Predictor-based safety control for systems with multiple time-varying delays ⋆

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Abstract: Control barrier functions (CBFs) have been recently considered for ensuring safety of nonlinear input-affine systems by means of appropriately designed controllers rendering a desired superlevel set of the CBF function forward invariant. In this work, we consider the safety control problem for nonlinear input-affine systems with multiple time-varying input delays. In order to ensure safety, we first design a set of predictors that estimate the state of the system at different future times by utilizing the control laws designed to ensure safety of the delay-free system. Under the assumption of perfect estimation of the future states, we show that under the designed controller, the closed-loop performance of the systems with and without the input delays is the same by the time the input with the largest delay acts on the system with delays for the first time.

Keywords: safety control, control barrier functions, input delay systems, multiple time-varying delays.

1. INTRODUCTION

Over the last decades, continuous technological advances have increased the number of applications autonomous systems can contribute at. In the vast majority of these applications, autonomous systems often need to perform a variety of complex tasks in highly uncertain environments while ensuring their safety e.g., avoiding collisions with respect to existing (static or dynamic) obstacles in the area or avoid entering at unsafe regions, where for example humans are present. Recently, safety has been expressed for general nonlinear input-affine systems as the problem of ensuring forward invariance of a known set $\mathcal{C} \subset \mathbb{R}^n$, called the *safety set*. This set is often defined as a superlevel set of a known, nonlinear function called a *control barrier function* (CBF).

CBF-based approaches have attracted much interest as they offer a direct control design based on which a state-dependent controller is found as a solution to a convex quadratic problem (QP). Initially proposed in the context of safety control in Wieland and Allgöwer (2007) and later in Ames et al. (2017), control barrier functions have been extended to other forms including but are not limited to fixed or finite time (Sharifi and Dimarogonas (2021); Srinivasan et al. (2018)), time-varying (Xu (2018)) or non-smooth CBFs (Glotfelter et al. (2017)). In addition, they have been applied to a variety of applications spanning from safety (collision or obstacle avoidance)Wang et al. (2017) to connectivity maintenance (Capelli and Sabattini (2020)) and satisfaction of spatio-temporal constraints (Garg and Panagou (2019); Lindemann and Dimarogonas (2018); Charitidou and Dimarogonas (2021)).

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Control barrier functions for systems with state or input delays have been considered in Orosz and Ames (2019); Kiss et al. (2021); Ren (2021) and Singletary et al. (2020); Jankovic (2018); Abel et al. (2019, 2020, 2021) respectively. In Orosz and Ames (2019) the notion of safety functionals is introduced, and Krasovskii-like conditions are obtained to ensure stability and safety of systems with time delays. In Kiss et al. (2021) the discretization of the time-delay system is proposed to simplify the design of the corresponding safety sets while in Ren (2021) Razumikhin type conditions are obtained for safety and control of nonlinear systems with time delays. Control barrier functions for systems with a single, constant input delay have been considered for the first time - to the authors' best knowledge - in Jankovic (2018), where the predictor-feedback approach Krstić (2009) is utilized. Closer to our work are the approaches presented in Abel et al. (2019, 2020), where systems with multiple, constant and distinct input delays are considered. While in Abel et al. (2019) the system with the multiple delays achieves the same closed loop performance with the system without delays after the larger delay of the system with delays is compensated, in Abel et al. (2020) authors aim towards ensuring safety when a smaller number of inputs acts on the system.

Recently, safety of systems with a single, known, time-varying input delay has been addressed in Abel et al. (2021). Nevertheless, to the best of our knowledge, ensuring safety in the presence of multiple, possibly different, time-varying input delays is still an open problem. Motivated by Abel et al. (2019), in this work we consider the problem of designing a feedback controller that ensures safety of the system with the multiple, time-varying delays. Under a monotonicity assumption on the time-varying delays acting at the system, we design a set of state-predictors considering the safety controllers designed for the nominal system and propose applying elementwise the delay-free nominal controller at the current predicted states to the system with the delays. Then, under the assumption of accurate prediction, we show that the closed-loop performance of the system under consideration is identical to the one of the nominal system, when the largest delay of the system is compensated for the first time.

The remainder of the paper is organized as follows: Section 2 includes notation and required background knowledge, Section 3 introduces the problem formulation, Section 4 presents the control design approach, Section 5 provides a numerical example and Section 6 provides conclusions and directions of future research.

