Multi-agent estimation and control based on a novel k-hop Distributed Prescribed Performance Observer

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***Abstract*— We propose a k-hop Distributed Prescribed Performance Observer (k-hop DPPO) for state estimation in multi-agent systems. The observer allows each agent to estimate the state of those agents that are 2-hop or more distant by communicating only with 1-hop neighbors, while guaranteeing that transient estimation errors satisfy prescribed performance defined a priori. Furthermore, we demonstrate that if the controller with perfect state knowledge drives the system towards the goal and the estimation based closed-loop system is set-Input to State Stable (set-ISS) with respect to the set describing the goal, then the state estimates can be adapted to achieve the team's objective. Simulation results are provided to demonstrate the effectiveness of the proposed results.**

***Index Terms*— Multi-agent systems, Distributed control, Prescribed Performance Observer.**

I. Introduction

A Multi-agent system consists of multiple autonomous agents that interact, collaborate, or compete to achieve individual or collective goals [1]. On one hand, these systems offer advantages over single-agent systems, such as increased redundancy and flexibility. On the other hand, they require more complex coordination and communication, which can limit their efficiency in environments with restricted sensing or communication capabilities. For this reason, enabling agents to estimate the states of those beyond their immediate neighbors can be beneficial.

Various results on distributed observers can be found in the literature [2]–[5]. In [2], a decentralized observer is presented for a multi-agent system with discrete-time dynamics, where model knowledge and local information are used to estimate the plant state using a consensus-based filter. Conversely, [5] proposes an asymptotic consensus-based observer, enabling each agent to estimate the global state by interacting with its 1-hop neighbors. A key limitation of these methods is the requirement for each agent to estimate the full system state, regardless of the specific information it actually needs. This makes them impractical for large-scale networks, where agents often only require partial state information, as their scalability decreases with the number of agents.

For this purpose, in our previous work [6], we proposed a finite time convergent k-hop distributed observer for nonlinear systems that allows each agent in the network to estimate only the states and inputs of those agents which are 2-hop or more distant by exploiting only local information through sensing and interaction with 1-hop neighbors. However, the main drawback of this approach is its inability to enforce a desired transient behavior on state estimation errors at the observer design stage. In many estimation-based control applications, where the plan must account for worst-case estimation errors, it is beneficial to have prior knowledge and control over the transient behavior of the maximum state estimation errors, without the need for post-design evaluation.

Motivated by this and inspired by Prescribed Performance Control techniques (PPC) mainly used in multi-agent systems to achieve consensus and formation control [7]–[11], we propose a k-hop Distributed Prescribed Performance Observer (k-hop DPPO) for the state estimation in multi-agent systems. This k-hop DPPO allows each agent in the network to estimate the state of those agents that are 2-hop or more distant while guaranteeing the state estimation errors to satisfy predefined transient performance. Compared to our previous work in [6] and to the other observers available in the literature, the proposed solution enables to define desired performance in advance and allows the observer to be tuned accordingly. The key advantage of this design lies in the explicit dependence of the observer's gain on the specified performance criteria, making it possible to tailor the observer based on the specific requirements and to guarantee estimation errors convergence within a predefined time frame regardless of the input structure. Furthermore, we demonstrate that if a feedback controller guarantees set-ISS stability with respect to the set representing the goal and it ensures closed-loop stability with perfect state knowledge, the system can reach the team objective even when estimation information, instead of the true states, is used.

The remainder of the paper is organized as follows: Section II presents the notation, preliminary information and the problem formulation. Section III introduces the proposed

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k-hop Distributed Prescribed Performance Observer. Section IV presents the feedback control structure and provides the conditions that guarantee the convergence of a k-hop estimation-based feedback controller towards the team objective. Section V provides a simulation result to demonstrate the convergence towards the goal when k-hop estimations are used in the feedback controller, and Section VI provides final remarks and future work.

II. Preliminaries and Problem Setting

Notation: We denote by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, the set of real, non-negative and positive real numbers. Let $|S|$ and S^c be the cardinality and the complement of a set S, $\mathbb{R}^n$ be an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ be a space of real matrices with n rows and m columns. Denote by I_n the identity matrix of size n and by 1_n the vector of ones of size n. Given $B \in \mathbb{R}^{n\times n}$, we represent with $\sigma_{\max}(B)$ the maximum singular value of B and with $\lambda_i(B)$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ respectively the i-th, minimum and maximum eigenvalues of B. We use $B \succ 0$ to denote a positive definite matrix B. Given $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^\top x}$. Let $\mathrm{diag}(a_1, \dots, a_n)$ be the diagonal matrix with diagonal elements $a_1, ..., a_n$ and let $\otimes$ be the Kronecker product. Let $\max_{i\in\{1,\dots,n\}}\{s_i\}$ be the maximum element in a set $S = \{s_1, \dots, s_n\}$. We define with $\mathcal{K}$ and $\mathcal{KL}$ the following: $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \gamma$ is continuous, strictly increasing and $\gamma(0) = 0\}$; $\mathcal{KL}=\{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$: for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty\}$.

A. Multi-Agent Systems

Consider a multi-agent system composed of a set of N interacting agents $\mathcal{V} = \{1, 2, \dots, N\}$. Suppose each agent $i \in \mathcal{V}$ behaves according to the nonlinear dynamics:

$$\dot{x}_i(t) = f(x_i) + Ax_i + u_i, \tag{1}$$

where $x_i \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is known, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a known Lipschitz continuous function with Lipschitz constant l_f, and $u_i \in \mathbb{R}^n$ is a measurable function. Denote with $\boldsymbol{x}$ and $\boldsymbol{u}$ the stacked vector of agents' states and inputs:

$$\boldsymbol{x} = \left[x_1^\top, \dots, x_N^\top\right]^\top, \ \boldsymbol{u} = \left[u_1^\top, \dots, u_N^\top\right]^\top. \tag{2}$$

Communication graph: Communication among the agents is represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of edges indicating established communication links between pairs of agents.

A path between two agents $i, j \in \mathcal{V}$ is defined as a sequence of non-repeating edges that allows j to be reached from i. Then, a k-hop path between agents i and j is a path consisting of k edges connecting i to j.

The set of k-hop neighbors of an agent $i \in \mathcal{V}$ is denoted as $\mathcal{N}_i^{k\text{-hop}}$ and it consists of all nodes $j \in \mathcal{V}$ for which there exists a p-hop path from j to i with $2 \leq p \leq k$. The elements of this set are represented as $\mathcal{N}_i^{k\text{-hop}} = \{N_1^i, \dots, N_{\eta_i}^i\}$, where each $N_j^i \in \mathcal{V}$ for $j \in \{1, \dots, \eta_i\}$ corresponds to the global index of the j-th k-hop neighbor of i. The number of such neighbors is denoted as $\eta_i = |\mathcal{N}_i^{k\text{-hop}}|$. For simplicity, we denote the set of direct (1-hop) neighbors of agent i as $\mathcal{N}_i$.

