(D_-\boldsymbol{v}(t_1, \boldsymbol{\epsilon}) - D_-\boldsymbol{m}(t_1)) = \boldsymbol{a}^\top(\frac{d}{dt}\boldsymbol{v}(t, \boldsymbol{\epsilon})|_{t=t_1} - D_-\boldsymbol{m}(t_1)) = \boldsymbol{a}^\top(\boldsymbol{p}(t_1, \boldsymbol{v}(t_1, \boldsymbol{\epsilon})) + \boldsymbol{\epsilon} - D_-\boldsymbol{m}(t_1)) > \boldsymbol{a}^\top(\boldsymbol{p}(t_1, \boldsymbol{v}(t_1, \boldsymbol{\epsilon})) - D_-\boldsymbol{m}(t_1)) \geq 0$, which is contradiction. Hence the proof is complete. □

Now we proceed to our analysis of the high-order terms in (2). First we note that for a given set of α_k functions, each ψ_{k-1} is governed by the system dynamics (1). We can however rearrange these equations as follows:

$$\begin{bmatrix} \dot{\psi}_0 \\ \dot{\psi}_1 \\ \dots \\ \dot{\psi}_{r-1} \end{bmatrix} = \begin{bmatrix} -\alpha_1(\psi_0) + \psi_1 \\ -\alpha_2(\psi_1) + \psi_2 \\ \dots \\ -\alpha_r(\psi_{r-1}) + \psi_r \end{bmatrix} \tag{10}$$

Interpreting (10) as a nonautonomous system with state variable $\boldsymbol{\psi} = (\psi_0, \psi_1, \cdots, \psi_{r-1})^\top$ and the time-varying term $\psi_r(\boldsymbol{x}(t))$, we can re-write (10) as

$$\frac{d}{dt}\boldsymbol{\psi} := \boldsymbol{p}(t, \boldsymbol{\psi}), \quad \boldsymbol{\psi}(t_0) := \boldsymbol{v}_0. \tag{11}$$

A key observation is that the function $\boldsymbol{p} : \mathbb{R}_+ \times \mathbb{R}^r \to \mathbb{R}^r$ is quasimonotone nondecreasing. This can be seen from the fact that, for any $i = 1, 2, \cdots, r - 1$, $p_i(t, \boldsymbol{\psi})$, the ith component of $\boldsymbol{p}(t, \boldsymbol{\psi})$, only contains two terms and is increasing with respect to ψ_i, the $(i + 1)$th component of the vector $\boldsymbol{\psi}$; for $i = r, p_r(t, \boldsymbol{\psi})$ only contains ψ_{r-1}, the rth component of the vector $\boldsymbol{\psi}$. A direct application of Lemma 1 yields:

Proposition 2. *Let $\boldsymbol{m} \in C^1(\mathbb{R}_+, \mathbb{R}^r)$, and let $\boldsymbol{\psi}(t)$ be the solution of (11). Then*

$$\frac{d}{dt}\boldsymbol{m}(t) \preceq \boldsymbol{p}(t, \boldsymbol{m}) \text{ for } t \geq t_0, \text{ and } \boldsymbol{m}(t_0) \preceq \boldsymbol{v}_0 \tag{12}$$

implies that

$$\boldsymbol{m}(t) \preceq \boldsymbol{\psi}(t) \text{ for } t \geq t_0. \tag{13}$$

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Proof. Since $\boldsymbol{p} : \mathbb{R}_+ \times \mathbb{R}^r \to \mathbb{R}^r$ is quasimonotone nondecreasing and $\boldsymbol{\psi}(t)$ exists for $t \in [0, \infty)$ (as the system (1) is forward complete), (13) follows directly from Lemma 1. $\square$

We next introduce an auxiliary system

$$\begin{bmatrix} \dot{m}_0 \\ \dot{m}_1 \\ \dots \\ \dot{m}_{r-1} \end{bmatrix} = \begin{bmatrix} -\alpha_1(m_0) + m_1 \\ -\alpha_2(m_1) + m_2 \\ \dots \\ -\alpha_r(m_{r-1}) \end{bmatrix}, \quad \boldsymbol{m}(t_0) := \boldsymbol{v}_0. \quad (14)$$

with the system state $\boldsymbol{m} = (m_0, m_1, \cdots, m_{r-1})^\top$. Note for h to be a HOBF, we require $\psi_r(\boldsymbol{x}) \geq 0$. Thus, the solution of the auxiliary system $\boldsymbol{m}(t)$ satisfies the conditions in Proposition 2, and $\boldsymbol{m}(t) \preceq \boldsymbol{\psi}(t)$ for all $t \geq t_0$.

Proposition 3. *Let $h : \mathscr{D} \to \mathbb{R}$ be a differentiable function defined on an open set $\mathscr{D} \subset \mathbb{R}^n$. If h is an HOBF for the system* (1)*, then the set $\mathscr{C} := \bigcap_{k=1}^r \mathscr{C}_{\psi_{k-1}}$ is asymptotically stable.*

Proof. We first show the following claim.

Claim 1: The origin of (14) is globally asymptotically stable.

Proof. The system (14) has a cascaded structure. We define a class of systems $\Sigma_k, k \in \{1, 2, ..., r-1\}$,

$$\Sigma_k : \begin{cases} \dot{m}_{k-1} = -\alpha_k(m_{k-1}) + m_k, \\ \dot{m}_k = -\alpha_{k+1}(m_k) + m_{k+1}, \\ \cdots, \\ \dot{m}_{r-1} = -\alpha_r(m_{r-1}) \end{cases}$$

with the system states $\boldsymbol{m}_k = (m_{k-1}, m_k, \cdots, m_{r-1})^\top$ and the initial value drawn from the corresponding components of $\boldsymbol{v}_0$. It is clear that the auxiliary system (14) is exactly the system Σ_1.

We prove Claim 1 in an inductive manner. First we show that the system $\Sigma_r : \dot{m}_{r-1} = -\alpha_r(m_{r-1})$ is globally asymptotically stable. Then we show that if the system Σ_k is globally asymptotically stable, so is the system Σ_{k-1}.