2. NOTATION AND PRELIMINARIES

Scalars and vectors are denoted by non-bold and bold letters respectively. The partial derivative of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ evaluated at $\mathbf{x}'$ is abbreviated by $\frac{\partial h(\mathbf{x}')}{\partial \mathbf{x}} = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}\big|_{\mathbf{x}=\mathbf{x}'}$ and is considered to be a row vector. The interior of the superlevel set of $h(\mathbf{x})$, denoted by $\mathcal{C}$, is defined as $int(\mathcal{C}) = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) > 0\}$. A class $\mathcal{K}$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a strictly increasing continuous function with $\alpha(0) = 0$. An extended class $\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a locally Lipschitz continuous and strictly increasing function with $\alpha(0) = 0$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function (Khalil, 1996, Def 4.3) if, for each fixed s, the mapping $\beta(s, r)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r, $\beta(s, r)$ is decreasing with respect to s and $\beta(s, r) \to 0$ as $s \to \infty$.

2.1 Control barrier functions for systems without delays

In this section we summarize some relevant results on safety control for systems without delays. Consider the input-affine dynamical system:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ is the state and control input respectively and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz functions. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Based on $h(\mathbf{x})$ we may define its zero superlevel set as follows:

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \geq 0\}. \tag{2}$$

Definition 1. (Xu et al., 2015, Def. 6) Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined in (2) for a continuously differentiable $h : \mathbb{R}^n \to \mathbb{R}$, the function h is called a *control barrier function (CBF)* defined on a set $\mathcal{D}$ with $\mathcal{C} \subset \mathcal{D} \subseteq \mathbb{R}^n$, if there exists an extended class $\mathcal{K}$ function α such that:

$$\sup_{\mathbf{u}\in\mathbb{R}^m} \left[\frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \alpha(h(\mathbf{x}))\right] \geq 0, \quad \forall \mathbf{x} \in \mathcal{D}.$$

Given a CBF $h : \mathbb{R}^n \to \mathbb{R}$, for all $\mathbf{x} \in \mathcal{D}$ define the set:

$$K_{cbf}(\mathbf{x}) = \left\{\mathbf{u} \in \mathbb{R}^m : \frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \alpha(h(\mathbf{x})) \geq 0\right\}.$$

Theorem 2. (Xu et al., 2015, Cor. 7) Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2) for a continuously differentiable function h, if h is a CBF on $\mathcal{D}$, then any locally Lipschitz continuous controller $\mathbf{u} : \mathcal{D} \to \mathbb{R}^m$ such that $\mathbf{u}(\mathbf{x}) \in K_{cbf}(\mathbf{x})$ will render the set $\mathcal{C}$ forward invariant.

Let $\mathbf{u}_{nom} : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz controller that is designed offline for the system to ensure a higher specification such as stabilization as in Ames et al. (2019). We will refer to $\mathbf{u}_{nom}(\mathbf{x})$ as the *nominal* controller. Then, the following result can be deduced:

Theorem 3. (Xu et al., 2015, Thm. 8) Assume that vector fields f and g in the control system (1) are both locally Lipschitz continuous, and that $h : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz continuous CBF. Let $\mathbf{u}_{nom} : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous nominal controller. Suppose furthermore that $\frac{\partial h}{\partial \mathbf{x}} g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \mathcal{D}$. Then, $\mathbf{u}^* = \mathbf{q}(\mathbf{x})$ defined as:

$$\mathbf{u}^* = \underset{\mathbf{u}\in\mathbb{R}^m}{\arg\min} \|\mathbf{u} - \mathbf{u}_{nom}(\mathbf{x})\|_2^2, \tag{3}$$

subject to:

$$\frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \alpha(h(\mathbf{x})) \geq 0 \tag{3a}$$

is locally Lipschitz continuous for $\mathbf{x} \in \mathcal{D}$.

The controller $\mathbf{q}(\mathbf{x}), \mathbf{x} \in \mathcal{D}$ found as a solution to (3) can be written in closed form as follows:

$$\mathbf{q}(\mathbf{x}) = \begin{cases} \mathbf{u}_{nom}(\mathbf{x}), & a(\mathbf{x}) > 0 \\ \mathbf{u}_{nom}(\mathbf{x}) - \dfrac{a(\mathbf{x})}{\|b(\mathbf{x})\|_2^2} b^T(\mathbf{x}), & \text{otherwise} \end{cases}, \tag{4}$$

where $a(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}_{nom}(\mathbf{x})) + \alpha(h(\mathbf{x}))$ and $b(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}} g(\mathbf{x})$. In the following we will denote the j-th element of $\mathbf{q}(\mathbf{x})$ by $q_j(\mathbf{x})$.