Denote with $\boldsymbol{x}^i$ and $\boldsymbol{u}^i$ the state and input vectors containing, respectively, the state $x_{N_j^i}$ and input $u_{N_j^i}$ of the k-hop neighbors $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$:

$$\boldsymbol{x}^i = \left[x_{N_1^i}^\top, \dots, x_{N_{\eta_i}^i}^\top\right]^\top, \ \boldsymbol{u}^i = \left[u_{N_1^i}^\top, \dots, u_{N_{\eta_i}^i}^\top\right]^\top \tag{3}$$

and let:

$$\hat{\boldsymbol{x}}^i = \left[\hat{x}_{N_1^i}^{i\ \top}, \dots, \hat{x}_{N_{\eta_i}^i}^{i\ \top}\right]^\top, \hat{\boldsymbol{u}}^i = \left[\hat{u}_{N_1^i}^{i\ \top}, \dots, \hat{u}_{N_{\eta_i}^i}^{i\ \top}\right]^\top \tag{4}$$

be the vectors containing their estimate carried out by the agent i, i.e., $\hat{x}_{N_j^i}^i$ and $\hat{u}_{N_j^i}^i$ for $j \in \{1, \dots, \eta_i\}$ are the estimates of the state $x_{N_j^i}$ and input $u_{N_j^i}$ of agent $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ done by i. Furthermore, denote with $\tilde{\boldsymbol{x}}^i$ and $\tilde{\boldsymbol{u}}^i$ the estimation errors:

$$\tilde{\boldsymbol{x}}^i = \left[\tilde{x}_{N_1^i}^{i\ \top}, \dots, \tilde{x}_{N_{\eta_i}^i}^{i\ \top}\right]^\top, \tilde{\boldsymbol{u}}^i = \left[\tilde{u}_{N_1^i}^{i\ \top}, \dots, \tilde{u}_{N_{\eta_i}^i}^{i\ \top}\right]^\top, \tag{5}$$

with $\tilde{x}_{N_j^i}^i = \hat{x}_{N_j^i}^i - x_{N_j^i}$ and $\tilde{u}_{N_j^i}^i = \hat{u}_{N_j^i}^i - u_{N_j^i}$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$. Indicate with $\boldsymbol{x}_i$ and $\boldsymbol{u}_i$ the vectors defined as:

$$\boldsymbol{x}_i = 1_{\eta_i} \otimes x_i, \quad \boldsymbol{u}_i = 1_{\eta_i} \otimes u_i, \tag{6}$$

and denote with $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{u}}_i$ the stacked vector of the estimates of x_i and u_i computed by the k-hop neighbors $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, $j \in \{1, \dots, \eta_i\}$:

$$\hat{\boldsymbol{x}}_i = \left[\hat{x}_i^{N_1^i\top}, \dots, \hat{x}_i^{N_{\eta_i}^i\ \top}\right]^\top, \hat{\boldsymbol{u}}_i = \left[\hat{u}_i^{N_1^i\top}, \dots, \hat{u}_i^{N_{\eta_i}^i\ \top}\right]^\top. \tag{7}$$

Similar to (5), define the estimation errors on $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{u}}_i$, computed by each $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, as $\tilde{\boldsymbol{x}}_i = \hat{\boldsymbol{x}}_i - \boldsymbol{x}_i$ and $\tilde{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_i - \boldsymbol{u}_i$, respectively, i.e.:

$$\tilde{\boldsymbol{x}}_i = \left[\tilde{x}_i^{N_1^i\top}, \dots, \tilde{x}_i^{N_{\eta_i}^i\ \top}\right]^\top, \tilde{\boldsymbol{u}}_i = \left[\tilde{u}_i^{N_1^i\top}, \dots, \tilde{u}_i^{N_{\eta_i}^i\ \top}\right]^\top, \tag{8}$$

with $\tilde{x}_i^{N_j^i} = \hat{x}_i^{N_j^i} - x_i$ and $\tilde{u}_i^{N_j^i} = \hat{u}_i^{N_j^i} - u_i$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$.

Suppose the following holds for the communication graph:

Assumption 1: $\mathcal{G}$ is a time invariant graph and each $i \in \mathcal{V}$ knows which are the agents belonging to $\mathcal{N}_i$ and $\mathcal{N}_i^{k\text{-hop}}$.

Assumption 2: Each agent $i \in \mathcal{V}$ has access and can propagate at each time instant the state of the agents $j \in \mathcal{N}_i$ to its 1-hop neighbors $\mathcal{N}_i$.

Assumption 3: $\|\tilde{\boldsymbol{u}}_i(0)\|$ is known for all $i \in \mathcal{V}$.

Note that: Assumption 1 is reasonable, as distributed neighborhood discovery algorithms have been investigated in the sensor network community [12]; Assumption 2 is valid if we assume that each agent can measure its 1-hop neighbors' state by means of sensors and can propagate this information to its neighbors exploiting communication; Assumption 3 is not restricting, as it implies only the knowledge of initialization quantities.

To reduce the complexity of the notation, without loss of generality we will assume $n = 1$ in the following sections. However, note that the results can be extended to higher dimensions with appropriate use of the Kronecker product.

B. k-Hop Distributed Prescribed Performance Observer

Inspired by the applications of PPC in multi-agent systems [7]–[11], we propose a k-hop DPPO that aims at prescribing the state estimation error $\tilde{x}^i_{N^i_j}$ within predefined regions for all $i \in \mathcal{V}$ and all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$. In particular:

$$-\rho^i_{N^i_j}(t) < \tilde{x}^i_{N^i_j}(t) < \rho^i_{N^i_j}(t) \tag{9}$$

must hold for all $t \geq 0$, $i \in \mathcal{V}$ and $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$, where each $\rho^i_{N^i_j}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is defined as:

$$\rho^i_{N^i_j}(t) = (\rho^i_{N^i_j,0} - \rho^i_{N^i_j,\infty})e^{-l^i_{N^i_j}t} + \rho^i_{N^i_j,\infty}, \tag{10}$$

where $\rho^i_{N^i_j,0}$, $\rho^i_{N^i_j,\infty}$, $l^i_{N^i_j}$ are positive parameters chosen such that $\rho^i_{N^i_j,0} > |\tilde{x}^i_{N^i_j}(0)|$ and $\rho^i_{N^i_j,0} > \rho^i_{N^i_j,\infty} > 0$. Denote with $\epsilon^i_{N^i_j} = \rho^i_{N^i_j}(t)^{-1}\tilde{x}^i_{N^i_j}$ the normalization of $\tilde{x}^i_{N^i_j}(t)$ with respect to $\rho^i_{N^i_j}(t)$. Then, by introducing a strictly increasing mapping $T : (-1, 1) \to \mathbb{R}$, with $T(0) = 0$, the transformed normalized state estimation errors $e^i_{N^i_j}$ are defined for all $i \in \mathcal{V}$ and $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$ as:

$$e^i_{N^i_j} = T(\epsilon^i_{N^i_j}) = T(\rho^i_{N^i_j}(t)^{-1}\tilde{x}^i_{N^i_j}). \tag{11}$$

In this work, we define $T(\epsilon) = \ln(\frac{1+\epsilon}{1-\epsilon})$ and denote its derivative with respect to ϵ as $J_T(\epsilon) = \frac{2}{1-\epsilon^2}$, which is positive by construction.

Remark 1: Since $T(\cdot)$ is sign preserving, $e^i_{N^i_j}$ and $\epsilon^i_{N^i_j}$ have the same sign.

Similar to (4) and (7), by stacking (11) as:

$$\boldsymbol{e}^i = \left[e^i_{N^i_1}, \dots, e^i_{N^i_{\eta_i}}\right]^\top \text{ and } \boldsymbol{e}_i = \left[e^{N^i_1}_i, \dots, e^{N^i_{\eta_i}}_i\right]^\top, \tag{12}$$

we can respectively define the transformed normalized estimation error on the i-th agent's estimate of all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$ state and the one on the i-th agent state computed by every $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$.