For the radially unbounded, positive definite Lyapunov function $V_r(m_{r-1}) = m_{r-1}^2/2$, we obtain $\dot{V}_r = -m_{r-1}\alpha_r(m_{r-1})$, which is is negative definite. Thus, the system Σ_r is globally asymptotically stable.

Assume system Σ_k is globally asymptotically stable. As a result, the system trajectory $\boldsymbol{m}_k(t)$ is bounded. For the Lyapunov candidate $V_{k-1}(m_{k-2}) = m_{k-2}^2/2$. Differentiation of V_{k-1} yields $\dot{V}_{k-1}(m_{k-2}) = -m_{k-2}\alpha_{k-1}(m_{k-2}) + m_{k-2}m_{k-1}$. Since $|m_{k-1}(t)| \leq \|\boldsymbol{m}_k(t)\|$ is bounded, $\lim_{t\to\infty} m_{k-1}(t) = 0$, we obtain that $m_{k-2}(t)$ is bounded. Thus $\boldsymbol{m}_{k-1}(t) = (m_{k-2}, m_{k-1}, m_k, \cdots, m_{r-1})^\top$ is again bounded. From [24, Corollary 10.3.3], since Σ_k is globally asymptotically stable, $\dot{m}_{k-2} = -\alpha_{k-1}(m_{k-2})$ is globally asymptotically stable, and the integral curve of the composite system Σ_{k-1} is forward complete and bounded, we conclude that the system Σ_{k-1} is also globally asymptotically stable.

By induction, the system Σ_1 is globally asymptotically stable at the origin, which completes the proof. $\square$

Now we proceed to show asymptotic stability of the set $\mathscr{C}$. Consider the following two cases:

1) $\boldsymbol{\psi}(t_0) \in \mathbb{R}_+^n$, i.e., $\|\boldsymbol{x}(t_0)\|_{\mathscr{C}} = 0$.
   From Proposition 1, it implies $\boldsymbol{x}(t) \in \mathscr{C}$ for $t \geq t_0$. Thus, $\|\boldsymbol{x}(t)\|_{\mathscr{C}} = 0, \forall t \geq t_0$.
2) $\boldsymbol{\psi}(t_0) \notin \mathbb{R}_+^n$, i.e., $\|\boldsymbol{x}(t_0)\|_{\mathscr{C}} > 0$.
   From Proposition 2, we have $\boldsymbol{\psi}(t) \succeq \boldsymbol{m}(t), \forall t \geq t_0$. Following Claim 1, we further obtain $\lim_{t\to\infty} \boldsymbol{\psi}(t) \succeq \lim_{t\to\infty} \boldsymbol{m}(t) = \boldsymbol{0}$, which is equivalent to $\|\boldsymbol{x}(t)\|_{\mathscr{C}} \to 0$ as $t \to \infty$.

Thus, the set $\mathscr{C}$ is asymptotically stable. $\square$

Proposition 3 generalizes the asymptotic stability results of the set $\mathscr{C}$ for relative-degree one ZBFs [12, Proposition 4] to HOBFs. This property is beneficial in practice because it indicates several different robustness properties. As discussed in [12], for the perturbed system $\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})$, if $\mathfrak{g}(\boldsymbol{x})$ is a vanishing perturbation, i.e., $\mathfrak{g}(\boldsymbol{x})$ is continuous and satisfies $\|\mathfrak{g}(\boldsymbol{x})\| \leq \sigma(\|\boldsymbol{x}\|_{\mathscr{C}})$ for $\boldsymbol{x} \in \mathscr{D} \setminus \mathscr{C}$ and some class $\mathcal{K}$ function $\sigma(\cdot)$, then the set $\mathscr{C}$ is still asymptotically stable. If $\mathfrak{g}(\boldsymbol{x})$ is not vanishing but sufficiently small, i.e., there exists a positive constant k such that $\|\mathfrak{g}(\boldsymbol{x})\|_\infty \leq k$, then a new asymptotically stable set containing $\mathscr{C}$ as well as asymptotic convergence to this new set can be established. Interested readers can refer to [12] and the references therein for more details.

III. HIGH-ORDER CONTROL BARRIER FUNCTIONS

Consider the nonlinear control affine system

$$\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})\boldsymbol{u}, \quad (15)$$

with the state $\boldsymbol{x} \in \mathbb{R}^n$, and the control input $\boldsymbol{u} \in U \subset \mathbb{R}^m$. We will consider the simplified case where $\mathfrak{f}$ and $\mathfrak{g}$ are locally Lipschitz functions in $\boldsymbol{x}$.

Definition 4 (Least relative degree)**.** *Given an arbitrary set $\mathscr{D} \subset \mathbb{R}^n$. A rth-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ has least relative degree r in $\mathscr{D}$ for system* (15) *if $L_{\mathfrak{g}} L_{\mathfrak{f}}^k h(\boldsymbol{x}) = \boldsymbol{0}, \forall \boldsymbol{x} \in \mathscr{D}$ for $k = 1, 2, \cdots, r-2$.*

The least relative degree condition is much weaker compared to the uniform relative degree condition [19], since the latter further requires $L_{\mathfrak{g}} L_{\mathfrak{f}}^{r-1} h(\boldsymbol{x}) \neq \boldsymbol{0}, \forall \boldsymbol{x} \in \mathscr{D}$.

Formally, a high-order control barrier function is defined as follows:

Definition 5 (High-order (zeroing) control barrier function (HOCBF))**.** *Consider control system* (15)*, and a rth-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. The function h is called a **high-order (zeroing) control barrier function** (of order r), if there exist differentiable extended class $\mathcal{K}$ functions α_k, $k \in \{1, ..., r\}$, and an open set $\mathscr{D}$ with $\mathscr{C} := \bigcap_{k=1}^r \mathscr{C}_{\psi_{k-1}} \subset \mathscr{D} \subset \mathbb{R}^n$, where ψ_k is given in* (2)*, such that*

1) h is of least relative order r in $\mathscr{D}$;
2) for all $\boldsymbol{x} \in \mathscr{D}$,

$$\sup_{\boldsymbol{u} \in U} \psi_r(\boldsymbol{x}) = \sup_{\boldsymbol{u} \in U}[L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) + L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x})\boldsymbol{u}$$
$$+ \alpha_r(\psi_{r-1}(\boldsymbol{x}))] \geq 0. \quad (16)$$

When letting $r = 1$, an HOCBF yields the zeroing control barrier function of [12]. This definition is also more general to its counterparts in [18] and [19] since : 1) α_k in [18] is

4

restricted to the set of linear functions, while α_k in [19] is restricted to the set of class $\mathcal{K}$ functions. We note that class-$\mathcal{K}$ functions are not well-defined for $\boldsymbol{x} \in \mathscr{D} \setminus \mathscr{C}$, and thus the robustness results presented here cannot be applied to the barriers of [19]; 2) the uniform relative degree r condition is not needed here, and thus our formulation is less restrictive than that of [19]; 3) while [18] and [19] both assume the closed-loop system (15) to be forward complete to ensure forward invariance, this is not required here. Hereafter we denote $\alpha = \alpha_r$ for notational brevity.

