Asymptotic Tracking of Second-Order Nonsmooth Feedback Stabilizable Unknown Systems with Prescribed Transient Response

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Abstract—This paper considers the asymptotic tracking control problem for a class of nonlinear systems subject to predefined constraints for the system response, such as maximum overshoot or minimum convergence rate. In particular, by employing discontinuous control protocols and nonsmooth analysis, we extend previous results on funnel control to guarantee at the same time asymptotic trajectory tracking. We consider 2nd-order systems that are affine in the control input and contain completely unknown nonlinear and nonsmooth vector fields, with no boundedness or approximation/parametric factorization assumptions. Simulation results verify the theoretical findings.

Index Terms—Uncertain systems, Robust adaptive control, Asymptotic stability, funnel control, prescribed performance control, nonlinear systems, non-smooth systems.

I. INTRODUCTION

CONTROL of uncertain systems is one of the main research topics in systems theory. Robust and adaptive control as well as neural network/fuzzy logic control are the dominant methodologies dealing with such systems [1], [2]. There exists a variety of works achieving both asymptotic and “practical” (ultimately bounded errors) stability under the presence of model uncertainties (e.g., [3]–[12]). The majority of the related works that achieve asymptotic stability assume parametric dynamic uncertainties and/or growth conditions and gain tuning [4], local Neural network approximations [11], or fuzzy logic controllers [12].

A well-studied special instance of adaptive control is funnel control, where the output of the system is confined to a predefined funnel [13]–[17]. It is a model-free control scheme of high-gain type, with numerous applications during the last years. Examples include electrical circuits [18], Lagrangian systems [19]–[21] and multi-agent systems [21]–[23]. The intuition behind funnel control is the incorporation of an adaptive gain in the control scheme, which increases (in absolute value) as the system’s output reaches the funnel’s boundary. In that way, the system’s output is “pushed” to always remain inside the funnel. Funnel control has been developed for both linear [18] and nonlinear systems [13] involving parametric [19] as well as structural [13], [14] dynamic uncertainties.

An important property that most related funnel-control works fail to achieve is that of asymptotic stability subject to unknown nonlinear dynamics. Traditional funnel control guarantees only confinement of the system output in a pre-specified funnel, and thus the closest property to asymptotic stability that can be achieved is that of “practical stability”, where the funnel converges arbitrarily close to zero. The latter, however, might yield undesired large inputs due to the small funnel values, and can be problematic in real-time systems. Such a scheme was developed in the works [24], [25] for first-order systems, where the funnel converges to zero. This, however, can create numerical ill-conditioning in the practical computation of the control input, since it involves the product of “large” and “small” quantities (the funnel reciprocal and the error signal) [25]. On the other hand, with potential guarantees of asymptotic stability, the funnel is not needed to converge close to zero, and can be used in order to encode just transient constraints for the system. Asymptotic tracking subject to transient constraints has been considered in several works [19], [22], [26], [27]; [22], [26], [27] consider linear systems (LTI and double integrator), whereas [19] assumes known model structure, with the uncertainties being only parametric; One can conclude that the aforementioned works cannot be extended in a straightforward manner to nonlinear systems where the dynamic terms have both parametric and structural uncertainties. In addition, a class of systems for which funnel control has not been taken into account in the related works is the non-smooth type, i.e., systems with discontinuous right-hand side. Such models are motivated by real-time systems, where several dynamic terms (e.g., friction) can be accurately modeled by discontinuous state functions.

In this paper, we consider the asymptotic tracking control problem subject to transient constraints imposed by a predefined funnel. We consider MIMO systems of the form

\begin{align}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= F(x, z, t) + G(x, z, t)u, & y &= x_1 \\
\dot{z} &= F_z(x, z, t),
\end{align}

where \( z \in \mathbb{R}^{n_z}, x := [x_1^T, x_2^T]^T \in \mathbb{R}^{2n}, \) with \( x_j := [x_{j1}, \ldots, x_{jn_j}]^T \in \mathbb{R}^{n_j}, \forall j \in [1, 2], \) are the system’s states, \( y := [y_1, \ldots, y_n]^T \in \mathbb{R}^n \) is the system’s output, which is required to track a desired trajectory \( y_d(t), \) and \( F : \mathbb{R}^{2n+n_z} \times [t_0, \infty) \to \mathbb{R}^{n}, \) \( F_z : \mathbb{R}^{2n+n_z} \times [t_0, \infty) \to \mathbb{R}^{n_z}, \) \( G : \mathbb{R}^{2n+n_z} \times [t_0, \infty) \to \mathbb{R}^{n \times n} \) are unknown vector fields, not necessarily continuous. We assume that \( x \) is available...
for measurement, whereas $z$ is not. In fact, the dynamics governing $z$ is called dynamic uncertainty and represents unmodeled dynamic phenomena that potentially affect the closed-loop response. The assumptions on the system dynamics are restricted to local boundedness and measurability as well as controllability conditions on $G$ and internal stability of $z$, without considering any uniform boundedness/growth condition or model approximation:

**Assumption 1:** The maps $(x, z) \rightarrow F(x, z, t) : \mathbb{R}^{2n+n_2} \rightarrow \mathbb{R}^n$, $(x, z) \rightarrow G(x, z, t) : \mathbb{R}^{2n+n_2} \rightarrow \mathbb{R}^{n \times n}$, $(x, z) \rightarrow F_z(x, z, t) : \mathbb{R}^{2n+n_2} \rightarrow \mathbb{R}^n$ are Lebesgue measurable and locally bounded for each fixed $t \in [t_0, \infty)$, uniformly in $t$, and the maps $t \mapsto F(x, z, t) : [t_0, \infty) \rightarrow \mathbb{R}^n$ and $t \mapsto G(x, z, t) : [t_0, \infty) \rightarrow \mathbb{R}^{n \times n}$ are Lebesgue measurable and uniformly bounded for each fixed $(x, z) \in \mathbb{R}^{2n+n_2}$, by unknown bounds.