3. PROBLEM FORMULATION

In this work we consider a system with multiple input delays defined as follows:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + \sum_{j=1}^{m} g_j(\mathbf{x}(t))u_j\big(\kappa_j(t)\big), \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g_j : \mathbb{R}^n \to \mathbb{R}^n, j \in \mathcal{J} = \{1, \ldots, m\}$ are locally Lipschitz functions, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} = [u_1 \ \ldots \ u_m]^T \in \mathbb{R}^m$ is the state and control input of (5) and $\kappa_j : \mathbb{R}_{\geq 0} \to \mathbb{R}, j \in \mathcal{J}$ are continuously differentiable functions incorporating the actuation delay that are assumed to satisfy the following inequality for every $t \in \mathbb{R}_{\geq 0}$:

$$\kappa_m(t) \leq \ldots \leq \kappa_1(t). \tag{6}$$

If $\kappa_j(t) = t - \tau_j$, for every $j \in \mathcal{J}$, where $\tau_j, j \in \mathcal{J}$ are known, positive constants, i.e., the delays acting at the system are constant, then (6) becomes $\tau_1 \leq \ldots \leq \tau_m$ which is identical to the condition considered in Abel et al. (2019, 2020). Here, we assume that (5) is forward complete for every locally bounded control input $u_j(\kappa_j(t)) \in \mathbb{R}, j \in \mathcal{J}, t \geq 0$, and for every initial condition $\mathbf{x}(0) = \mathbf{x}_0$, i.e., the system does not exhibit finite escape time. Similar to Krstić (2010), we make the following assumption on the delay functions $\kappa_j(t), j \in \mathcal{J}$:

Assumption 4. Every function $\kappa_j : \mathbb{R}_{\geq 0} \to \mathbb{R}, j \in \mathcal{J}$ is continuously differentiable and for every $t \geq 0$ satisfies the following:

$$\kappa_j(t) \leq t, \tag{7a}$$

$$t - \kappa_j(t) < \infty, \tag{7b}$$

$$\kappa_j'(t) = \frac{d\kappa_j(t)}{dt} > 0, \tag{7c}$$

Equation (7a) ensures that the system depends only on past or current inputs, i.e., it is causal. The second part of Assumption 4 guarantees that the j-th control input will be applied at the system in finite time while (7c) implies that $\kappa_j(t)$ is monotonically increasing for every $t \geq 0$. As a result of the latter, the inverse function of $\kappa_j(t), j \in \mathcal{J}, t \geq 0$, denoted by $\kappa_j^{-1}(t)$, exists and is well-defined in $\{\kappa_j(t) : t \geq 0\}$. We will call $\kappa_j(t)$ and $\kappa_j^{-1}(t)$ the j-th delay and j-th prediction time at t. For example, if $\kappa_j(t) = t - \tau_j$, where $\tau_j > 0$ is a constant delay, then, $\kappa_j^{-1}(t) = t + \tau_j$. Thus, in this case $\mathbf{x}(\kappa_j^{-1}(t)) = \mathbf{x}(t + \tau_j)$ is the state of the system at the future time $t + \tau_j$, for every $t \geq 0$.

Consider a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and define its zero superlevel set as in (2). The set $\mathcal{C} \subset \mathbb{R}^n$ is called the *safety set* and is assumed to be non-empty with no isolation points, i.e., $int(\mathcal{C}) \neq \emptyset$ and $\overline{int(\mathcal{C})} = \mathcal{C}$. Based on the above, we may state the problem considered in this paper as follows:

Problem 5. Consider the multi-input delay system (5) and the set $\mathcal{C}$ defined in (2). Design the control inputs $u_j(\kappa_j(t)), j \in \mathcal{J}$, for every $t \geq 0$ such that $\mathcal{C}$ is rendered forward invariant, i.e., if $\mathbf{x}(0) \in \mathcal{C}$, then $\mathbf{x}(t) \in \mathcal{C}$, for every $t > 0$.