By differentiating $\boldsymbol{e}^i$ as in (12) with respect to time:

$$\dot{\boldsymbol{e}}^i = \boldsymbol{J}^i\boldsymbol{P}^i(t)^{-1}(\dot{\tilde{\boldsymbol{x}}}^i - \dot{\boldsymbol{P}}^i(t)\boldsymbol{P}^i(t)^{-1}\tilde{\boldsymbol{x}}^i), \tag{13}$$

where $\boldsymbol{J}^i = \text{diag}\left(J_T(\epsilon^i_{N^i_1}), \dots, J_T(\epsilon^i_{N^i_{\eta_i}})\right)$, $\boldsymbol{P}^i(t) = \text{diag}\left(\rho^i_{N^i_1} \dots, \rho^i_{N^i_{\eta_i}}\right)$ and $\dot{\boldsymbol{P}}^i(t) = \text{diag}\left(\dot{\rho}^i_{N^i_1}, \dots, \dot{\rho}^i_{N^i_{\eta_i}}\right)$. Similarly, by differentiating $\boldsymbol{e}_i$ defined as in (12):

$$\dot{\boldsymbol{e}}_i = \boldsymbol{J}_i\boldsymbol{P}_i^{-1}(t)(\dot{\tilde{\boldsymbol{x}}}_i - \dot{\boldsymbol{P}}_i(t)\boldsymbol{P}_i^{-1}(t)\tilde{\boldsymbol{x}}_i), \tag{14}$$

where $\boldsymbol{J}_i = \text{diag}\left(J_T(\epsilon^{N^i_1}_i), \dots, J_T(\epsilon^{N^i_{\eta_i}}_i)\right)$, $\boldsymbol{P}_i(t) = \text{diag}\left(\rho^{N^i_1}_i, \dots, \rho^{N^i_{\eta_i}}_i\right)$ and $\dot{\boldsymbol{P}}_i(t) = \text{diag}\left(\dot{\rho}^{N^i_1}_i, \dots, \dot{\rho}^{N^i_{\eta_i}}_i\right)$.

Remark 2: Given (11), if $\boldsymbol{e}^i$ is bounded, then $\epsilon^i_{N^i_j}$ is constrained within the interval $(-1, 1)$. As a result, $\tilde{\boldsymbol{x}}^i_{N^i_j}$ for all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$ evolves satisfying (9).

C. Problem Formulation

In this work, we aim at designing a distributed k-hop DPPO for the multi-agent system (1) to allow each agent to estimate the state of its k-hop neighbors with predefined performance requirements on the estimation errors. With the notation presented, the problem can be formalized as:

***Problem** 1:* Given the multi-agent system (1), with communication graph $\mathcal{G}$ and with prescribed performance functions $\rho^i_{N^i_j}(t)$ as in (10) for all $i \in \mathcal{V}$ and all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$, design a distributed k-hop observer such that $-\boldsymbol{P}^i(t)1_{\eta_i} < \tilde{\boldsymbol{x}}^i(t) < \boldsymbol{P}^i(t)1_{\eta_i}$ holds for all $i \in \mathcal{V}$ and all $t \in \mathbb{R}_{\geq 0}$, with $\tilde{\boldsymbol{x}}^i(t)$ as in (5).

III. k-hop Distributed Prescribed Performance Observer Design

In this section a k-hop DPPO to solve Problem 1 is introduced. For this purpose, assume the state estimate $\hat{\boldsymbol{x}}^i$ is updated according to (15):

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}^i &= \boldsymbol{f}(\hat{\boldsymbol{x}}^i) + \boldsymbol{A}^i\hat{\boldsymbol{x}}^i + \hat{\boldsymbol{u}}^i - \boldsymbol{\Omega}^i H^{\text{kc}}_i\boldsymbol{P}^i(t)^{-1}\boldsymbol{J}^i\boldsymbol{e}^i - \boldsymbol{K}^i\boldsymbol{\xi}^i \\ \boldsymbol{\xi}^i &= \sum_{j\in\mathcal{N}_i}[\hat{\boldsymbol{W}}_i(\hat{\boldsymbol{W}}_j^\top\boldsymbol{W}_j\hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{x}}^i - \hat{\boldsymbol{W}}_i^\top\boldsymbol{W}_i\hat{\boldsymbol{W}}_j^\top\hat{\boldsymbol{x}}^j) \\ &\quad + \hat{\boldsymbol{W}}_i(\boldsymbol{W}_j^\top\boldsymbol{W}_j\hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{x}}^i - \hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{W}}_i\boldsymbol{W}_j^\top\boldsymbol{W}_j\boldsymbol{x})],\end{aligned} \tag{15}$$

where $\hat{\boldsymbol{W}}_i \in \mathbb{R}^{\eta_i \times N}$ is a binary matrix selecting the states estimated by agent i, i.e $\boldsymbol{x}^i = \hat{\boldsymbol{W}}_i\boldsymbol{x}$, $\boldsymbol{W}_i \in \mathbb{R}^{|\mathcal{N}_i|\times N}$ is a binary matrix selecting the state of agents $j \in \mathcal{N}_i$, $\boldsymbol{P}^i(t)$, $\boldsymbol{J}^i$ are defined as per (13), $\boldsymbol{f}(\hat{\boldsymbol{x}}^i)$, $\boldsymbol{A}^i$, H^{kc}_i, $\boldsymbol{\Omega}^i$, $\boldsymbol{K}^i$ are such that:

$$\boldsymbol{f}(\hat{\boldsymbol{x}}^i) = \left[f(\hat{x}^i_{N^i_1}), \dots, f(\hat{x}^i_{N^i_{\eta_i}})\right]^\top, \quad \boldsymbol{A}^i = I_{\eta_i} \otimes A, \tag{16}$$

$$H^{\text{kc}}_i = \text{diag}(|\mathcal{N}_{N^i_1} \cap \mathcal{N}_i|, \dots, |\mathcal{N}_{N^i_{\eta_i}} \cap \mathcal{N}_i|), \tag{17}$$

$\boldsymbol{\Omega}^i = \text{diag}(\omega_{N^i_1}, \dots, \omega_{N^i_{\eta_i}})$ and $\boldsymbol{K}^i = \text{diag}(K_{N^i_1}, \dots, K_{N^i_{\eta_i}})$, where $\omega_j \in \mathbb{R}_{\geq 0}$ and $K_j \in \mathbb{R}_{\geq 0}$ are observer parameters to be tuned $\forall j \in \mathcal{V}$.

In (15), each agent $i \in \mathcal{V}$ updates $\hat{\boldsymbol{x}}^i$ by comparing, in $\boldsymbol{\xi}^i$, its estimates with the real state information $x_{N^i_l}$ shared by the agents $j \in \mathcal{N}_i \cap \mathcal{N}_{N^i_l}$, $\forall N^i_l \in \mathcal{N}^{2\text{-hop}}_i$: $\hat{\boldsymbol{W}}_i(\boldsymbol{W}_j^\top\boldsymbol{W}_j\hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{x}}^i - \hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{W}}_i\boldsymbol{W}_j^\top\boldsymbol{W}_j\boldsymbol{x})$ and with the estimations $\hat{x}^j_{N^i_l}$ shared by those $j \in \mathcal{N}_i \cap \mathcal{N}^{k\text{-hop}}_{N^i_l}$, $\forall N^i_l \in \mathcal{N}^{k\text{-hop}}_i$: $\hat{\boldsymbol{W}}_i(\hat{\boldsymbol{W}}_j^\top\hat{\boldsymbol{W}}_j\hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{x}}^i - \hat{\boldsymbol{W}}_i^\top\hat{\boldsymbol{W}}_i\hat{\boldsymbol{W}}_j^\top\hat{\boldsymbol{x}}^j)$. Furthermore, the term $\boldsymbol{\Omega}^i H^{\text{kc}}_i\boldsymbol{P}^i(t)^{-1}\boldsymbol{J}^i\boldsymbol{e}^i$ is added to $\hat{\boldsymbol{x}}^i$ dynamics to guarantee the prescribed performance satisfaction.

Note that the observer is independent of the input structure, and only the input estimate $\hat{\boldsymbol{u}}^i$ is involved in (15).