**Assumption 2:** The matrix

$$
\bar{G}(x, z, t) := G(x, z, t) + G(x, z, t)\mathbf{T}
$$

is positive definite, $\forall(x, z, t) \in \mathbb{R}^{2n+n_2} \times [t_0, \infty)$, i.e., it holds that $\lambda_{\min}(\bar{G}(x, z, t)) > 0$, where $\lambda_{\min}(\bar{G}(x, z, t))$ is its unknown minimum eigenvalue.

**Assumption 3:** There exists a sufficiently smooth function $U_z : \mathbb{R}^n \rightarrow \mathbb{R}_\geq 0$ and class $K_\infty$ functions $\gamma_\pm(\cdot)$, $\tilde{\gamma}(\cdot)$, $\gamma(\cdot)$ such that $\gamma_\pm(\|z\|) \leq U_z(z) \leq \tilde{\gamma}(\|z\|)$, and

$$
(\frac{\partial U_z}{\partial z})^T F_z(x, z, t) \leq -\tilde{\gamma}(\|z\|) + \pi_z(x, z, t),
$$

where $x \mapsto \pi_z(x, z, t) : \mathbb{R}^n \rightarrow \mathbb{R}_\geq 0$ is continuous and class $K_\infty$ for each fixed $(z, t) \in \mathbb{R}^n \times [t_0, \infty)$, and $(z, t) \mapsto \pi_z(x, z, t) : \mathbb{R}^n \times [t_0, \infty) \rightarrow \mathbb{R}_{\geq 0}$ is uniformly bounded for each fixed $x \in \mathbb{R}^{2n}$.

**Assumption 4:** The state $x$ is available for measurement.

**Assumption 5:** The desired trajectory and its derivatives are bounded by finite and unknown constants $\bar{y}_d_0, \bar{y}_d_1 > 0$, i.e., $\|y_d(t)\| < \bar{y}_d_0 \leq \tilde{y}_d, \|y_d(t)\| < \bar{y}_d_1 \leq \tilde{y}_d, \forall t \in [t_0, \infty)$, where $\tilde{y}_d := \max_{\zeta \in [0, 1]} \{\tilde{y}_d\}$. Note that Assumption 2 is a sufficient controllability condition and Assumption 3 suggests that $z$ is input-to-state practically stable with respect to $z, x, t$ implying stable zero (internal) dynamics [13]. Note also that the vector fields $F(\cdot)$ and $G(\cdot)$ are not required to be continuous everywhere.

According to the authors’ best knowledge, this is the first work that guarantees asymptotic stability for a 2nd-order system subject to funnel constraints and under the considered assumptions 1-5, from all initial conditions that are compliant with the funnel and independent of the system model. In fact, according to the authors’ best knowledge, asymptotic stability (even without funnel constraints) for 2nd-order systems has not been guaranteed in the related literature under these assumptions (i.e., unknown nonlinear discontinuous dynamics and no boundedness/growth assumptions on $F(\cdot)$) and from initial conditions independent of the system model. Regarding the latter, note that, if the initial value of the funnel is a design parameter, it can be always set larger than the initial value of the error to be confined in the funnel, rendering thus the results global. This work extends our preliminary results [28], which considered the special instance of time-invariant 2nd-order Lagrangian systems, to systems with more general dynamic terms $F, G$ and relaxation of the positive definitiveness of $G$.

The rest of the paper is structured as follows. Section II introduces some preliminary background and the notation followed throughout the article. Section III provides the proposed control protocol as well as the corresponding stability analysis. Finally, simulation results are given in Section IV and Section V concludes the paper.

II. NOTATION AND PRELIMINARIES

A. Notation

The sets of real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_\geq 0$, respectively. $\|x\|$ denotes the 2-norm of a vector $x \in \mathbb{R}^n$; The open and closed balls with respect to the 2-norm and with radius $\delta$, centered at $x \in \mathbb{R}^n$, are denoted by $B(x, \delta)$ and $\bar{B}(x, \delta)$, respectively.

B. Nonsmooth Analysis

Consider the following differential equation with a discontinuous right-hand side:

$$
\dot{x} = f(x, t), \quad (2)
$$

where $f : D \times [t_0, \infty) \rightarrow \mathbb{R}^n$, $D \subset \mathbb{R}^n$, is Lebesgue measurable and locally essentially bounded, uniformly in $t$. The Filippov regularization of $f$ is defined as [29]

$$
K[f](x, t) := \bigcap_{\delta > 0} \overline{\mathbb{B}}(f(B(x, \delta) \setminus \bar{N}), t), \quad (3)
$$

where $\bigcap_{\delta > 0} \overline{\mathbb{B}}(E) = \overline{\mathbb{B}}(E)$ is the convex closure of the set $E$.

**Definition 1 (Def. 1 of [30]):** A function $x : [t_0, t_1) \rightarrow \mathbb{R}^n$, with $t_1 > t_0$, is called a Filippov solution of (2) on $[t_0, t_1)$ if $x(t)$ is absolutely continuous and if, for almost all $t \in [t_0, t_1)$, it satisfies $\dot{x} \in K[f](x, t)$, where $K[f](x, t)$ is the Filippov regularization of $f(x, t)$.