Similar to Proposition 1, the following result guarantees the forward invariance of $\mathscr{C}$. Given an HOCBF h, for all $\boldsymbol{x} \in \mathscr{D}$, we define the set

$$
\begin{aligned}
K_{HOCBF}(\boldsymbol{x}) = \{\boldsymbol{u} \in U : & L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) \\
& + L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x})\boldsymbol{u} + \alpha(\psi_{r-1}(\boldsymbol{x})) \geq 0\}. \quad (17)
\end{aligned}
$$

Theorem 1. *Consider an HOCBF h, $\psi_{k-1}, 1 \leq k \leq r$ defined in (2). Then any locally Lipschitz continuous controller $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\boldsymbol{u}(\boldsymbol{x}) \in K_{HOCBF}$ will render the set $\mathscr{C} := \bigcap_{k=1}^{r} \mathscr{C}_{\psi_{k-1}}$ forward invariant for the system (15).*

Proof. The proof follows directly from Proposition 1. $\qquad\square$

Remark 2. *If there exists an HOCBF h and a locally Lipschitz continuous controller $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\boldsymbol{u}(\boldsymbol{x}) \in K_{HOCBF}$ and (15) forward complete, then the set $\mathscr{C}$ is asymptotically stable. This follows directly from the proof of Proposition 3. This property is useful in practice because, for example, when the system starts outside of the safe set $\mathscr{D} \setminus \mathscr{C}$, we know the system state will asymptotically reach the set $\mathscr{C}$.*

Remark 3. *Consider the perturbed system*

$$
\dot{\boldsymbol{x}} = \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}(\boldsymbol{x})\boldsymbol{u} + \mathfrak{p}(x)\boldsymbol{\omega}, \quad (18)
$$

where $\boldsymbol{\omega} \in \mathbb{R}^v$ is an external disturbance, while $\mathfrak{p}(x)\boldsymbol{\omega}$ represents a structured disturbance/uncertainty that is nether vanishing nor sufficiently small. If $L_{\mathfrak{p}}L_{\mathfrak{f}}^k h(\boldsymbol{x}) = \boldsymbol{0}, \forall \boldsymbol{x} \in \mathscr{D}$ for $k = 1, 2, \cdots, r-2$ (i.e., h has the same least relative degree with respect to $\boldsymbol{\omega}$ as with respect to $\boldsymbol{x}$), then we could robustify the HOCBF condition using a similar technique to [13] by requiring $L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) + L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x})\boldsymbol{u} + \|L_{\mathfrak{p}}\psi_{r-1}(\boldsymbol{x})\|\bar{\omega} + \alpha_r(\psi_{r-1}(\boldsymbol{x})) \geq 0$, where $\bar{\omega}$ is the known upper bound of $\boldsymbol{\omega}(t)$. If this condition holds, then the set $\mathscr{C}$ is again rendered forward invariant for the perturbed system. The proof also follows directly from Proposition 1.

Motivated by existing methods [25], we define a point-wise minimum-invasive controller. Suppose that a nominal control input $\boldsymbol{u}_{\text{nom}} : \mathscr{D} \to \mathbb{R}^m$ Lipschitz continuous in $\boldsymbol{x}$, has been designed, and we need to modify the control input online to account for the safety constraints. The modified controller is given by the quadratic program below:

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}) = \arg\min_{\boldsymbol{u} \in U} & \|\boldsymbol{u} - \boldsymbol{u}_{\text{nom}}\|_2^2 \\
\text{s.t.} \quad & L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x})\boldsymbol{u} + L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) + \alpha(\psi_{r-1}(\boldsymbol{x})) \geq 0.
\end{aligned} \quad (19)
$$

This formulation is known as "safety-critical" in that constraint satisfaction is prioritized over the nominal control law.

IV. Singularity-free, Performance-critical HOCBFs

In the previous section, the existence of an HOCBF ensures safety of the overall system. However the construction of the HOCBF is not straightforward in general. Following a similar analysis to Section 3.1 of [12], for any r^{th}-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, if $U = \mathbb{R}^m$ and $L_{\mathfrak{g}}L_{\mathfrak{f}}^{r-1}h(\boldsymbol{x}) \neq \boldsymbol{0}, \forall \boldsymbol{x} \in \mathscr{D}$ (i.e., h is of uniform relative degree r in $\mathscr{D}$), then (19) is feasible for all $\boldsymbol{x} \in \mathscr{D}$ and h is an HOCBF. Moreover, the resulting controller is locally Lipschitz continuous in $\mathscr{D}$. In the following section, we will study the case when $U = \mathbb{R}^m, L_{\mathfrak{g}}L_{\mathfrak{f}}^{r-1}h(\boldsymbol{x}) = \boldsymbol{0}$ for some $\boldsymbol{x} \in \mathscr{D}$ (i.e., h is of *least relative degree r in $\mathscr{D}$*).

A. Singularity-free HOCBF design

One notable difference between Definition (5) and the existing constructions [18]–[20] is that an HOCBF candidate does not need to have uniform relative degree r. The motivation for this comes from the fact that even the double integrator dynamics with circular region constraints will violate this assumption, as shown in the following example.