***Example** 1:* Consider a path graph with $N = 4$ agents. Denote the global state with $\boldsymbol{x} = [x_1\ x_2\ x_3\ x_4]^\top$. Then, if $\mathcal{N}^{3\text{-hop}}_1 = \{3, 4\}$ and $\mathcal{N}_1 = \{2\}$, $\boldsymbol{W}_1 = [0\ 1\ 0\ 0]$, $\hat{\boldsymbol{W}}_1 = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$ and $\boldsymbol{\xi}^1 = \begin{bmatrix}\hat{x}^1_3 - x_3\\\hat{x}^1_4 - \hat{x}^2_4\end{bmatrix}$.

Remark 3: For any graph $\mathcal{G}$ and any value of k, there always exists a full rank permutation matrix R of proper dimension [13, pp.32] such that:

$$\left[(\tilde{\boldsymbol{x}}^1)^\top, \dots, (\tilde{\boldsymbol{x}}^N)^\top\right]^\top = R\left[\tilde{\boldsymbol{x}}_1^\top, \dots, \tilde{\boldsymbol{x}}_N^\top\right]^\top. \tag{18}$$

Therefore, the convergence of $\tilde{\boldsymbol{x}}^i(t)$ as per (5) is implied by the one of $\tilde{\boldsymbol{x}}_i(t)$ as per (8).

As a result, Problem 1 can be reformulated as:

***Problem* 2:** Given the multi-agent system (1), with communication graph $\mathcal{G}$ and with prescribed performance functions $\rho^i_{N^i_j}(t)$ as in (10) for all $i \in \mathcal{V}$ and all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$, design a distributed k-hop observer such that $-\boldsymbol{P}_i(t)1_{\eta_i} < \tilde{\boldsymbol{x}}_i(t) < \boldsymbol{P}_i(t)1_{\eta_i}$ holds for all $i \in \mathcal{V}$ and all $t \in \mathbb{R}_{\geq 0}$, with $\tilde{\boldsymbol{x}}_i(t)$ as in (8).

Consider $\tilde{x}^{N^i_j}_i$ and $\tilde{u}^{N^i_j}_i$ with $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$ as per (8), i.e., the estimation errors on the i-th agent state and input when the estimate is performed by agent $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$. Then, from (8) and (15), the dynamics of $\tilde{x}^{N^i_j}_i$ for $j \in \{1, \ldots, \eta_i\}$ is:

$$
\begin{aligned}
\dot{\tilde{x}}^{N^i_j}_i &= f(\hat{x}^{N^i_j}_i) - f(x_i) + A\tilde{x}^{N^i_j}_i + \tilde{u}^{N^i_j}_i - K_i \xi^{N^i_j}_i \\
&\quad - \rho^{N^i_j}_i(t)^{-1}\omega_i |\mathcal{N}_i \cap \mathcal{N}_{N^i_j}| J_T(\epsilon^{N^i_j}_i) e^{N^i_j}_i, \\
\xi^{N^i_j}_i &= \sum_{l \in (\mathcal{N}_{N^i_j} \cap \mathcal{N}^{k\text{-hop}}_i)} (\tilde{x}^{N^i_j}_i - \tilde{x}^l_i) + \sum_{l \in (\mathcal{N}_{N^i_j} \cap \mathcal{N}_i)} \tilde{x}^{N^i_j}_i,
\end{aligned} \tag{19}
$$

where $\xi^{N^i_j}_i$ denotes the i-th component of the update $\boldsymbol{\xi}^{N^i_j}$ in (15) when the estimates computed by agent N^i_j are taken into account.

Define $\boldsymbol{\xi}_i := [\xi^{N^i_1}_i, \ldots, \xi^{N^i_{\eta_i}}_i]^\top$. Then, by stacking $\xi^{N^i_j}_i$ as per (19) for all $N^i_j \in \mathcal{N}^{k\text{-hop}}_i$ it results:

$$
\boldsymbol{\xi}_i = (L^{\text{kc}}_i + H^{\text{kc}}_i)\tilde{\boldsymbol{x}}_i = M^{\text{kc}}_i \tilde{\boldsymbol{x}}_i, \tag{20}
$$

where the matrix L^{kc}_i is the Laplacian matrix of the sub-graph $\mathcal{G}_i = (\mathcal{N}^{k\text{-hop}}_i, \mathcal{E}_i)$ induced by the k-hop neighbors of agent i, with $\mathcal{E}_i = \{(p,q) \in \mathcal{E} : \{p,q\} \in \mathcal{N}^{k\text{-hop}}_i\}$ [1, pp. 24], $H^{\text{kc}}_i \in \mathbb{R}^{\eta_i \times \eta_i}$ is defined as per (17) and $M^{kc}_i \in \mathbb{R}^{\eta_i \times \eta_i}$ is defined as $M^{\text{kc}}_i = L^{\text{kc}}_i + H^{\text{kc}}_i$.

***Example* 2:** Consider the same network as in Example 1. Then, $L^{\text{kc}}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $H^{\text{kc}}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Lemma 1 ([6]): If $\mathcal{G}$ is connected, $M^{\text{kc}}_i \succ 0 \; \forall i \in \mathcal{V}$ with $\mathcal{N}^{k\text{-hop}}_i \neq \emptyset$.

Denote with $\delta^{\tilde{x}}_i \in \mathbb{R}_{>0}$ the desired maximum steady state estimation error and assume each agent runs a convergent input observer, e.g. the k-hop input observer in [6, Th. 2]. Given the convergence, $\|\tilde{\boldsymbol{u}}_i(t)\| \leq \delta^{\tilde{u}}_i$ holds with $\delta^{\tilde{u}}_i = \|\tilde{\boldsymbol{u}}_i(0)\|$ and the state observer behavior can be stated as:

Theorem 1: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed observer (15). For all $i \in \mathcal{V}$, assume that the input estimation error $\|\tilde{\boldsymbol{u}}_i(t)\|$ is upper bounded by $\delta^{\tilde{u}}_i \in \mathbb{R}_{\geq 0}$. Then, $\tilde{\boldsymbol{x}}_i(t)$ as per (8) reaches the set $\mathcal{X}_i := \{\tilde{\boldsymbol{x}}_i(t) : \|\tilde{\boldsymbol{x}}_i(t)\| \leq \delta^{\tilde{x}}_i\}$ while guaranteeing $-\boldsymbol{P}_i(t)1_{\eta_i} < \tilde{\boldsymbol{x}}_i(t) < \boldsymbol{P}_i(t)1_{\eta_i}$ provided that $|\tilde{\boldsymbol{x}}_i(0)| < \boldsymbol{P}_i(0)1_{\eta_i}$, $\delta^{\tilde{x}}_i$ is chosen to satisfy $\delta^{\tilde{x}}_i \leq \min_{N^i_j \in \mathcal{N}^{k\text{-hop}}_i}\{\rho^{N^i_j}_{i,\infty}\}$ and K_i is tuned for each $i \in \mathcal{V}$ such that:

$$
K_i \geq \frac{\delta^{\tilde{u}}_i/\delta^{\tilde{x}}_i + l_f + \sigma_{\max}(\boldsymbol{A}^i) + \delta_i}{\lambda_{\min}(M^{\text{kc}}_i)}, \tag{21}
$$

where $\delta_i = \max_{N^i_j \in \mathcal{N}^{k\text{-hop}}_i}\{l^i_{N^i_j}(\rho^i_{N^i_j,0} - \rho^i_{N^i_j,\infty})/\rho^i_{N^i_j,0}\}$.