4. CONTROL APPROACH

In this section we present the design of a feedback controller, found as a solution to a quadratic program (QP), that ensures safety for systems with multiple delays. To achieve this and given (6), we first design a set of state-predictors at times $\kappa_j^{-1}(t), j \in \mathcal{J}$ for (5). These predictors will be later considered in the design of the individual inputs $u_j(t), j \in \mathcal{J}$ acting on the system with different delays.

Assume for now the existence of a set of a-priori known controllers:

$$u_j(t) = q_j(\mathbf{z}_j(t)), \qquad j \in \mathcal{J} \tag{8}$$

where $q_j : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function for every $j \in \mathcal{J}$. As will be later shown, these feedback controllers are defined by (4) with respect to the predicted state at $\kappa_j^{-1}(t), j \in \mathcal{J}$, denoted by $\mathbf{z}_j(t)$. Following a similar procedure to Bekiaris-Liberis and Krstić (2017) and considering the time-varying nature of the input delays (Krstić (2010)), we may define the future state predictors as follows:

Proposition 6. Consider the system dynamics (5) with input delay functions $\kappa_j(t), j \in \mathcal{J}, t \geq 0$ satisfying (6). Let Assumption 4 hold and assume the existence of locally Lipschitz functions $q_j : \mathbb{R}^n \to \mathbb{R}, j \in \mathcal{J}$. Let $\phi_{j,j'}(t) = \kappa_j(\kappa_{j'}^{-1}(t))$, for any $j, j' \in \mathcal{J}$ and define $\mathbf{z}_j(t) = \mathbf{x}(\kappa_j^{-1}(t)), j \in \mathcal{J}$. Then, the functions $\mathbf{z}_j(t), j \in \mathcal{J}$ satisfy the following equations:

$$\mathbf{z}_1(t) = \mathbf{x}(t) + \int_{\kappa_1(t)}^{t} \frac{1}{\kappa_1'(\kappa_1^{-1}(s))}\Big(f(\mathbf{z}_1(s)) + \sum_{j=1}^{m} g_j(\mathbf{z}_1(s))u_j(\phi_{j,1}(s))\Big)ds, \tag{9a}$$

$$\mathbf{z}_{k+1}(t) = \mathbf{z}_k(t) + \int_{\phi_{k+1,k}(t)}^{t} \frac{1}{\kappa_{k+1}'(\kappa_{k+1}^{-1}(s))}\Big(f(\mathbf{z}_{k+1}(s)) + \sum_{j=1}^{k} g_j(\mathbf{z}_{k+1}(s))q_j(\mathbf{z}_{k+1}(s)) + \sum_{j=k+1}^{m} g_j(\mathbf{z}_{k+1}(s))u_j(\phi_{j,k+1}(s))\Big)ds, \; k \in \mathcal{J}\backslash\{m\} \tag{9b}$$

with initial conditions:

$$\mathbf{z}_1(\theta) = \mathbf{x}(0) + \int_{\kappa_1(0)}^{\theta} \frac{1}{\kappa_1'(\kappa_1^{-1}(s))}\Big(f(\mathbf{z}_1(s)) + \sum_{r=1}^{m} g_r(\mathbf{z}_1(s))u_r(\phi_{r,1}(s))\Big)ds, \quad \text{for} \quad \kappa_1(0) \leq \theta \leq 0 \tag{10a}$$

$$\mathbf{z}_j(\theta) = \mathbf{z}_{j-1}(0) + \int_{\phi_{j,j-1}(0)}^{\theta} \frac{1}{\kappa_j'(\kappa_j^{-1}(s))}\Big(f(\mathbf{z}_j(s)) + \sum_{r=1}^{j-1} g_r(\mathbf{z}_j(s))q_r(\mathbf{z}_j(s)) + \sum_{r=j}^{m} g_r(\mathbf{z}_j(s))u_r(\phi_{r,j}(s))\Big)ds,$$
$$\text{for} \quad \phi_{j,j-1}(0) \leq \theta \leq 0, \; j \in \mathcal{J}\backslash\{1\}. \tag{10b}$$