Example 1. *Consider the double integrator dynamics $\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{v}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ I_2 \end{pmatrix}\boldsymbol{u}$ with $\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^2$, $\boldsymbol{x} = (\boldsymbol{p}, \boldsymbol{v})$. Let $b(\boldsymbol{p}, \boldsymbol{v}) := d^2 - \|\boldsymbol{p}\|^2$ defining a circular region in $\mathbb{R}^2$ with radius d. $\mathscr{C}_b = \{(\boldsymbol{p}, \boldsymbol{v}) : b(\boldsymbol{x}) \geq 0\}$. With straightforward calculation, we obtain $L_{\mathfrak{g}}b = \boldsymbol{0}, L_{\mathfrak{g}}L_{\mathfrak{f}}b = -2\boldsymbol{p}^\top$. Thus, $L_{\mathfrak{g}}L_{\mathfrak{f}}b(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \mathscr{E} = \{(\boldsymbol{p}, \boldsymbol{v}) : \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} \subset \mathscr{C}_b$, which does not satisfy the conditions from [19], [20]. We will show how the proposed HOCBF considered here addresses the singularity issue for application to more general systems/constraints.*

We now present a method to address the possible infeasibility of the quadratic program (19) due to the existence of singular points. In the following, we show that as long as the singular points are strictly bounded away from the boundary, a novel control barrier function can be constructed such that the constraints in (19) are always feasible.

Proposition 4. *Consider a smooth function $b : \mathbb{R}^n \to \mathbb{R}$ with the associated set $\mathscr{C}_b$ and an open set $\mathscr{D}$ with $\mathscr{C}_b \subset \mathscr{D}$. Let b have least relative degree r in $\mathscr{D}$ and define the set $\mathscr{E} := \{\boldsymbol{x} \in \mathscr{D} : L_{\mathfrak{g}}L_{\mathfrak{f}}^{r-1}b(\boldsymbol{x}) = \boldsymbol{0}\}$. Assume that there exists a scalar $\xi > 0$ such that*

$$
\mathscr{E} \subseteq \mathscr{C}_{b,\xi}. \quad (20)
$$

Define $h : \mathbb{R}^n \to \mathbb{R}$ as

$$
h(\boldsymbol{x}) = \chi\left(\frac{b(\boldsymbol{x})}{\xi}\right), \quad (21)
$$

with $\chi : \mathbb{R} \to \mathbb{R}$ a rth-order differentiable function satisfying

$$
\begin{cases}
\chi(0) = 0, \\
\chi(\tau) = 1, & \text{for } \tau \geq 1, \\
\frac{d\chi}{d\tau}(\tau) > 0, & \text{for } \tau < 1.
\end{cases} \quad (22)
$$

If $U = \mathbb{R}^m$, then the function h is an HOCBF.

Proof. It is trivial to verify that $\mathscr{C}_h = \mathscr{C}_b := \{\boldsymbol{x} \in \mathbb{R}^n : b(\boldsymbol{x}) \geq 0\}$, and r is also the least relative degree of function

h. We need to prove that there always exist a $\boldsymbol{u} \in \mathbb{R}^m$ and sufficiently smooth extended class $\mathcal{K}$ functions α_ks such that

$$L_{\mathfrak{g}}\psi_{r-1}\boldsymbol{u} + L_{\mathfrak{f}}\psi_{r-1} + \alpha(\psi_{r-1}) \geq 0 \qquad (23)$$

holds for all $\boldsymbol{x} \in \mathscr{D}$ with $\psi_{k-1}, k = 1, 2, \cdots, r$ defined in (2). Denote $\mathscr{C} := \bigcap_{k=1}^{r} \mathscr{C}_{\psi_{k-1}}$ as in Definition 5.

We first examine the properties of $L_{\mathfrak{g}}\psi_{r-1}$ and $L_{\mathfrak{f}}\psi_{r-1}$. With $h(\boldsymbol{x})$ defined in (21), we obtain $L_{\mathfrak{f}}h = \frac{d\chi}{d\tau}(b(\boldsymbol{x})/\xi)\frac{\partial b/\xi}{\partial \boldsymbol{x}}$. $\mathfrak{f} = \frac{1}{\xi}\frac{d\chi}{d\tau}L_{\mathfrak{f}}b, L_{\mathfrak{g}}h = \frac{1}{\xi}\frac{d\chi}{d\tau}L_{\mathfrak{g}}b$. If $r > 1$, then $L_{\mathfrak{g}}h = \mathbf{0}$. Note that $\psi_1 = L_{\mathfrak{f}}h + \alpha_1(h(\boldsymbol{x}))$, it derives

$$L_{\mathfrak{f}}\psi_1 = \frac{1}{\xi}\left(\frac{1}{\xi}\frac{d^2\chi}{d\tau^2}L_{\mathfrak{f}}bL_{\mathfrak{f}}b + \frac{d\chi}{d\tau}L_{\mathfrak{f}}^2b\right) + \frac{d\alpha_1}{dh}L_{\mathfrak{f}}h$$

$$L_{\mathfrak{g}}\psi_1 = \frac{1}{\xi}\left(\frac{1}{\xi}\frac{d^2\chi}{d\tau^2}L_{\mathfrak{f}}b\frac{\partial b}{\partial \boldsymbol{x}} \cdot \mathfrak{g} + \frac{d\chi}{d\tau}L_{\mathfrak{g}}L_{\mathfrak{f}}b\right) + \frac{d\alpha_1}{dh}L_{\mathfrak{g}}h \qquad (24)$$

$$= \frac{1}{\xi}\frac{d\chi}{d\tau}L_{\mathfrak{g}}L_{\mathfrak{f}}b$$

If $r > 2$, then $L_{\mathfrak{g}}\psi_1 = \mathbf{0}$ and we can iterate these calculations until ψ_{r-1} that gives us $L_{\mathfrak{g}}\psi_{r-1} = \frac{1}{\xi}\frac{d\chi}{d\tau}L_{\mathfrak{g}}L_{\mathfrak{f}}^{r-1}b$. Thus, in view of the properties of χ given in (22), we know

1) $L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x}) = \mathbf{0}$ if and only if $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$;
2) $L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$.