Furthermore, if there exists $t_u \in \mathbb{R}_{\geq 0}$ such that $\tilde{\boldsymbol{u}}_i(t) = 0_{\eta_i}$ for all $i \in \mathcal{V}$ and all $t \geq t_u$, then $\tilde{\boldsymbol{x}}_i(t)$ converges to zero for all $i \in \mathcal{V}$.

Proof: From (15) and (20), the vector form of (19) becomes:

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{x}}}_i &= (\boldsymbol{f}(\hat{\boldsymbol{x}}_i) - \boldsymbol{f}(\boldsymbol{x}_i)) + \boldsymbol{A}^i\tilde{\boldsymbol{x}}_i - K_i M^{\text{kc}}_i \tilde{\boldsymbol{x}}_i \\
&\quad - \omega_i H^{\text{kc}}_i \boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i \boldsymbol{e}_i + \tilde{\boldsymbol{u}}_i,
\end{aligned} \tag{22}
$$

where $\boldsymbol{f}(\hat{\boldsymbol{x}}_i) = [f(\hat{x}^{N^i_1}_i), \ldots, f(\hat{x}^{N^i_{\eta_i}}_i)]^\top$, $\boldsymbol{f}(\boldsymbol{x}_i) = 1_{\eta_i} \otimes f(x_i)$ and $\boldsymbol{P}_i(t)$, $\boldsymbol{J}_i$ are defined as in (14). By replacing (22) into (14):

$$
\begin{aligned}
\dot{\boldsymbol{e}}_i &= \boldsymbol{J}_i\boldsymbol{P}_i(t)^{-1}(\boldsymbol{f}(\hat{\boldsymbol{x}}_i) - \boldsymbol{f}(\boldsymbol{x}_i) + \tilde{\boldsymbol{u}}_i - K_i M^{\text{kc}}_i \tilde{\boldsymbol{x}}_i \\
&\quad + \boldsymbol{A}^i\tilde{\boldsymbol{x}}_i - \omega_i H^{\text{kc}}_i \boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i\boldsymbol{e}_i - \dot{\boldsymbol{P}}_i(t)\boldsymbol{P}^{-1}_i(t)\tilde{\boldsymbol{x}}_i).
\end{aligned} \tag{23}
$$

For all $i \in \mathcal{V}$ consider the candidate Lyapunov function $V(\boldsymbol{e}_i, \tilde{\boldsymbol{x}}_i) = \frac{1}{2}\boldsymbol{e}^\top_i \boldsymbol{e}_i + \frac{1}{2}\tilde{\boldsymbol{x}}^\top_i \tilde{\boldsymbol{x}}_i$, whose time derivative by means of (22) and (23) can be written as:

$$
\begin{aligned}
\dot{V} &= \boldsymbol{e}^\top_i \boldsymbol{J}_i \boldsymbol{P}^{-1}_i(t)\tilde{\boldsymbol{u}}_i + \boldsymbol{e}^\top_i \boldsymbol{J}_i \boldsymbol{P}^{-1}_i(t)(\boldsymbol{f}(\hat{\boldsymbol{x}}_i) - \boldsymbol{f}(\boldsymbol{x}_i) \\
&\quad + \boldsymbol{A}^i\tilde{\boldsymbol{x}}_i - K_i M^{\text{kc}}_i\tilde{\boldsymbol{x}}_i - \dot{\boldsymbol{P}}_i(t)\boldsymbol{P}^{-1}_i(t)\tilde{\boldsymbol{x}}_i) \\
&\quad - \omega_i \boldsymbol{e}^\top_i \boldsymbol{J}_i \boldsymbol{P}^{-1}_i(t) H^{\text{kc}}_i \boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i\boldsymbol{e}_i + \tilde{\boldsymbol{x}}^\top_i\tilde{\boldsymbol{u}}_i \\
&\quad + \tilde{\boldsymbol{x}}^\top_i\left(\boldsymbol{f}(\hat{\boldsymbol{x}}_i) - \boldsymbol{f}(\boldsymbol{x}_i) + \boldsymbol{A}^i\tilde{\boldsymbol{x}}_i - K_i M^{\text{kc}}_i\tilde{\boldsymbol{x}}_i\right) \\
&\quad - \omega_i\tilde{\boldsymbol{x}}^\top_i H^{\text{kc}}_i \boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i\boldsymbol{e}_i.
\end{aligned} \tag{24}
$$

Given the same sign of the components of $\boldsymbol{P}^{-1}_i(t)\tilde{\boldsymbol{x}}_i$ and $\boldsymbol{e}_i$ resulting from Remark 1, and the positive semi-definiteness of $\boldsymbol{J}_i$ and H^{kc}_i stemming from their definition in (13) and (17), $-\omega_i\boldsymbol{e}^\top_i\boldsymbol{J}_i\boldsymbol{P}^{-1}_i(t)H^{\text{kc}}_i\boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i\boldsymbol{e}_i - \omega_i\tilde{\boldsymbol{x}}^\top_i H^{\text{kc}}_i\boldsymbol{P}^{-1}_i(t)\boldsymbol{J}_i\boldsymbol{e}_i \leq 0$ for all $\tilde{\boldsymbol{x}}_i, \boldsymbol{e}_i \in \mathbb{R}^{\eta_i}$. Furthermore, since $\|\boldsymbol{f}(\hat{\boldsymbol{x}}_i) - \boldsymbol{f}(\boldsymbol{x}_i)\| \leq l_f\|\tilde{\boldsymbol{x}}_i\|$ holds due to the Lipschitz continuity of f [14, (3.2)], $\lambda_{min}(J)\|x\|^2 \leq x^\top Jx \leq \lambda_{max}(J)\|x\|^2$ holds for any symmetric matrix J, $x^\top Bx \leq \sigma_{max}(B)\|x\|^2$ holds for any square matrix B [13], $\lambda_{\max}(-\dot{\boldsymbol{P}}_i(t)\boldsymbol{P}^{-1}_i(t)) \leq \lambda_{\max}(-\dot{\boldsymbol{P}}_i(0)\boldsymbol{P}^{-1}_i(0))$ holds for all $t \in \mathbb{R}_{\geq 0}$ from the definitions of $\rho^i_{N^i_j}(t)$ and $\boldsymbol{P}_i(t)$ in (10) and (14), and since $\lambda_{\max}(-\dot{\boldsymbol{P}}_i(0)\boldsymbol{P}^{-1}_i(0)) = \max_{N^i_j \in \mathcal{N}^{k\text{-hop}}_i}\{l^i_{N^i_j}(\rho^i_{N^i_j,0} - \rho^i_{N^i_j,\infty})/\rho^i_{N^i_j,0}\} = \delta_i$, $\dot{V}$ as in (24) can be upper bounded as:

$$
\begin{aligned}
\dot{V} &\leq \|\tilde{\boldsymbol{x}}_i\|\|\tilde{\boldsymbol{u}}_i\| + \|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}^{-1}_i(t)\|\|\tilde{\boldsymbol{x}}_i\|\big(\sigma_{\max}(\boldsymbol{A}^i) \\
&\quad + l_f - K_i\lambda_{\min}(M^{\text{kc}}_i) + \delta_i\big) + \|\tilde{\boldsymbol{x}}_i\|^2\big(\sigma_{\max}(\boldsymbol{A}^i) \\
&\quad + l_f - K_i\lambda_{\min}(M^{\text{kc}}_i)\big) + \|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}^{-1}_i(t)\|\|\tilde{\boldsymbol{u}}_i\|.
\end{aligned} \tag{25}
$$