**Lemma 1 (Lemma 1 of [30]):** Let $x(t)$ be a Filippov solution of (2) and $V : D \times [t_0, t_1) \rightarrow \mathbb{R}$ be a locally Lipschitz, regular function [1]. Then $V(x(t), t)$ is absolutely continuous, $\dot{V}(x(t), t) = \frac{\partial}{\partial x} V(x(t), t)$ exists almost everywhere (a.e.), i.e., for almost all $t \in [t_0, t_1)$, and $\dot{V}(x(t), t) \in \mathbb{E} \bar{V}(x(t), t)$, where

$$
\bar{V} := \bigcap_{\xi \in \partial V(x(t), t)} \xi^T \left[ K[f](x, t) \right],
$$

and $\partial V(x(t), t)$ is Clarke’s generalized gradient at $(x, t)$ [30].

**Theorem 1 (Corollary 2 of [30]):** For the system given in (2), let $D \subset \mathbb{R}^n$ be an open and connected set containing $x = 0$ and suppose that $f$ is Lebesgue measurable and $x \mapsto f(x, t)$ is essentially locally bounded, uniformly in $t$. Let $V : D \times [t_0, t_1) \rightarrow \mathbb{R}$ be locally Lipschitz and regular such that $W_1(x) \leq V(x, t) \leq W_2(x)$, $\forall t \in [t_0, t_1)$, $x \in D$, and

$$
\dot{z} \leq -W(x(t)), \quad \forall z \in \bar{V}(x(t), t), \quad t \in [t_0, t_1), \quad x \in D,
$$

1See [30] for a definition of regular functions.
where \( W_1 \) and \( W_2 \) are continuous positive definite functions and \( W \) is a continuous positive semi-definite on \( D \). Choose \( r > 0 \) and \( c > 0 \) such that \( \mathcal{B}(0, r) \subset D \) and \( c < \min_{\|x\|=r} W_1(x) \). Then for all Filippov solutions \( x : [t_0, t_1) \rightarrow \mathbb{R}^n \) of (2), with \( x(t_0) \in D : \{ x \in \mathcal{B}(0, r) : W_2(x) \leq c \} \), it holds that \( t_1 = \infty \), \( x(t) \in D \), \( \forall t \in [t_0, \infty) \), and \( \lim_{t \to \infty} W(x(t)) = 0 \).

### III. MAIN RESULTS

The control objective is the asymptotic output tracking of a desired bounded trajectory \( \overset{\cdot}{y}_a := [y_{1, \dot{a}}, \ldots, y_{n, \dot{a}}] : [t_0, \infty) \rightarrow \mathbb{R}^n \), with bounded derivatives, as stated in Assumption 5. Moreover, as discussed in Section I, we aim at imposing a certain predefined behavior for the transient response of the system. More specifically, motivated by funnel control techniques [14], [31], [32], given \( n \) predefined funnels, described by the smooth functions (also called performance functions in [13]) \( \rho_p : [t_0, \infty) \rightarrow [\rho_{p_\inf}, \rho_{p_\sup}] \subset \mathbb{R}_{\geq 0} \), where \( \rho_{p_\inf}, \rho_{p_\sup} \in \mathbb{R}_{> 0} \) are positive lower and upper bounds, respectively, we aim at guaranteeing that 2\( -\rho_p(t) < y_i(t) - y_{i, d}(t) < \rho_p(t) \), \( \forall t \in [t_0, \infty) \), given that \( -\rho_{p_i}(t) < y_i(t) - y_{i, d}(t) < \rho_{p_i}(t) \), \( \forall t \in [t_0, \infty) \), and all closed loop signals remain bounded.

Our solution to Problem 1 is based on the error transformation proposed in [13], which converts the constrained error behavior \( -\rho_p(t) < y_i(t) - y_{i, d}(t) < \rho_p(t) \) to an unconstrained one. More specifically, we define the errors

\[
epsilon_p := [\epsilon_{p_1}, \ldots, \epsilon_{p_n}]^T := y - y_d,
\]

as well as the error transformations \( \epsilon_{p_i} \in \mathbb{R} \) according to:

\[
\epsilon_{p_i} = \rho_{p_i}(T(\epsilon_{p})) \quad \forall i \in \{1, \ldots, n\},
\]

where \( T : \mathbb{R} \rightarrow (-1, 1) \) is a smooth, strictly increasing analytic function, with \( T(0) = 0 \). Since \( T \) is increasing, the inverse mapping \( T^{-1} : (-1, 1) \rightarrow \mathbb{R} \) is well-defined, and it holds that

\[
\lim_{\zeta \to -\infty} T(\zeta) = -1, \quad \lim_{\zeta \to +\infty} T(\zeta) = 1
\]

and hence, if \( \epsilon_{p_i} \) remains bounded in a compact set, the desired funnel objective \( -\rho_p(t) < \epsilon_{p_i}(t) < \rho_p(t) \) is achieved, \( \forall i \in \{1, \ldots, n\} \). We further require that

\[
|\xi| < \frac{\partial T^{-1}(\zeta)}{\partial \zeta} T^{-1}(\zeta) \quad \forall \zeta \in (-1, 1).
\]

A possible choice that satisfies the aforementioned specifications is \( T(\zeta) = \exp(\zeta) - 1 \).

From (5), we obtain

\[
\dot{\epsilon}_{p_i} = \frac{T_p}{\rho_{p_i}} (x_{2_i} - \dot{y}_{i, d} - \dot{\rho}_p \epsilon_{p_i}) \quad \forall i \in \{1, \ldots, n\}
\]

or, in stack vector form,

\[
\dot{\epsilon}_p = r_p \rho_{p_i}^{-1} (x_2 - \dot{y}_d - \dot{\rho}_p \epsilon_p),
\]

where \( \epsilon_p := [\epsilon_{p_1}, \ldots, \epsilon_{p_n}]^T \), \( \dot{\rho}_p := \frac{\partial T^{-1}(\zeta)}{\partial \zeta} |\zeta| = \frac{\rho_{p_\sup} - \rho_{p_\inf}}{\rho_{p_\sup}} \). Due to the increasing property of \( T(\zeta) \), it holds that \( r_p \) is positive definite, and thus in order to render \( \dot{\epsilon}_p \) negative a straightforward choice for a desired value for \( x_{2, a} \) is

\[
x_{2, a} := y_d + \dot{\rho}_p \rho_{p_i}^{-1} \epsilon_p - k_p r_p \epsilon_p,
\]

where \( k_p \in \mathbb{R}_{> 0} \) is a positive and constant scalar gain. Since, however, \( x_{2, a} \) is not the system’s input, we follow a backstepping-like methodology and define the error

\[
epsilon_v := [e_{v_1}, \ldots, e_{v_n}]^T := x_2 - x_{2, a}.
\]