The condition in (23) is examined in two cases. For $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$, we derive that $L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x}) = \mathbf{0}$, $L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}) = 0, \alpha(\psi_{r-1}) = \alpha(\alpha_{r-1}(\psi_{r-2})) = \cdots = \alpha_r \circ \alpha_{r-1} \circ \cdots \alpha_1(h(\boldsymbol{x})) = \alpha_r \circ \alpha_{r-1} \circ \cdots \alpha_1(1) > 0$, thus the condition in (23) is trivially satisfied. For $\boldsymbol{x} \in \mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$, as $L_{\mathfrak{g}}\psi_{r-1} \neq \mathbf{0}$ and the condition in (23) imposes a linear constraint on $\boldsymbol{u}$. Thus, there always exists a $\boldsymbol{u} \in \mathbb{R}^m$ that satisfies (23) and h is an HOCBF via Definition 5. $\qquad \square$

Here we note that the Assumption in (20) is intuitive and easy-to-check as it requires all the singularity points to be inside $\mathscr{C}_{b,\xi}$ for some positive number ξ. In the double integrator example, this assumption is clearly fulfilled as $\mathcal{E} \subseteq \mathscr{C}_{b,\xi}$ for any $0 < \xi < d$. We further show that the resulting controller is locally Lipschitz continuous.

Proposition 5. *Assume the conditions in Proposition 4 hold and the nominal controller $\boldsymbol{u}_{nom} : \mathscr{D} \to \mathbb{R}^m$ is bounded and locally Lipschitz continuous in $\mathscr{D}$. With h given in (21) and ψ_k given in (2), assume furthermore that $L_{\mathfrak{g}}\psi_{r-1}$ and $L_{\mathfrak{f}}\psi_{r-1}$ are locally Lipschitz continuous. Then,*

1) the solution to the quadratic program (19) is locally Lipschitz continuous in $\mathscr{D}$;
2) the controller (19) renders the set $\mathscr{C} := \bigcap_{k=1}^{r} \mathscr{C}_{\psi_{k-1}}$ forward invariant for system in (15).

Proof. The feasibility of the linear inequality constraint on $\boldsymbol{u}$ is guaranteed in Proposition 4 for every $\boldsymbol{x} \in \mathscr{D}$. The solution to the quadratic program (19) has a closed-form solution, given by the KKT condition [6], as

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{\text{nom}}(\boldsymbol{x}) + \mu L_{\mathfrak{g}}^{\top}\psi_{r-1}(\boldsymbol{x}) \qquad (25)$$

with

$$\mu = \begin{cases} 0, & \text{if } L_{\mathfrak{g}}\psi_{r-1}\boldsymbol{u}_{\text{nom}} + \alpha(\psi_{r-1}) + L_{\mathfrak{f}}\psi_{r-1} \geq 0, \\ \dfrac{-L_{\mathfrak{g}}\psi_{r-1}\boldsymbol{u}_{\text{nom}} - \alpha(\psi_{r-1}) - L_{\mathfrak{f}}\psi_{r-1}}{\|L_{\mathfrak{g}}\psi_{r-1}\|^2}, & \text{otherwise.} \end{cases}$$

The derivation is straightforward considering whether the linear constraint on $\boldsymbol{u}$ in (19) is active or not and thus omitted

here. Recall that $L_{\mathfrak{g}}\psi_{r-1} = \mathbf{0}$ if and only if $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$, and $L_{\mathfrak{g}}\psi_{r-1}\boldsymbol{u}_{\text{nom}} + \alpha(\psi_{r-1}) + L_{\mathfrak{f}}\psi_{r-1} \geq 0$ is trivially satisfied for $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$. Thus μ and $\boldsymbol{u}(\boldsymbol{x})$ are well-defined in $\mathscr{D}$.

The solution in (25) can be viewed as $\boldsymbol{u}(\boldsymbol{x}) = \omega_1(\boldsymbol{x}) + \omega_2(\omega_3(\boldsymbol{x}))\omega_4(\boldsymbol{x})$ with $\omega_1(\boldsymbol{x}) = \boldsymbol{u}_{\text{nom}}(\boldsymbol{x}), \omega_2(v) = \begin{cases} 0, & \text{if } v \geq 0 \\ v, & \text{if } v < 0 \end{cases}, \omega_3(\boldsymbol{x}) = L_{\mathfrak{g}}\psi_{r-1}\boldsymbol{u}_{\text{nom}} + \alpha(\psi_{r-1}) + L_{\mathfrak{f}}\psi_{r-1}, \omega_4(\boldsymbol{x}) = \frac{-L_{\mathfrak{g}}^{\top}\psi_{r-1}}{\|L_{\mathfrak{g}}\psi_{r-1}\|^2}$. For $\boldsymbol{x} \in \mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$, $L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x}) \neq \mathbf{0}$, we obtain $\omega_1, \omega_2, \omega_3, \omega_4$ are locally Lipschitz continuous and thus $\boldsymbol{u}(\boldsymbol{x})$ is locally Lipschitz continuous in $\mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$. Furthermore, for $\boldsymbol{x} \in \mathscr{D} \cap \mathscr{C}_{b,\xi}$, we have $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{\text{nom}}(\boldsymbol{x})$ and thus $\boldsymbol{u}(\boldsymbol{x})$ is locally Lipschitz continuous in $\mathscr{D} \cap \mathscr{C}_{b,\xi}$.

Now we show that the control input $\boldsymbol{u}(\boldsymbol{x})$ is continuous at the boundary between $\mathscr{D} \cap \mathscr{C}_{b,\xi}$ and $\mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$. Assume a Cauchy sequence of points $\{\boldsymbol{x}_i\}_{i=1,2,3,\cdots} \subset \mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$ such that $\lim_{i \to \infty} \boldsymbol{x}_i = \boldsymbol{x}_0$ with $\boldsymbol{x}_0$ at the boundary between $\mathscr{D} \cap \mathscr{C}_{b,\xi}$ and $\mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$. From the closed-form solution (25) and the facts that $\boldsymbol{u}_{\text{nom}}(\boldsymbol{x}_i)$ is bounded, $\lim_{i \to \infty} L_{\mathfrak{g}}\psi_{r-1}(\boldsymbol{x}_i) = \mathbf{0}, \lim_{i \to \infty} L_{\mathfrak{f}}\psi_{r-1}(\boldsymbol{x}_i) = 0$, and $\lim_{i \to \infty} \alpha(\psi_{r-1}(\boldsymbol{x}_i)) = \alpha_r \circ \alpha_{r-1} \circ \cdots \alpha_1(1) > 0$, we obtain $\lim_{i \to \infty} \boldsymbol{u}(\boldsymbol{x}_i) = \boldsymbol{u}(\boldsymbol{x}_0)$. Together with local Lipschitz continuity in $\mathscr{D} \cap \mathscr{C}_{b,\xi}$ and $\mathscr{D} \setminus (\mathscr{D} \cap \mathscr{C}_{b,\xi})$, respectively, we conclude that the resulting controller from (19) is locally Lipschitz continuous. From Theorem 1, the resulting controller $\boldsymbol{u}$ guarantees forward invariance of $\mathscr{C}$. $\qquad \square$