To impose $\left(\sigma_{\max}(\boldsymbol{A}^i) + l_f - K_i\lambda_{\min}(M^{\text{kc}}_i)\right)\|\tilde{\boldsymbol{x}}_i\|^2 \leq -\alpha_i\|\tilde{\boldsymbol{x}}_i\|^2$ and $\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}^{-1}_i(t)\|\|\tilde{\boldsymbol{x}}_i\|(\sigma_{\max}(\boldsymbol{A}^i) + l_f - K_i\lambda_{\min}(M^{\text{kc}}_i) + \delta_i) \leq -\theta_i\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}^{-1}_i(t)\|\|\tilde{\boldsymbol{x}}_i\|$ for some design parameters $\theta_i, \alpha_i \in \mathbb{R}_{>0}$ to be chosen, K_i must be tuned such that $K_i \geq \max\{K^1_i, K^2_i\}$, where $K^1_i = \frac{\theta_i + l_f + \sigma_{\max}(\boldsymbol{A}^i) + \delta_i}{\lambda_{\min}(\boldsymbol{M}^{\text{kc}}_i)}$ and $K^2_i = \frac{\alpha_i + l_f + \sigma_{\max}(\boldsymbol{A}^i)}{\lambda_{\min}(\boldsymbol{M}^{\text{kc}}_i)}$. If K_i is tuned to satisfy $K_i \geq \max\{K^1_i, K^2_i\}$, (25) becomes:

$$
\begin{aligned}
\dot{V} &\leq -\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}^{-1}_i(t)\|(\theta_i\|\tilde{\boldsymbol{x}}_i\| - \|\tilde{\boldsymbol{u}}_i\|) \\
&\quad - (\alpha_i\|\tilde{\boldsymbol{x}}_i\| - \|\tilde{\boldsymbol{u}}_i\|)\|\tilde{\boldsymbol{x}}_i\|.
\end{aligned} \tag{26}
$$

Let's start by investigating (26) for $\tilde{\boldsymbol{x}}_i \in \mathcal{X}^c_i$, i.e. for those $\tilde{\boldsymbol{x}}_i$ such that $\|\tilde{\boldsymbol{x}}_i\| > \delta^{\tilde{x}}_i$. According to (26), and since $\|\tilde{\boldsymbol{u}}_i(t)\| \leq \delta^{\tilde{u}}_i$, if θ_i and α_i are chosen such that $\theta_i \geq \delta^{\tilde{u}}_i/\delta^{\tilde{x}}_i$ and $\alpha_i \geq \delta^{\tilde{u}}_i/\delta^{\tilde{x}}_i$, $\dot{V} < 0$ holds for all $\tilde{\boldsymbol{x}}_i \in \mathcal{X}^c_i$. Therefore, since $-\dot{\boldsymbol{P}}_i(t)\boldsymbol{P}^{-1}_i(t) \succ 0$ from the definition of $\boldsymbol{P}_i$, if α_i and θ_i are chosen to be equal, $\dot{V} < 0$ holds if K_i is tuned to satisfy (21), given that $K_i = \max\{K^1_i, K^2_i\} = K^1_i$.

When $\tilde{\boldsymbol{x}}_i \in \mathcal{X}_i$, K_i as per (21) does not anymore guarantee $\dot{V} < 0$ for all $\tilde{\boldsymbol{x}}_i \in \mathcal{X}_i$. Suppose $K_i = \frac{\delta_i^{\tilde{u}}/\delta_i^{\tilde{x}} + l_f + \sigma_{\max}(\boldsymbol{A}^i) + \delta_i}{\lambda_{\min}(M_i^{\mathrm{kc}})}$. Then, since (25) becomes:

$$
\begin{aligned}
\dot{V} \le & -\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}_i^{-1}(t)\| \left(\|\tilde{\boldsymbol{x}}_i\|\delta_i^{\tilde{u}}/\delta_i^{\tilde{x}} - \|\tilde{\boldsymbol{u}}_i\|\right) \\
& - \left(\|\tilde{\boldsymbol{x}}_i\|\delta_i^{\tilde{u}}/\delta_i^{\tilde{x}} - \|\tilde{\boldsymbol{u}}_i\|\right)\|\tilde{\boldsymbol{x}}_i\|,
\end{aligned} \tag{27}
$$

$\dot{V} \le (\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}_i^{-1}(t)\| + \delta_i^{\tilde{x}})\delta_i^{\tilde{u}} \le c$, for some $c \in \mathbb{R}_{>0}$, for all $\tilde{\boldsymbol{x}}_i \in \{\tilde{\boldsymbol{x}}_i : \|\tilde{\boldsymbol{x}}_i\| < \delta_i^{\tilde{x}}\}$ and $\dot{V} \le 0$ for all $\tilde{\boldsymbol{x}}_i \in \{\tilde{\boldsymbol{x}}_i : \|\tilde{\boldsymbol{x}}_i\| = \delta_i^{\tilde{x}}\}$. As a result, from the continuity of the solution of (1), we can conclude the convergence of the system towards the set $\mathcal{X}_i$, that results to be an invariant set for the system when K_i is chosen as in (21) [14, Th. 4.4]. Furthermore, since (27) for $\|\tilde{\boldsymbol{u}}_i\| = 0$ results into $\dot{V} \le -\frac{\delta_i^{\tilde{u}}}{\delta_i^{\tilde{x}}}\|\boldsymbol{e}_i\|\|\boldsymbol{J}_i\|\|\boldsymbol{P}_i^{-1}(t)\|\|\tilde{\boldsymbol{x}}_i\| - \frac{\delta_i^{\tilde{u}}}{\delta_i^{\tilde{x}}}\|\tilde{\boldsymbol{x}}_i\|^2 < 0$ for all $(\boldsymbol{e}_i, \tilde{\boldsymbol{x}}_i) \neq 0_{\eta_i^2}$, convergence of the state estimation error to zero holds if there exists $t_u \in \mathbb{R}_{\ge 0}$ such that $\tilde{\boldsymbol{u}}_i(t) = 0_{\eta_i}$ for all $t \ge t_u$.

Given the convergence of the system towards $\mathcal{X}_i$ and the boundedness of $\boldsymbol{e}_i$ resulting from $\dot{V} < 0$ for all $\tilde{\boldsymbol{x}}_i(t) \in \mathcal{X}_i^c$ and all $\boldsymbol{e}_i \in \mathbb{R}^{\eta_i}$, if $\delta_i^{\tilde{x}}$ is chosen to satisfy $\delta_i^{\tilde{x}} \le \min_{N_j^i \in \mathcal{N}_i^{k\text{-hop}}}\{\rho_{i,\infty}^{N_j^i}\}$ and $|\tilde{\boldsymbol{x}}_i(0)| < \boldsymbol{P}_i(0)1_{\eta_i}$, Remark 2 guarantees $-\boldsymbol{P}_i(t)1_{\eta_i} < \tilde{\boldsymbol{x}}_i(t) < \boldsymbol{P}_i(t)1_{\eta_i}$ for all $t \in \mathbb{R}_{\ge 0}$.

Note that ω_i results to be a free design parameter. ■

IV. k-hop DPPO-Based Feedback Controller

The introduced k-hop DPPO provides each agent $i \in \mathcal{V}$ with an estimate of the state of agents $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, enabling the design of an estimation-based closed-loop controller to achieve the team's objective.

Consider the vectorized multi-agent dynamics from (1):

$$
\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + (I_N \otimes A)\boldsymbol{x} + \boldsymbol{u}, \tag{28}
$$

where $\boldsymbol{x}$ is defined as in (2), and the input vector $\boldsymbol{u}$ is a general nonlinear state-feedback function of the form:

$$
\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}) = \left[q_1(\bar{\boldsymbol{x}}_1, \boldsymbol{x}^1), \ldots, q_N(\bar{\boldsymbol{x}}_N, \boldsymbol{x}^N)\right]^\top, \tag{29}
$$

where $\boldsymbol{x}^i$ for each $i \in \mathcal{V}$ is defined as in (3), and $\bar{\boldsymbol{x}}_i$ contains the state information of agent i and of all $j \in \mathcal{N}_i$, i.e. $\bar{\boldsymbol{x}}_i = [x_i, \boldsymbol{x}^\top \boldsymbol{W}_i^\top]^\top$.