Next, we proceed in a similar manner and define a funnel for each \( e_{v_i} \), \( i \in \{1, \ldots, n\} \), described by the functions \( \rho_{v_i} : [t_0, \infty) \rightarrow [\rho_{v_{i, \inf}}, \rho_{v_{i, \sup}}] \subset \mathbb{R}_{\geq 0} \), where \( \rho_{v_{i, \inf}}, \rho_{v_{i, \sup}} \in \mathbb{R}_{> 0} \) are the positive lower and upper bounds, respectively, with the constraint \( \rho_{v_{i, \inf}}(t) > |e_{v_i}(t)|, \quad \forall t \in (t_0, \infty) \). Note that \( e_{v_i}(t) = x_2(t) - x_{2, a}(t) \) can be calculated at \( t = t_0 \) since it is a function of the state, the funnel functions and the desired trajectory profile. Then, we define the open set

\[
D_{u, t} := \{(x, t) \in \mathbb{R}^{2n} \times [t_0, \infty) : \rho_p(t)^{-1} \epsilon_p(t) \in (-1, 1)^n, \quad \rho_v(t)^{-1} e_v(t) \in (-1, 1)^n\}
\]

and design the control law \( u : D_{u, t} \rightarrow \mathbb{R}^n \) as:

\[
u = -k_{v_2} \rho_{v_1}^{-1} (k_{v_3} \|r_p \epsilon_p\| + k_v \dot{d}) s_v - k_{v_4} \rho_{v_1}^{-1} r_v e_v \quad \forall (x, t) \in D_{u, t}
\]

where

\[
s_v := \begin{cases} \frac{r_v e_v}{\|r_v e_v\|}, & \text{if } \|r_v e_v\| \neq 0, \\ 0, & \text{otherwise}, \end{cases}
\]

\[
\rho_v := \text{diag}\{\rho_{v_1}, \ldots, \rho_{v_n}\}, \quad e_v := [e_{v_1}, \ldots, e_{v_n}]^T, \quad \epsilon_v := T^{-1}(\epsilon_v), \quad \epsilon_v := [\rho_{v_1}, \ldots, \rho_{v_n}]^T, \quad r_v := \frac{\partial T^{-1}(\zeta)}{\partial \zeta} |\zeta| = \frac{\rho_{v_{i, \sup}} - \rho_{v_{i, \inf}}}{\rho_{v_{i, \sup}}}, \quad k_{v_i} \in \mathbb{R}_{> 0}, i \in \{1, \ldots, 4\} \text{ are positive constant scalar gains, and } \dot{d} \text{ is an adaptive variable gain, subject to the constraint } \dot{d}(t) \geq 0, \quad \text{and dynamics}
\]

\[
\dot{d} = \gamma_d |r_v e_v|,
\]
where $\gamma_d \in \mathbb{R}_{>0}$ is a positive constant gain.

**Remark 1:** The control design procedure follows closely the prescribed performance backstepping-like methodology of the previous works [21], [23], [33], introduced in [34]. The desired signals and control laws in these works consist only of proportional terms with respect to the transformed errors $\varepsilon_p$, $\varepsilon_v$, i.e., $-k_p\varepsilon_p$ and $-k_v\rho_v\varepsilon_v$ in (11) and (14), respectively, which are guaranteed to be ultimately bounded. In this work, we incorporate (a) the extra terms in (11) that would render (10) exponentially stable, and (b) the discontinuous term in (14), which, as will be shown in the sequel, enforces convergence of the transformed errors to zero, guaranteeing thus asymptotic stability. A similar discontinuous term was employed in [25] to achieve asymptotic stability, by enforcing however the funnel functions to converge to zero, which is not required in the proposed framework. Finally, we note that the adaptive term $d = \gamma_d \int_0^t \| r_e(s) \varepsilon_v(s) \| ds$ in (14) is similar to the integrator term in PI-funnel control [16], introduced to attempt asymptotic stability, without, however, providing theoretical guarantees for the class of systems considered here.

**Remark 2:** Note that no information regarding the dynamic model is incorporated in the control protocol (4)-(15). All the necessary signals consist of the funnel terms $\rho_p$, $\rho_v$ and of known functions of the state and the desired trajectory $y_d$. Furthermore, no a-priori gain tuning is needed and, as the next theorem states, the solution of Problem 1 is guaranteed from all initial conditions that satisfy $-\rho_p(t_0) < y_i(t_0) - y_{i,d}(t_0) < \rho_p(t_0),$ $\forall i \in \{1, \ldots, n\}$. As will be revealed subsequently, the adaptive gain $\hat{d}$ compensates the unknown dynamic terms, which are proven to be bounded due to the confinement of the state in the prescribed funnels.

The correctness of the control protocol (4)-(15) is shown in the next theorem.

**Theorem 2:** Consider a system subject to the dynamics (1), Assumptions 1-5, as well as a desired trajectory $y_d$ and funnels as described in Problem 1 satisfying $-\rho_p(t_0) < y_i(t_0) - y_{i,d}(t_0) < \rho_p(t_0),$ $\forall i \in \{1, \ldots, n\}$. Then the control protocol (4)-(15) guarantees the existence of at least one local Filippov solution of the closed-loop system (1)-(14) that solves Problem 1. Moreover, every such local solution can be extended to a global solution and all closed-loop signals remain bounded, for all $t \geq t_0$.