Proof. The validity of our claim will be shown by induction. We will first show the design of the predictors at times $\kappa_j^{-1}(t), j = 1, 2$ and then we will generalize to any predictor $\mathbf{x}(\kappa_j^{-1}(t)), j \in \{3, \ldots, m\}$. In the following we use the fact that $\frac{d\kappa_j^{-1}(t)}{dt} = \frac{1}{\kappa_j'(\kappa_j^{-1}(t))}$. Observe that the division with $\kappa_j'(\kappa_j^{-1}(t))$ is well defined for every t due to (7c) of Assumption 4. Let $s = \kappa_1^{-1}(t)$ for all $t \geq \kappa_1(0)$. Hence, $\mathbf{z}_1(t) = \mathbf{x}(\kappa_1^{-1}(t)) = \mathbf{x}(s)$. Then, it holds:

$$\begin{aligned}\frac{d\mathbf{z}_1(t)}{dt} &= \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \big(f(\mathbf{x}(s)) + \sum_{j=1}^{m} g_j(\mathbf{x}(s))u_j(\kappa_j(s))\big)\frac{ds}{dt} \\ &= \big(f(\mathbf{z}_1(t)) + \sum_{j=1}^{m} g_j(\mathbf{z}_1(t))u_j(\phi_{j,1}(t))\big)\frac{d\kappa_1^{-1}(t)}{dt} \\ &= \frac{1}{\kappa_1'(\kappa_1^{-1}(t))}\big(f(\mathbf{z}_1(t)) + \sum_{j=1}^{m} g_j(\mathbf{z}_1(t))u_j(\phi_{j,1}(t))\big).\end{aligned}$$

Integrating the above equation from $\kappa_1(t)$ to t and from $\kappa_1(0)$ to $\theta \leq 0$ we get (9a) and (10a), respectively. Next, let $\tau = \phi_{1,2}(t)$ for every $t \geq \phi_{2,1}(0)$. Observe that $\mathbf{z}_2(t) = \mathbf{x}(\kappa_2^{-1}(t)) = \mathbf{z}_1(\tau)$ for all $t \geq \phi_{2,1}(0)$. Then, we have:

$$\frac{d\mathbf{z}_2(t)}{dt} = \big(f(\mathbf{z}_2(t)) + g_1(\mathbf{z}_2(t))q_1(\mathbf{z}_2(t)) + \sum_{j=2}^{m} g_j(\mathbf{z}_2(t))u_j(\phi_{j,2}(t))\big)\frac{d\kappa_2^{-1}(t)}{dt},$$

where we considered that $u_1(\phi_{1,2}(t)) = q_1(\mathbf{z}_1(\phi_{1,2}(t))) = q_1(\mathbf{z}_2(t))$ and the fact that $\mathbf{z}_2(t) = \mathbf{z}_1(\phi_{1,2}(t))$. Integrating from $\phi_{2,1}(t)$ to t and $\phi_{2,1}(0)$ to $\theta \leq 0$ we get (9b) and (10b) for $k = 1$ and $j = 2$ respectively. For any $k \geq 2$, denote by $\mathbf{z}_k(t)$ the predictor of $\mathbf{x}$ at $\kappa_k^{-1}(t)$. Then, $\mathbf{z}_k(t)$ evolves according to the following differential equation:

$$\frac{d\mathbf{z}_k(t)}{dt} = \big(f(\mathbf{z}_k(t)) + \sum_{j=1}^{k-1} g_j(\mathbf{z}_k(t))q_j(\mathbf{z}_k(t)) + \\ + \sum_{j=k}^{m} g_j(\mathbf{z}_k(t))u_j(\phi_{j,k}(t))\big)\frac{d\kappa_k^{-1}(t)}{dt}.$$

Define $w = \phi_{k,k+1}(t)$. Then, $\mathbf{z}_{k+1}(t) = \mathbf{z}_k(w) = \mathbf{x}(\kappa_{k+1}^{-1}(t))$, for every $t \geq \phi_{k+1,k}(0)$. Using this definition, for every $t \geq \phi_{k+1,k}(0)$, we have:

$$\frac{d\mathbf{z}_{k+1}(t)}{dt} = \big(f(\mathbf{z}_{k+1}(t)) + \sum_{j=1}^{k} g_j(\mathbf{z}_{k+1}(t))q_j(\mathbf{z}_{k+1}(t)) + \\ + \sum_{j=k+1}^{m} g_j(\mathbf{z}_{k+1}(t))u_j(\phi_{j,k+1}(t))\big)\frac{d\kappa_{k+1}^{-1}(t)}{dt}, \tag{11}$$

where as in the case with $k = 1$ we considered that $u_j(\phi_{k,k+1}(t)) = q_j(\mathbf{z}_{k+1}(t)), j \in \{1, \ldots, k\}, k \leq m-1$. Integrating from $\phi_{k+1,k}(t)$ to t we derive (9b). Additionally, integrating (11) from $\phi_{k+1,k}(0)$ to $\theta \leq 0$, we get (10b) for $j = k$. Since $\mathbf{z}_{k+1}(t)$ satisfies (9b) for every $k \in \mathcal{J}\backslash\{m\}$, we can deduce the result.