Since the exact state information of the k-hop neighbors $\boldsymbol{x}^i$ for all $i \in \mathcal{V}$ is not locally available, the controller in (29) is implemented using their estimated states $\hat{\boldsymbol{x}}^i$, i.e.:

$$
\boldsymbol{u} = \boldsymbol{q}(\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}) = \left[q_1(\bar{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}^1), \ldots, q_N(\bar{\boldsymbol{x}}_N, \hat{\boldsymbol{x}}^N)\right]^\top. \tag{30}
$$

Furthermore, since $\bar{\boldsymbol{x}}_i$ consists of components of $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}^i = \boldsymbol{x}^i + \tilde{\boldsymbol{x}}^i$, by introducing $\tilde{\boldsymbol{x}} = [\tilde{\boldsymbol{x}}^1, \ldots, \tilde{\boldsymbol{x}}^N]^\top$, $\boldsymbol{u}$ can be rewritten as $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x} + \tilde{\boldsymbol{x}})$.

By defining $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \boldsymbol{f}(\boldsymbol{x}) + (I_N \otimes A)\boldsymbol{x} + \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x} + \tilde{\boldsymbol{x}})$, (28) results into $\dot{\boldsymbol{x}} = \Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, where $\tilde{\boldsymbol{x}}$ is regarded as an input disturbance affecting the nominal unforced system.

Definition 1 ([15]): A system $\dot{x} = f(x, u)$, with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is set-Input to State Stable (set-ISS) with respect to a set $\mathcal{A}$ if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for each initial condition and any locally essentially bounded input u satisfying $\sup_{t \ge 0}\|u(t)\| \le \infty$, $\|x(t)\|_{\mathcal{A}} \le \beta(\|x(0)\|_{\mathcal{A}}, t) + \gamma\left(\sup_{0 \le \tau \le t}\|u(\tau)\|\right)$ holds with $\|x(t)\|_{\mathcal{A}} = \mathrm{dist}(x, \mathcal{A}) = \inf_{a \in \mathcal{A}}\{\|x - a\|\}$.

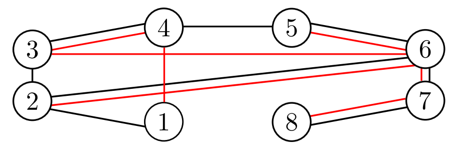


Fig. 1. Graphs $\mathcal{G}_C$ and $\mathcal{G}_T$, respectively in black and red.

Denote with $\delta^{\tilde{\boldsymbol{x}}}$ the upper bound on the norm of the steady state estimation error $\tilde{\boldsymbol{x}}$, i.e. $\delta^{\tilde{\boldsymbol{x}}} = \|[\delta_1^{\tilde{x}}, \ldots, \delta_N^{\tilde{x}}]^\top\|$. Then:

Theorem 2: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed observer (15). Suppose each agent runs the local control input (30). Then, the multi-agent system reaches $\mathcal{A}_e = \{\boldsymbol{x} : \|\boldsymbol{x}\|_{\mathcal{A}} \le \gamma(\delta^{\tilde{\boldsymbol{x}}})\}$ if $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is set-ISS with respect to the set $\mathcal{A}$ representing the desired team objective and the feedback controller in (29) ensures convergence of the multi-agent system towards $\mathcal{A}$. Furthermore, if there exists $t_u \in \mathbb{R}$ such that $\tilde{\boldsymbol{u}}_i(t) = 0$ for all $i \in \mathcal{V}$ and all $t \ge t_u$, $\mathcal{A}_e \equiv \mathcal{A}$.

Proof: From Theorem 1, if $K_i > \frac{\delta_i^{\tilde{u}}/\delta_i^{\tilde{x}} + l_f + \sigma_{\max}(\boldsymbol{A}^i) + \delta_i}{\lambda_{\min}(M_i^{\mathrm{kc}})}$, there exists a finite time $t_x \in \mathbb{R}_{\ge 0}$ such that $\|\tilde{\boldsymbol{x}}_i(t)\| \le \delta_i^{\tilde{x}}$ for all agents $i \in \mathcal{V}$ and all $t \ge t_x$. Therefore, since $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is set-ISS and from $t = t_x$ the multi-agent system evolves from $\boldsymbol{x}(t_x)$ under the dynamics $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ with $\|\tilde{\boldsymbol{x}}\| \le \delta^{\tilde{\boldsymbol{x}}}$, $\|\boldsymbol{x}(t)\|_{\mathcal{A}} \le \beta(\|\boldsymbol{x}(t_x)\|_{\mathcal{A}}, t - t_x) + \gamma(\delta^{\tilde{\boldsymbol{x}}}), \forall t \ge t_x$. As a result, thanks to the convergence of $\beta(\|\boldsymbol{x}(t_x)\|_{\mathcal{A}}, t - t_x)$ to zero resulting from $\mathcal{KL}$ function definition, the system converges to $\mathcal{A}_e = \{\boldsymbol{x} : \|\boldsymbol{x}\|_{\mathcal{A}} \le \gamma(\delta^{\tilde{\boldsymbol{x}}})\}$. Furthermore, since Theorem 1 guarantees asymptotic convergence of $\|\tilde{\boldsymbol{x}}_i(t)\|$ if there exists $t_u \in \mathbb{R}_{\ge 0}$ such that $\tilde{\boldsymbol{u}}_i(t) = 0_{\eta_i}$ for all $i \in \mathcal{V}$ and all $t \ge t_u$, and since $\|\boldsymbol{x}(t)\|_{\mathcal{A}} \le \beta(\|\boldsymbol{x}(t_u)\|_{\mathcal{A}}, t - t_u) + \gamma(\|\tilde{\boldsymbol{x}}(t_u)\|)$ holds $\forall t \ge t_u$, the system is guaranteed to converge towards $\mathcal{A}$ as t_u approaches infinity, as both $\beta(\|\boldsymbol{x}(t_u)\|_{\mathcal{A}}, t - t_u)$ and $\gamma(\|\tilde{\boldsymbol{x}}(t_u)\|)$ converge to zero. ■

V. Simulations

Consider a multi-agent system composed of $N = 8$ agents communicating according to the graph $\mathcal{G}_C = (\mathcal{V}, \mathcal{E}_C)$ in Fig. 1. Assume each agent behaves according to $\dot{x}_i = f(x_i) + Ax_i + u_i$, with $x_i = [x_{i,1}\ x_{i,2}]^\top$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $f(x_i) = \begin{bmatrix} \tanh(x_{i,1}) & \sin(x_{i,2}) \end{bmatrix}^\top$ and where u_i is designed to drive the agents towards consensus by exploiting only the edges of the graph $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$ in Fig. 1, i.e. $u_i = -f(x_i) - Ax_i + k_c v_i$, where $v_i = \sum_{j \in (\mathcal{N}_i^C \cap \mathcal{N}_i^T)}(x_j - x_i) + \sum_{j \in \mathcal{N}_i^T / \mathcal{N}_i^C}(\hat{x}_j^i - x_i)$, $k_c \in \mathbb{R}_{>0}$ is a control design parameter used to speedup convergence and $\mathcal{N}_i^C$ and $\mathcal{N}_i^T$ are respectively the i-th agent's neighbors in graph $\mathcal{G}_C$ and $\mathcal{G}_T$. The estimation term $\hat{x}_j^i$, used in $\sum_{j \in \mathcal{N}_i^T / \mathcal{N}_i^C}(\hat{x}_j^i - x_i)$, is introduced in v_i to compensate for the lack of local information available to agent i about the state of all agents $j \in \mathcal{N}_i^T / \mathcal{N}_i^C$. This necessity arises from the requirement to achieve consensus using only edges of $\mathcal{G}_T$, a condition specifically set to test the proposed observer. Under the chosen input, the system dynamics becomes $\dot{x}_i = v_i$. The choice of canceling part of the dynamics doesn't affect the observer design, which is independent of the input's structure and is still designed for the nonlinear system.