**Proof:** The intuition of the subsequent proof is as follows: We first show the existence of at least one Filippov solution of the closed loop system in $D_c$ for a time interval $I \subseteq [t_0, \infty)$. Next, we prove that for any of these solutions, the state remains bounded in $I$ by bounds independent of the endpoint of $I$. Hence, the dynamic terms of (1) are also upper bounded by a term, which we aim to compensate via the adaptation gain $\hat{d}$.

We start by defining some terms that will be used in the subsequent analysis: $M_p := \max_{i \in \{1, \ldots, n\}} \{ \rho_{p,i} \}$, $m_p := \min_{i \in \{1, \ldots, n\}} \{ \rho_{p,i} \}$, $M_v := \max_{i \in \{1, \ldots, n\}} \{ \rho_{v,i} \}$, $m_v := \min_{i \in \{1, \ldots, n\}} \{ \rho_{v,i} \}$, $\Delta := \rho_{v,1}^{-1} G(z, t) \rho_{p,1}^{-1}$, $\beta := (k_v k_{pv})^{-1}, \text{ and } \xi_p := \inf_{\varepsilon \in (-1, 1)} \frac{\partial T(\varepsilon)}{\partial \varepsilon}$. Note that all the aforementioned terms are strictly positive. In particular, $\Delta$ is strictly positive due to the definition of the funnels $\rho_p$ and Assumption 2, and $\xi_p$ is strictly positive due to the strictly increasing property of $T(\cdot)$ and hence of $T^{-1}(\cdot).$ Moreover, in view of (6), it holds that $\arg \inf_{\varepsilon \in (-1, 1)} \frac{\partial T(\varepsilon)}{\partial \varepsilon} \in (-1, 1)$.

By employing (14), (15), we can write the closed loop system
\begin{align}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &\in \mathbb{K}[F](x, z, t), \\
\dot{x}_3 &\in \mathbb{K}[G](x, z, t) \mathbb{K}[u](x, t), \\
\dot{\hat{d}} &= \gamma_d \| r_e \varepsilon_v \|,
\end{align}
where $\mathbb{K}[F](x, z, t)$, $\mathbb{K}[G](x, z, t)$, $\mathbb{K}[u](x, t)$ are the Filippov regularizations (see (3)) of the respective terms. For $u$ specifically, $\mathbb{K}[u](x, t)$ is formed by substituting the term $s_v$ with its reguralized term, which is $S_v := \frac{r_v \varepsilon_v}{\| r_v \varepsilon_v \|}$ if $\| r_v \varepsilon_v \| \neq 0$, and $S_v = \| r_v \varepsilon_v \|$ otherwise. Note that, in any case, it holds that $(r_v \varepsilon_v)\mathbb{K}[S](x, z, t) = \| r_v \varepsilon_v \|$.

Define now $\mathcal{D}_c := \{ (\mathcal{X}, t), t \in \mathbb{R}^{2n+2n+1} \}$ and consider the open set $\mathcal{D}_c := \{ (\mathcal{X}, t), t \in \mathbb{R}^{2n+2n+1} \}$. Since $\rho_p(t_0) > |e_p(t_0)|$ and $\rho_v(t_0) > |e_v(t_0)|$, $\forall i \in \{1, \ldots, n\}$, the set $\mathcal{D}_c$ is nonempty. Moreover, since $T(.)$, and hence its derivative, are analytic, their zero sets have zero measure [35] and thus the right hand-side of (16) is Lebesgue measurable and locally essentially bounded in $\mathcal{X}$ over the set $\mathcal{X} := (\mathcal{X}, t) \in \mathcal{D}_c$, and Lebesgue measurable in $t$ over the set $\{ t : (\mathcal{X}, t) \in \mathcal{D}_c \}$. Hence, according to Prop. 3 of [36], for each initial condition $(\mathcal{X}, t_0) \in \mathcal{D}_c$, there exists at least one Filippov solution $\tilde{x}(t)$ of (16), defined in $I := [t_0, t_{\infty})$, where $t_{\infty} > t_0$ such that $(\tilde{x}(t), t) \in \mathcal{D}_c$, $\forall t \in I$. By applying (8), we conclude the existence of the respective Filippov solutions $\varepsilon_p(t), e_v(t) \in \mathbb{R}^n$, $\forall t \in I$. Let now $\tilde{x}(t_0)$ denote the initial condition of the system (16) satisfying $(\tilde{x}(t_0), t_0) \in \mathcal{D}_c$ and consider the family of Filippov solutions starting from $\tilde{x}(t_0)$ denoted by the set $\mathcal{X}$. Note that, although not explicitly stated, $t_{\infty}$ and $I$ might be different for each solution in $\mathcal{X}$.

We aim to prove that all $\varepsilon_p(t), e_v(t)$ are bounded and converge to zero, for all $(\mathcal{X}, t) \in \mathcal{X}$.