Throughout this work, we will assume that the predictors defined in (9a)-(9b), (10a)-(10b) are computed analytically and perfectly estimate the systems' future state, i.e., $\mathbf{z}_j(t) = \mathbf{x}(\kappa_j^{-1}(t)), j \in \mathcal{J}$. For a numerical approximation of the predictors, the reader may refer to Karafyllis (2011); Karafyllis and Krstić (2017).

Let's consider (5) and assume for a moment that every input acts at the system with the same delay. Assume without loss of generality this delay to be $\kappa_1(t)$. Observe that no input acts at the system in the time interval $[0, \kappa_1^{-1}(0)]$. However, for any $t \geq 0$, safety can be enforced by means of the following constraint:

$$\frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{x}(\kappa_1^{-1}(t))) + \sum_{j\in\mathcal{J}} g_j(\mathbf{x}(\kappa_1^{-1}(t)))u_j(t)) \\ \geq -\alpha(h(\mathbf{x}(\kappa_1^{-1}(t)))), \tag{12}$$

where $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is an extended class $\mathcal{K}$ function. By Proposition 6, $\mathbf{x}(\kappa_1^{-1}(t)) = \mathbf{z}_1(t)$. Based on that, (12) can be written as follows:

$$\frac{\partial h}{\partial \mathbf{x}}(f(\mathbf{z}_1(t)) + \sum_{j\in\mathcal{J}} g_j(\mathbf{z}_1(t))u_j(t)) \geq -\alpha(h(\mathbf{z}_1(t))).$$

This constraint resembles (3a) with $\mathbf{z}_1(t)$ instead of $\mathbf{x}(t)$. Hence, invoking the solution of (3), where $\mathbf{u}_{nom} : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz, nominal controller, we may obtain the optimal control inputs $u_j^*(t) = q_j(\mathbf{z}_1(t)), j \in \mathcal{J}$, according to (4). When multiple input delays are considered, it is not straightforward how (12) can be modified to account for the different future states $\mathbf{x}(\kappa_j^{-1}(t)), j \in \mathcal{J}$. Motivated by Abel et al. (2019), we propose solving (3) sequentially for every $j \in \mathcal{J}$, assuming that at the j-th iteration the inputs act at the system with the same delay $\kappa_j(t)$. Then, omitting the remaining part of the solution of the j-th iteration, we apply only $q_j(\mathbf{z}_j(t))$ to (5). As a result, using the state-predictors of Proposition 6, we may define $u_j^*(t), j \in \mathcal{J}$ as follows:

$$u_j^*(t) = q_j(\mathbf{z}_j(t)), \quad j \in \mathcal{J}. \tag{13}$$

Due to the existence of the time-varying delays, as in the case of a single delay, discussed above, the system (5) remains uncontrolled for the first $\kappa_1^{-1}(0)$ time units. Therefore, in order to ensure the safety of the system at all times, considering the multiple time-varying delays of (5), we pose the following assumption:

Assumption 7. Consider the system (5) with input delay functions $\kappa_j(t), j \in \mathcal{J}, t \geq 0$ satisfying (6) and let Assumption 4 hold. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF function for (1). Then, the initial conditions $\mathbf{x}(0)$ and $u_j(t), t \in [\kappa_j(0), \phi_{j,1}(0)), j \in \mathcal{J}$ of (5) ensure that $h(\mathbf{x}(t)) > 0$ holds, for every $t \in [0, \kappa_1^{-1}(0)]$.