B. Performance-Critical HOCBF

In many applications, it would be favorable to know in advance when the nominal controller is implemented without any modifications,i.e., $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{nom}(\boldsymbol{x})$ in some pre-defined set. This is useful, for example, when training a learning-based controller or performing high-precision motion control during spacecraft rendezvous and docking. We refer to these instances as "performance-critical" because, to ensure satisfaction of the task, the designers have to know *a priori* when the nominal control will always be implemented.

To formally address the performance-critical tasks, we denote the *safety region*[2], inside which the system states should always evolve, and the *performance-critical region*, inside which the nominal control signal should be utilized, as the respective superlevel sets of smooth functions $b, s : \mathbb{R}^n \to \mathbb{R}$. Intuitively, as long as the performance-critical region lies strictly inside the safety region, with the transformation in (21), the nominal control signal is recovered in the performance-critical regions while safety is always guaranteed.

Theorem 2. *Consider the control affine system (15). Let $b, s : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, and let b have least relative degree r in an open set $\mathscr{D}$ with $\mathscr{C}_b \subset \mathscr{D}$. Assume the conditions in Proposition 5 hold. Assume furthermore that $\mathscr{C}_s$ is strictly bounded away from the safety boundary, i.e.,*

$$\mathscr{C}_s \subseteq \mathscr{C}_{b,\xi}. \qquad (26)$$

[2]Note that the safety region may not be the same as the safe set. In the double integrator example, the safe region is the circular region $\mathscr{C}_b = \{(\boldsymbol{p}, \boldsymbol{v}) : d^2 - \|\boldsymbol{p}\|^2 \geq 0\}$ that only constrains the state $\boldsymbol{p}$, while the safe set is a subset of $\mathscr{C}_b$ that will be rendered forward invariant.

Then, with h given in (21) and ψ_k given in (2),

1) *h is an HOCBF;*
2) *the controller (19) renders the set $\mathscr{C} := \bigcap_{k=1}^{r} \mathscr{C}_{\psi_{k-1}}$ forward invariant for system in (15);*
3) *$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{nom}(\boldsymbol{x})$ for states $\boldsymbol{x} \in \mathscr{C}_s$.*

Proof. Point 1) and Point 2) follow from Proposition 4 and Proposition 5, respectively. For $\boldsymbol{x} \in \mathscr{C}_{b,\xi}$, the constraint in the quadratic program (19) is trivially satisfied, thus $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{nom}(\boldsymbol{x})$ for states $\boldsymbol{x} \in \mathscr{C}_s$. $\qquad\square$

When b has exact relative degree r for all states in the safe set, we obtain the following corollary.

Corollary 1. *Consider the control affine system (15). Let $b, s : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, and let b have uniform relative degree r in an open set $\mathscr{D}$ with $\mathscr{C}_b \subset \mathscr{D}$. Assume that there exists a scalar $\xi > 0$ such that*

$$\mathscr{C}_s \subseteq \mathscr{C}_{b,\xi}. \tag{27}$$

Then, with h given in (21) and ψ_k given in (2),

1) *h is an HOCBF;*
2) *the controller (19) renders the set $\mathscr{C} := \bigcap_{k=1}^{r} \mathscr{C}_{\psi_{k-1}}$ forward invariant for system in (15);*
3) *$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{nom}(\boldsymbol{x})$ for states $\boldsymbol{x} \in \mathscr{C}_s$.*

V. AN APPLICATION TO RIGID-BODY ATTITUDE DYNAMICS

In this section, we apply the proposed high-order control barrier function methodology to rigid-body attitude dynamics. A similar formulation was proposed in our previous work [26]. The main difference is that here we exploit the proposed HOCBF framework to construct a safe, stabilizing control law from a simple nominal stabilizing controller. The method in [26] on the other hand uses a more complicated nominal control design. The simulations presented here show that the use of the HOCBF framework allows for modular, safe, stabilizing control design.

The attitude dynamics of a rigid-body with states consisting of orientation and angular velocity $(R, \boldsymbol{\omega})$ ((1) in [26]) can be written in a control affine form as

$$\dot{\boldsymbol{x}} := \mathfrak{f}(\boldsymbol{x}) + \mathfrak{g}\boldsymbol{u}, \tag{28}$$

where $\boldsymbol{x} = (r_{11}, r_{12}, \cdots, r_{33}, \omega_1, \omega_2, \omega_3) \in \mathbb{R}^{12}, \mathfrak{f}(\boldsymbol{x}) = (r_{12}\omega_3 - r_{13}\omega_2; r_{13}\omega_1 - r_{11}\omega_3; r_{11}\omega_2 - r_{12}\omega_1; r_{22}\omega_3 - r_{23}\omega_2; r_{23}\omega_1 - r_{21}\omega_3; r_{21}\omega_2 - r_{22}\omega_1; r_{32}\omega_3 - r_{33}\omega_2; r_{33}\omega_1 - r_{31}\omega_3; r_{31}\omega_2 - r_{32}\omega_1; J^{-1}(-[\boldsymbol{\omega}]_{\times} J\boldsymbol{\omega})) \in \mathbb{R}^{12}, \mathfrak{g} = \begin{pmatrix} \mathbf{0}_{9\times 3} \\ J^{-1} \end{pmatrix}$.