In view of the definition of $\mathcal{D}_c$ (see also (13)), for all $(\mathcal{X}, t) \in \mathcal{X}$ it holds that
\begin{align}
|e_p(t)| < \bar{\rho}_{p,i}, |e_v(t)| < \bar{\rho}_{v,i}, \quad (17)
\end{align}
$\forall t \in I$, where $\hat{\rho}_{p,i}$ and $\hat{\rho}_{v,i}$ are the upper bounds of $\rho_{p,i}(t)$ and $\rho_{v,i}(t)$, respectively, $\forall i \in \{1, \ldots, n\}$. Consider now the
Lyapunov function $V_p := \frac{1}{2} \|\epsilon_p\|^2$, for which it holds, in view of (10), (11), (12), and (17)\n
$$V_p = \epsilon_p^T r_p \rho_p^{-1} (x_2 - \bar{y}_d - \rho_p \rho_p^{-1} \epsilon_p) = -k_p \epsilon_p^T r_p \rho_p^{-1} r_p \epsilon_p + \epsilon_p^T r_p \rho_p^{-1} e_v < -k_p M_p \|r_p \epsilon_p\|^2 + M_v M_p \|r_p \epsilon_p\|,$$

\forall t \in I. Hence, we conclude that $\dot{V}_p < 0$ when $\|r_p \epsilon_p\| > \frac{M_v M_p}{k_p M_p}$. Since $r_p$, is positive definite, $\forall t \in I$. Moreover, the derivative of $T(\cdot)$ and $T^{-1}(\cdot)$ are smooth, the derivative approaches infinity only when $\epsilon \to \pm 1$. Therefore, in view of the definition of $y_d$ in (10), we conclude the existence of a finite $\bar{r}_p > 0$ such that $\|r_p \epsilon_p\| \leq \bar{r}_p, \forall t \in I$. Further, (5) implies that $\|r_p \epsilon_p\| \leq \bar{\epsilon}_p := \frac{M_p T(\epsilon_p)^{1/2}}{\sqrt{n}}, \forall t \in I$. Hence, we conclude that $\|x_{2,d}(t)\| \leq \bar{x}_d := \bar{y}_d + M_p \epsilon_p + k_p \rho_p \epsilon_p, \forall t \in I$, where $\bar{y}_d$ is the uniform bound of the desired trajectory, introduced in Assumption 5. We also conclude that $\|x_1(t)\| \leq \bar{x}_1 := \bar{e}_p + \bar{y}_d, \forall t \in I$. In addition, by employing $x_2 = v + x_{2,d}$ and (17), we conclude that $\|x_2(t)\| < \bar{x}_2 := M_v \sqrt{n} + \bar{x}_d, \forall t \in I$.

Finally, by differentiating $x_{2,d}$, employing the smoothness and boundedness of $\rho_p$ and its derivatives, the smoothness of $T(\cdot)$, the boundedness of $\bar{y}_d$ as well as the aforementioned bounds, we can conclude the existence of a bound $\bar{v}_d$ such that $\|\dot{x}_{2,d}(t)\| \leq \bar{v}_d, \forall t \in I$.

Furthermore, the boundedness of $x(t)$ and Assumption 3 imply the existence of a positive definite constant $\tilde{z}$ such that $\|z(t)\| \leq \tilde{z}, \forall t \in I$. Hence, since $F(x, z, t)$ is Lebesgue measurable and locally essentially bounded in $\mathbb{R}^{2n} + n$ and $\|x_1(t)\| \leq \tilde{x}_1 < \infty, \|x_2(t)\| < \tilde{x}_2 < \infty, \|z(t)\| \leq \tilde{z}, \forall t \in I$, there exists some positive $F$, such that $\|F(x(t), z(t), t)\| \leq F, \forall t \in I$, and hence, for each $(x, z)$, since $K[F]$ is formed by the convex closure of $F$, it holds that $\max_{\zeta \in K[F]} \{F(x(t), z(t), t)\} \leq F, \forall t \in I$ and $\tilde{x}(t) \in \tilde{x}$. Note that, in view of the aforementioned discussion, $F$ depends solely on the initial conditions and the parameters of the funnel functions. Define now the finite constant term $d \in \mathbb{R}_{>0}$ as\n
$$d := \frac{\beta}{m_v} \left( \bar{F} + \bar{v}_d + M_p \sqrt{n} \right).$$

(18)

Note that the term in the parenthesis of (18) is an upper bound for the term $\|F(x(t), z(t), t) - x_{2,d}(t) - \rho_u(t) \rho_1(t)^{-1} e_v(t)\|$, for all $\tilde{x}(t) \in \tilde{x}$ and almost all $t \in I$.

Define also the signal $\tilde{d} := \tilde{d} - d$, where $\tilde{d}$ is the adaptive gain introduced in (14). Consider now the function\n
$$V(\tilde{z}) := \alpha V_p + \frac{\beta_2}{2} \|\tilde{e}_v\|^2 + \frac{1}{2 \tilde{\gamma}_d} d^2,$$

where $\tilde{\gamma} := \tilde{\gamma}_p, \tilde{\gamma}_v, \tilde{d}$, and $\alpha > 0$ is a positive constant to be defined, $V(\tilde{z})$ satisfies $W_1(\tilde{z}) \leq V(\tilde{z}) \leq W_2(\tilde{z})$, for $W_1(\tilde{z}) := \min \left\{ \frac{\alpha}{2}, \frac{\beta_2}{2} \frac{1}{\tilde{\gamma}_d} \right\} \|\tilde{z}\|^2$ and $W_2(\tilde{z}) := \max \left\{ \frac{\alpha}{2}, \frac{\beta_2}{2} \frac{1}{\tilde{\gamma}_d} \right\} \|\tilde{z}\|^2$.