Given the connectivity of $\mathcal{G}_T$, the feedback control assuming perfect state knowledge, i.e. $v_i^{\mathrm{ideal}} := \sum_{j \in \mathcal{N}_i^T}(x_j - x_i)$, guarantees the convergence of the multi-agent system towards consensus [1]. Furthermore, since v_i can be rewritten as $v_i = \sum_{j\in\mathcal{N}_i^T}(x_j - x_i) + w_i$, with $w_i = \sum_{j\in\mathcal{N}_i^T/\mathcal{N}_i^{CT}} \tilde{x}_j^i$ bounded, and the vectorized system dynamics results into $\dot{\boldsymbol{x}} = -k_c(L_T \otimes I_2)\boldsymbol{x} + k_c\boldsymbol{w}$, where L_T is the Laplacian matrix of the graph $\mathcal{G}_T$ and $\boldsymbol{w} = [w_1^\top, \ldots, w_N^\top]^\top$, it is possible to show that $\|\boldsymbol{x}(t)\|_{\mathcal{A}} \leq e^{-k_c\lambda_2(L_T)t}\|\boldsymbol{x}(0)\|_{\mathcal{A}} + \frac{k_c}{\lambda_2(L_T)}\sup_{0\leq\tau\leq t}\|\boldsymbol{w}(\tau)\|$, where $\lambda_2(L_T)$ is the smallest positive eigenvalue of L_T. Since the upper bound on $\|\boldsymbol{x}(t)\|_{\mathcal{A}}$ is consistent with the set-ISS definition in Definition 1, we can conclude the validity of Theorem 2 for the case study.

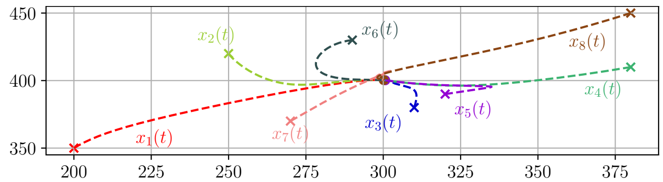
For simulation purposes, a sampling time equal to $dt = 10^{-4}$ s and parameters satisfying (21) have been chosen. In particular, $K_1 \approx K_7 \approx 837$, $K_2 \approx K_4 \approx 640$, $K_3 \approx 523$, $K_5 \approx 665$, $K_6 \approx 245$, $K_8 = 873$, $\omega_i = 1$ and $\tilde{\boldsymbol{u}}_i = 0_{2\eta_i}$ for all $i \in \mathcal{V}$. To guarantee prescribed performance, each component $\rho^i_{N_j^i,h}(0)$ of $\rho^i_{N_j^i}(0)$ has been tuned such that $|\tilde{x}^i_{N_j^i,h}(0)| < \rho^i_{N_j^i,h}(0)$ holds for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $h \in \{1,2\}$. Without loss of generality, $\rho^i_{N_j^i,h}(t) = \rho(t) = 49.3e^{-160t} + 0.7$ for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $h \in \{1,2\}$. Fig. 2 shows the results when the system is controlled by exploiting the state estimations computed by the proposed k-hop DPPO. As per Theorem 1, and as can be deduced from the plot of $\max_{i\in\mathcal{V},N_j^i\in\mathcal{N}_i^{k\text{-hop}},h\in\{1,2\}}\{|\tilde{x}^i_{N_j^i,h}(t)|\}$ in Fig. 2b, given the existence of a time $t_u \in \mathbb{R}_{\geq 0}$ for which $\tilde{\boldsymbol{u}}_i(t) = 0_{2\eta_i}$ for all $t \geq t_u$, each state estimation error converges towards zero while remaining within the funnel. Furthermore, as per Theorem 2, convergence of the agents' state towards consensus is guaranteed as shown in Fig. 2a.

To show the advantage of the proposed k-hop DPPO with respect to other observers with no performance guarantees, Fig. 2b presents also the evolution of the maximum absolute state estimation error achieved by a consensus based k-hop distributed observer where each agent i updates its estimate $\hat{\bar{x}}^i_{N_j^i}$ of all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, as $\dot{\hat{\bar{\boldsymbol{x}}}}^i = \boldsymbol{f}(\hat{\bar{\boldsymbol{x}}}^i) + \boldsymbol{A}^i\hat{\bar{\boldsymbol{x}}}^i + \hat{\bar{\boldsymbol{u}}}^i - \boldsymbol{K}^i\boldsymbol{\xi}^i$. To highlight the advantage resulting from $-\boldsymbol{\Omega}^i H_i^{\text{kc}}\boldsymbol{P}^i(t)^{-1}\boldsymbol{J}^i\boldsymbol{e}^i$ in (15), K_i, $\hat{\bar{\boldsymbol{u}}}^i$ are chosen equal to those used in the k-hop DPPO, and $\boldsymbol{\xi}^i$ is defined as in (15), but with $\hat{\bar{\boldsymbol{x}}}^i$ instead of $\hat{\boldsymbol{x}}^i$. Although the consensus based observer guarantees estimation convergence, the transient behavior of some estimation errors $\tilde{\bar{x}}^i_{N_j^i,h} = \hat{\bar{x}}^i_{N_j^i,h} - x_{N_j^i,h}$ violates the prescribed performance, as $\rho(t) < \max_{i\in\mathcal{V},N_j^i\in\mathcal{N}_i^{k\text{-hop}},h\in\{1,2\}}\{|\tilde{\bar{x}}^i_{N_j^i,h}(t)|\}$ for some t.

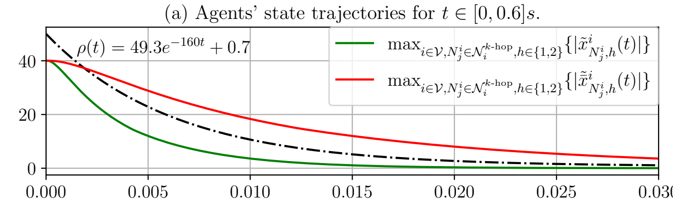
VI. Conclusion and Future Work

We proposed a k-hop DPPO in which each agent exploits the communication with its neighbors to estimate the states of those agents k-hop away while guaranteeing predefined transient behavior specified a priori. The resulting estimations are independent of the input structure and are guaranteed to converge towards a set characterized by the desired maximum steady state estimation error. Furthermore, we proved that under set-ISS condition of the feedback control law, the state estimations can be used to drive the agents towards an equilibrium representing the team objective.

In future work we will study the effect of time varying graphs, directed graphs, disturbances and model uncertainties on the observer design.



(a) Agents' state trajectories for $t \in [0, 0.6]s$.



(b) Maximum absolute state estimation error achieved with the k-hop DPPO and with consensus based k-hop observer with no PPO guarantees.

Fig. 2. (a) Agents' state evolution; initial and final states are respectively represented by crosses and points. (b) Comparison between the maximum absolute state estimation error (across each estimation, each agent and each component) achieved with the proposed $\boldsymbol{k}$-hop DPPO (green) and the $\boldsymbol{k}$-hop observer with no performance guarantees (red). The performance bound is indicated by a dashed line.

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