We denote $C_{TSO(3)} := \{ \boldsymbol{x} \in \mathbb{R}^{12} : \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \in SO(3) \}$. In the following, $(R, \boldsymbol{\omega})$ and $\boldsymbol{x}$ are used interchangeably.

Given some sample orientations $R_i \in SO(3), i \in \mathcal{N}$, we define the safe region $\cup_{i\in\mathcal{N}} S_i$, where $S_i = \{ R \in SO(3) : r_i(R) \geq 0 \}, r_i(R) = \epsilon - \|R - R_i\|_F^2 / 2$. Assume that the safe region is connected. To measure the margin of the attitude trajectory to the safe region $\cup_{i\in\mathcal{N}} S_i$, we define $b(\boldsymbol{x}) = \sum_{i\in\mathcal{N}} s(r_i(R)/\epsilon) - \delta$, where $\delta > 0$ is a constant, and a smooth transition function $s(v) = \begin{cases} 0 & v\in(-\infty, 0), \\ \frac{\rho(v)}{\rho(v)+\rho(1-v)} & v\in[0,1), \\ 1 & v\in[1,\infty) \end{cases}$

with $\rho(v) := (1/v)e^{-1/v}$. The associated constrained set is $\mathscr{C}_b(\boldsymbol{x}) := \{ \boldsymbol{x} \in \mathscr{C}_{TSO(3)} : b(\boldsymbol{x}) \geq 0 \}$. To ensure that the trajectory evolves within $\cup_{i\in\mathcal{N}} S_i$, we conservatively require $b(\boldsymbol{x}(t)) \geq 0$ for $t \geq 0$.

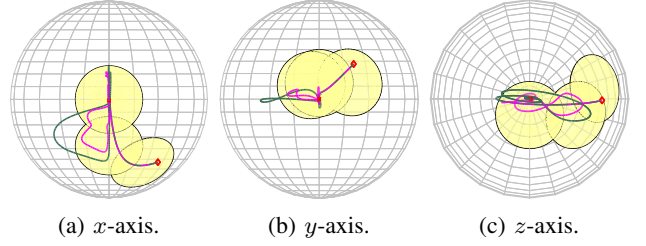(a) x-axis.　　　(b) y-axis.　　　(c) z-axis.

Fig. 1: Comparison of the attitude trajectories in body-fixed xyz axes with additive control signals. The square point and the cross point represent the starting attitude R_0 and the target attitude R_f, respectively, and the yellow region is the safe region. The purple and green lines represent the results wherein the barrier function is in use or not in use, respectively.
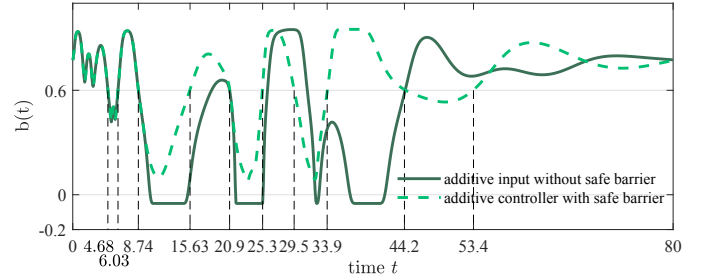
(a) Time histories of $b(t)$ when additive control signals exist. For all t with $b(t) > \xi = 0.6$, the system state is in the performance-critical region where the nominal control signal is used.
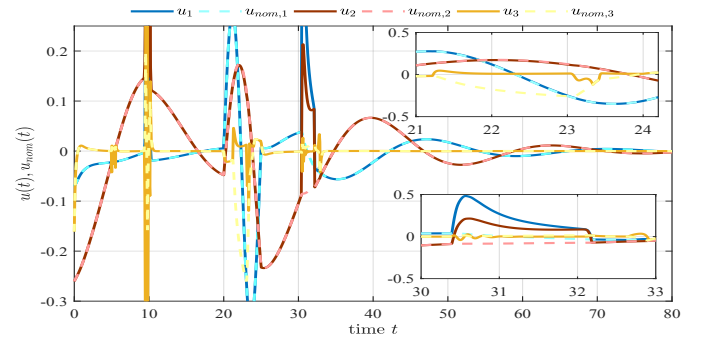
(b) The time history of the nominal and actual control inputs when the barrier function is in use. The discrepancies between $\boldsymbol{u}(t)$ and $\boldsymbol{u}_{nom}(t)$ occur in the time interval $t \in [4.8, 5.6] \cup [9.4, 10.4] \cup [21, 24] \cup [30, 33]$ and $b(t) < \xi = 0.6$ for all t in this interval.

Fig. 2: Attitude stabilization with additive control signals.

Following the analysis in [26], we know that $b(\boldsymbol{x})$ is of least relative degree $r = 2$. Moreover, the singular points, at which the exact relative degree is greater than 2, lie on the geodesics between the sampling points $R_i, i \in \mathcal{N}$ [26, Proposition 3], and thus are bounded away from the boundary of the safe region. This fact satisfies the assumption in Proposition 4. Applying the results in this paper, we obtain: 1) $h(\boldsymbol{x}) = \chi\left(\frac{b(\boldsymbol{x})}{\xi}\right)$ is an HOCBF; 2) with $h(\boldsymbol{x})$ given, the set $\mathscr{C} = \mathscr{C}_{\psi_0} \cap \mathscr{C}_{\psi_1}$ is forward invariant; 3) the nominal control

signal will be implemented in any subset of $\mathscr{C}_{b,\xi}$ (performance-critical set).