Then, according to Lemma 1, $\dot{V}(\tilde{z}(t)) \in \tilde{V}(\tilde{z}(t))$ with $\tilde{V}(\tilde{z})$ is continuously differentiable, its generalized gradient reduces to the standard gradient and thus it holds that $\dot{V} = \nabla V^T K(\tilde{z})$, where $\nabla V = [\alpha \tilde{e}_p, \beta \tilde{e}_v, \frac{1}{\tilde{v}} \tilde{d}]^T$. After using (11), (14), (15), and $x_2 = x_{2,d} + e_v$, one obtains\n
$$\dot{V} \in \tilde{W}_s := -\alpha k_p \tilde{r}_p \rho_p^{-1} \tilde{r}_p \tilde{e}_p + \alpha \tilde{e}_p \tilde{r}_p \rho_p^{-1} e_v - \beta \kappa_v \tilde{e}_v \tilde{r}_v \rho_v^{-1} K[G](x, t) \tilde{r}_v \tilde{e}_v + d \|r_v e_v\| + \beta \tilde{e}_v \tilde{r}_v \rho_v^{-1} (K[F](x, t) - \tilde{x}_{2,d} - \rho_u(t) \rho_1(t)^{-1} e_v) - \beta \kappa_v \tilde{e}_v \tilde{r}_v \rho_v^{-1} K[G](x, t) \rho_v^{-1} I_0 \left( k_v \|r_p \epsilon_p\| + k_u \tilde{d} \right).$$

Note that, since $\tilde{d}(t_0) \geq 0$, (15) implies that $\tilde{d}(t) \geq 0, \forall t \in I$. Moreover, since the Filippov regularization (3) is defined as a closed set and $\tilde{V} \in \tilde{W}_s$, it holds that $\max_{\zeta \in \tilde{W}_s} \{\zeta\} \leq \max_{\zeta \in \tilde{W}} \{\zeta\}$. By substituting $G = \frac{G^+ G_2^+}{2} + \frac{G^+ G_2}{2}$ and employing the skew-symmetry of the second term, we obtain in view of Assumption 2 and the definition of $d$ in (18):

$$\max_{\tilde{\gamma} \in \tilde{V}} \zeta \leq \max_{\tilde{\gamma} \in \tilde{W}_s} \zeta \leq -\frac{\alpha k_p}{M_p} \|r_p \epsilon_p\|^2 - \kappa_v \beta \lambda \|r_v e_v\|^2 - \kappa_v \kappa_u \beta \lambda \|r_v e_v\| \|r_v e_v\| \|d \|r_v e_v\| + \|r_v e_v\| \|d \|r_v e_v\| + \alpha \|\tilde{e}_v\| \|d \|r_v e_v\| + \alpha \|\tilde{e}_v\| \|d \|r_v e_v\|,$$

for all solutions $\tilde{x}(t) \in \tilde{x}$. By setting $\zeta = T(\tilde{e}_v)$ in (7), we obtain $|T(\tilde{e}_v)| \leq \|r_v e_v\| \|d \|r_v e_v\| + \|r_v e_v\| \|d \|r_v e_v\|,$ and hence by employing $e_v = \rho_u(t) \tilde{e}_v, i \in \{1, \ldots, n\}$, we obtain that

$$\alpha \|\tilde{e}_p\| \|r_p \epsilon_p\| \|e_v\| \leq \frac{M_v}{m_p} \|\tilde{r}_p \epsilon_p\| \|r_v e_v\|.$$

Therefore, by setting $\alpha = \frac{k_v k_u \kappa u \beta \lambda}{M_v}$, employing $d = \tilde{d} - d$ and in view of the fact that $\beta = (k_v k_u \kappa u \beta \lambda)^{-1}$, we obtain

$$\max_{\zeta \in \tilde{V}} \zeta \leq -\frac{\alpha k_p}{M_p} \|r_p \epsilon_p\|^2 - \kappa_v \beta \lambda \|r_v e_v\|^2 =: -W(\tilde{z}),$$

\forall t \in I, $\tilde{x}(t) \in \tilde{x}$, where $W$ is continuous and positive semi-definite on $\mathbb{R}^{2n+1}$, since $r_v$ and $r_p$ are positive definite. Hence, we conclude that $\zeta \leq -W(\tilde{z}), \forall \zeta \in \tilde{V}(\tilde{z}(t)), \forall t \in I$ and all $\tilde{x}(t) \in \tilde{x}$. Choose now any finite $r > 0$ and let c <
min∥ϕ∥=W_{1}(ϕ). Note that all the conditions of Theorem 1 are satisfied and hence, all Filippov solutions starting from the connection points of the spring and damper, and

\[ d := \sqrt{0.25 + 0.25 \sin(x_{11} - x_{12}) + 0.125(1 - \cos(x_{12} - x_{11}))} \]

Note that no boundedness assumptions or compact sets \( \forall t \in [0, \infty) \), implying that \( \lim_{t \to \infty} \|\tilde{\epsilon}(t)\| = 0 \), and hence, \( \lim_{t \to \infty} \|\tilde{\epsilon}(t)\| = 0 \). Moreover, the response of the system is bounded and \( \forall \epsilon \in \{1, \ldots, n\} \). We conclude that the aforementioned analysis is still valid and \( \forall \epsilon \in \tilde{\epsilon} \) remains bounded. Moreover, the response of the system is bounded and \( \forall \epsilon \in \{1, \ldots, n\} \).

**Remark 3:** It is straightforward to apply the proposed methodology to the simpler case of first-order systems, i.e., when (1a) and (1b) are replaced by \( \dot{x} = F(x, z, t) + G(x, z, t) u \). The control law then takes the form \( u = -k_2 \rho_{p,1}^{-1} d_{s,0} - k_1 \rho_{p,1}^{-1} r_p \epsilon_p \), with \( d = \gamma d ||r_p \epsilon_p|| \), and \( s_0 \) defined using \( r_p \epsilon_p \). The boundedness of the solutions in the maximal interval of existence \( I \) implies the boundedness of \( F(\cdot) \) and \( \epsilon_p \), and, by following a similar procedure with \( V \) and \( \dot{V} \), the derivative of \( V_{p,1} := \frac{\alpha_1}{2} ||\epsilon_p||^2 + \frac{1}{2} d^2 \) yields \( \max_{\epsilon_p, \tilde{\epsilon}_p} \{\} \leq -k_3 ||r_p \epsilon_p||^2 \) for some positive constants \( \alpha_1, k_3 \), implying \( \lim_{t \to \infty} \|\epsilon_p(t)\| = 0 \).

**Remark 4:** Note that no boundedness assumptions or growth conditions are needed for the vector fields \( F(x, z, t) \) and \( G(x, z, t) \). In particular, the effect of \( F(x, z, t) \) is canceled by the introduced adaptive signal \( \hat{d} \), which increases according to (15). It is proved, nevertheless, that this adaptive signal remains bounded. Moreover, the response of the system is solely determined by the funnel functions \( \rho_p \) and \( \rho_{s,1} \) (and \( \rho_{s,1} \), in the second control scheme), isolated from the system dynamics and the control gains selection. Nevertheless, we note that appropriate gain tuning might be needed to suppress chattering in real life scenarios. Similarly, note that the region of attraction (initial conditions) of \( (\epsilon_p, \tilde{\epsilon}_p) = (0, 0) \) is independent from the system dynamics and the control gain selection, and depends only on the choice of the funnel functions \( \rho_p, \forall i \in \{1, \ldots, n\} \). In particular, if \( \rho_p(t_0) \) are design parameters, we can always choose them such that \( -\rho_p(t_0) < \epsilon_{p,1}(t_0) \).
is the distance between these points; \( \theta \) is defined as
\[
\theta := \tan^{-1}\left( \frac{0.25(\cos(x_{12}) - \cos(x_{11}))}{0.5 + 0.25(\sin(x_1) - \sin(x_2))} \right)
\]
and \( T_1, T_2 \) are friction terms on the motors evolving according to
\[
\dot{T}_i = d_{a_i}(t) + \tau_i + \hat{\tau}_i + \hat{x}_1, \quad \text{with}
\]
\[
\dot{x}_i = \frac{\parallel \hat{x}_i \parallel}{1 + \exp\left(-\frac{\parallel \hat{x}_i \parallel}{0.1}\right)}
\]
and \( d_{a_i}(t) := (-1)^{i-1}\cos(t)^2 \), for \( t \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{7\pi}{2}, \frac{11\pi}{2}\right] \cup \left[\frac{21\pi}{2}, \frac{23\pi}{2}\right] \cup \left[\frac{33\pi}{2}, 50\right] \) and 0 otherwise, \( i \in \{1, 2\} \) being an additional disturbance. The time varying signal \( \sigma(t) \) is:
\[
\sigma(t) = \begin{cases} 
1 & \text{if } t \in [0, 3) \cup [3.5, \infty), \\
0.5 & \text{if } t \in [3, 3.5)
\end{cases}
\]
modeling a loss of effectiveness of the second motor when \( t \in [3, 3.5) \). We also choose \( g_y = 9.81 \) as the gravity constant and \( J_1 = 0.5, J_2 = 0.625 \). The initial conditions are \( t_0 = 0, x(0) = [0, 0, 0, 0]^\top, \tau_1(0) = 0 = \tau_2(0) \) and the desired trajectory \( y_d = [2 \cos(t), \frac{7}{2} - 2 \sin(t)]^\top \). The prescribed funnel functions are chosen as \( \rho_p(t) = 2.5 \exp(-0.1t) + 2.5, \forall i \in \{1, 2\} \), which converge to 2.5. We also choose \( \rho_v(t) = (||e_v(0)|| - 2) \exp(-0.1t) + 2.5 \), as well as the gains \( k_p = 10, k_{v_1} = 2 \cdot 10^3, k_{v_2} = 0.1, k_{v_3} = 0.025, k_{v_4} = 0.05, \) and \( \gamma_d = 50 \). The simulation results are depicted in Figs. 1-4 for \( t \in [0, 55] \) sec. More specifically, Fig. 1 depicts the errors \( e_p(t), e_v(t) \) along with the performance functions \( \rho_p(t), \rho_v(t) \). One can conclude that \( e_p(t) \) and \( e_v(t) \) not only respect their imposed funnels but also converge asymptotically to zero, without the need of arbitrarily small values for \( \lim_{t \to \infty} \rho_p(t) \) and \( \lim_{t \to \infty} \rho_v(t) \). This can be verified also by Fig. 2, which depicts the evolution of the transformed errors \( \epsilon_p(t), \epsilon_v(t), \forall t \in [0, 55] \) sec, and shows their asymptotic convergence to zero. Finally, Figs. 3 and 4 illustrate the inputs \( u(t) \) as well as the adaptation signal \( \hat{d}(t), \forall t \in [0, 55] \) sec. One can conclude the convergence of \( \hat{d}(t) \) to a constant value as well as the boundedness of the control input \( u(t) \), as was proved in the theoretical analysis.

V. CONCLUSION AND FUTURE WORK

This paper presents a novel control scheme that guarantees asymptotic stability subject to funnel constraints for a class of 2nd-order systems with unknown, nonlinear, and possibly discontinuous dynamics, from all initial conditions that satisfy the funnel constraints. We design a control protocol based on adaptive and discontinuous control methodologies and the correctness of the proposed scheme is independent from the control gain selection. Future efforts will be devoted towards extending the proposed scheme to more general systems with potential controllability relaxations.

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**Fig. 1.** The evolution of the errors \( e_p(t) \) (top), \( e_v(t) \) (bottom), depicted with blue, along with the performance functions \( \rho_p(t), \rho_v(t) \), depicted with red, \( \forall t \in [0, 55] \) sec.

**Fig. 2.** The evolution of the transformed errors \( \epsilon_p(t), \epsilon_v(t), \forall t \in [0, 55] \) sec.

**Fig. 3.** The evolution of the control inputs \( u(t) = [u_1(t), u_2(t)]^\top, \forall t \in [0, 55] \) sec.
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