We consider an attitude stabilization scenario from $(R_0, \mathbf{0})$ to $(R_f, \mathbf{0})$, $J = \begin{pmatrix} 5.5 & 0.06 & -0.03 \\ 0.06 & 5.5 & 0.01 \\ -0.03 & 0.01 & 0.1 \end{pmatrix} \text{kg} \cdot \text{m}^2$. We set $R_f = I$, the sampling orientations $R_3 = \exp(10°/180° \times \pi[\mathbf{e}_1]_\times)$, $R_2 = \exp(30°/180° \times \pi[\mathbf{e}_2]_\times)R_3$, $R_1 = \exp(30°/180° \times \pi[0, 0.447, 0.894]_\times)R_2$, the initial attitude $R_0 = \exp(10°/180° \times \pi[\mathbf{e}_1]_\times)R_1$, and $\epsilon = 0.1206$, which corresponds to cell radius 0.3491 rad ($20°$). We use the saturated stabilizing controller from [27] as the nominal controller:

$$\mathbf{u}_{nom}(R, \boldsymbol{\omega}) = -k_1(R - R^\top)^\vee - k_2 \tanh(\boldsymbol{\omega}), \qquad (29)$$

where $\tanh(\cdot)$ is the element-wise hyperbolic tangent function. The controller parameters are set as $k_1 = k_2 = 0.2$. The parameters in the control barrier function are chosen as $\delta = 0.05, \xi = 0.6, \alpha_1(v) = \alpha_2(v) = v, \chi(v) = \begin{cases} (v-1)^3+1, & \text{if } v \leq 1; \\ 1, & \text{if } v > 1. \end{cases}$

We simulate an attitude stabilization scenario where the control signal in (29) is augmented with an additive signal $\mathbf{u}_{add} = 0.3 * (\sin(2\pi\frac{t-20}{5}), \sin(\pi\frac{t-20}{5}), -\sin(\pi\frac{t-20}{5}))$ for the time interval $t \in [20, 25]$ and view their sum as the nominal control signal in the quadratic program (19). This control signal simulates, for example, a human input to the system that leads to a deviation from the previous trajectory and may drive the states out of the safe region. The trajectories are shown in Fig. 1. When the barrier function is in use, the resulting trajectory evolves within the safe region. Moreover, from Fig. 2, we see that the actual control signal coincides with the nominal control signal whenever $b(t) \geq \xi = 0.6$, which validates the performance-critical property.

Compared to the simulation results in [26], we note that similar results are obtained here with a simple nominal stabilizing control law. This shows the effectiveness and modularity of the proposed HOCBF framework.

VI. CONCLUSION

In this paper, we formulate high-order (zeroing) barrier functions and their controlled equivalent for nonlinear dynamical systems. This formulation generalizes the concept of zeroing barrier functions and similar concepts in the literature. Our results do not require forward completeness of the system to show forward invariance of the set. More importantly, we show for the first time that the intersection of superlevel sets associated with the high-order barrier function, is asymptotically stable. Thanks to this property, our method generalizes the robustness results of the standard zeroing barrier function formulation. We also provide a remedy to handle the singular states that arise when implementing the minimally-invasive control law, while ensuring safety of the overall system. Finally, we derive a performance-critical property so that one can define the performance-critical regions *a priori*. The proposed formulation is implemented on the non-trivial case study of rigid-body attitude dynamics.

REFERENCES

[1] C. E. Garcia, D. M. Prett, and M. Morari, "Model predictive control: theory and practice—a survey," *Automatica*, vol. 25, no. 3, pp. 335–348, 1989.

[2] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[3] E. F. Camacho and C. B. Alba, *Model predictive control*. Springer Science & Business Media, 2013.

[4] M. Z. Romdlony and B. Jayawardhana, "Uniting control lyapunov and control barrier functions," in *53rd IEEE Conference on Decision and Control*. IEEE, 2014, pp. 2293–2298.

[5] K. P. Tee, S. S. Ge, and E. H. Tay, "Barrier lyapunov functions for the control of output-constrained nonlinear systems," *Automatica*, vol. 45, no. 4, pp. 918–927, 2009.

[6] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[7] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[8] A. D. Ames, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to adaptive cruise control," in *Proc. IEEE Conf. on Decision and Control*, New York, 2014, pp. 6271–6278.

[9] S. Prajna, "Barrier certificates for nonlinear model validation," *Automatica*, vol. 42, no. 1, pp. 117–126, 2006.

[10] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[11] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 462–467, 2007.

[12] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control," in *Proc. IFAC Conf. Anal. Design Hybrid Syst.*, vol. 48, 2015, pp. 54–61.

[13] M. Jankovic, "Robust control barrier functions for constrained stabilization of nonlinear systems," *Automatica*, vol. 96, pp. 359–367, 2018.

[14] P. Glotfelter, J. Cortés, and M. Egerstedt, "Nonsmooth barrier functions with applications to multi-robot systems," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 310–315, 2017.

[15] L. Wang, A. Ames, and M. Egerstedt, "Safety barrier certificates for collisions-free multirobot systems," *IEEE Transactions on Robotics*, vol. 33, no. 3, pp. 661–674, 2017.

[16] L. Lindemann and D. V. Dimarogonas, "Control barrier functions for signal temporal logic tasks," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 96–101, 2018.

[17] Q. Nguyen and K. Sreenath, "Exponential control barrier functions for enforcing high relative-degree safety-critical constraints," in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 322–328.

[18] X. Xu, "Constrained control of input–output linearizable systems using control sharing barrier functions," *Automatica*, vol. 87, pp. 195–201, 2018.

[19] W. Xiao and C. Belta, "Control barrier functions for systems with high relative degree," in *2019 IEEE Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 27–34.

[20] W. Shaw Cortez and D. V. Dimarogonas, "Correct-by-design control barrier functions for Euler-Lagrange systems with input constraints," in *2020 American Control Conference (ACC)*, 2020.

[21] R. Redheffer, "The theorems of Bony and Brezis on flow-invariant sets," *The American Mathematical Monthly*, vol. 79, no. 7, pp. 740–747, 1972.

[22] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*, ser. Systems & Control : Foundations & Applications. Birkhäuser, 2015.

[23] V. Lakshmikantham, S. Leela, and A. A. Martynyuk, *Stability analysis of nonlinear systems*. Springer, 1989.

[24] A. Isidori, *Nonlinear control systems II*. Springer Science & Business Media, 1999.

[25] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *Proc. European Control Conf.*, 2019, pp. 3420–3431.

[26] X. Tan and D. V. Dimarogonas, "Construction of control barrier function and C^2 reference trajectory for constrained attitude maneuvers," in *59th IEEE Conference on Decision and Control*. IEEE, 2020, pp. 3329–3334.

[27] T. Lee, "Robust adaptive attitude tracking on $SO(3)$ with an application to a quadrotor uav," *IEEE Transactions on Control Systems Technology*, vol. 21, no. 5, pp. 1924–1930, 2012.