Assumption 7 ensures that the system remains safe at the first $\kappa_1^{-1}(0)$ time units, when no control is applied to the system. After $\kappa_1^{-1}(0)$ time units, the control inputs $u_j^*(t)$, defined in (13), are applied to the system from $\kappa_j^{-1}(0)$ onwards. As a result, over the time intervals $[\kappa_j^{-1}(0), \kappa_{j+1}^{-1}(0)]$ the system is controlled only with the first j control inputs $u_{j'}$ respectively, i.e., $j' \in \mathcal{J}\backslash\{j+1, \ldots, m\}$. As these controllers are pre-computed without information on the past inputs acting at the system at each time t, it is possible for the system to be steered outside the safety set $\mathcal{C}$. Therefore, in order to ensure the safety of the system at the intervals $[\kappa_j^{-1}(0), \kappa_{j+1}^{-1}(0)], j \in \mathcal{J}\backslash\{m\}$ we pose the following assumption:

Assumption 8. Consider the system (5) with input delay functions $\kappa_j(t), j \in \mathcal{J}, t \geq 0$ satisfying (6) and let Assumptions 4 and 7 hold. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF function for (1). Consider further the locally Lipschitz functions $q_j : \mathbb{R}^n \to \mathbb{R}$, defined in (13), where $\mathbf{u}_{nom} : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz, offline designed, nominal controller. Then, $\mathbf{x}(t) \in \mathcal{C}$, for every $t \in [\kappa_j^{-1}(0), \kappa_{j+1}^{-1}(0)], j \in \mathcal{J}\backslash\{m\}$, when $\mathbf{x} ::= \mathbf{x}(t)$ evolves according to:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \sum_{r=1}^{j} g_r(\mathbf{x})q_r(\mathbf{x}) + \sum_{r=j+1}^{m} g_r(\mathbf{x})u_r(\kappa_r(t)).$$

Assumption 8 ensures that the system controlled by the control inputs $q_{j'}(\mathbf{z}_{j'}(t)), j' \in \mathcal{J}\backslash\{j+1, \ldots, m\}$, defined in (13), remains always safe in the time interval $[\kappa_j^{-1}(0), \kappa_{j+1}^{-1}(0)]$, i.e., the state evolves inside $\mathcal{C}$. When $\kappa_j(t) = t - \tau_j$, for every $j \in \mathcal{J}$, where $\tau_j > 0$ is a known, constant delay, then Assumption 8 recovers the assumption considered in (Abel et al., 2019, Prop. 2). A similar assumption to Assumption 8 has also been considered in Bekiaris-Liberis and Krstić (2017) in the context of stabilization, where the system is assumed to not exhibit finite escape time when controlled with the first $1, \ldots, j$ inputs, where $j \in \mathcal{J}$. Having ensured initial safety (assumption 7) and safety of the partially controlled system (assumption 8), we may ensure the safety of (5) at all times as follows:

Theorem 9. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and assume that $h(\mathbf{x})$ is a CBF function for (1). Consider further the system (5) with input delay functions $\kappa_j(t), j \in \mathcal{J}, t \geq 0$ satisfying (6) and let Assumptions 4,7 and 8 hold. Then, the control inputs defined by (13) for every $j \in \mathcal{J}$ render $\mathcal{C}$ forward invariant for (5).

Proof. By Assumptions 7-8 the safety of the system is ensured at the intervals $[0, \kappa_1^{-1}(0)]$ and $[\kappa_j^{-1}(0), \kappa_{j+1}^{-1}(0)], j \in$

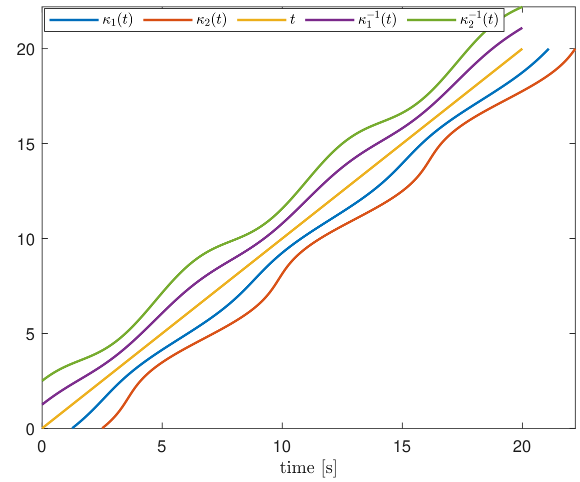


Fig. 1. The delay functions $\kappa_j(t), j \in \mathcal{J}$ and their corresponding inverse functions $\kappa_j^{-1}(t)$.

$\mathcal{J} \backslash \{m\}$. Hence, it remains to be shown that $\mathbf{x}(t) \in \mathcal{C}$, for every $t \geq \kappa_m^{-1}(0)$. Observe that for every $t \geq \kappa_m^{-1}(0)$, all inputs are acting to the system, hence, under the assumption that the state-predictors defined in (9a)-(9b) accurately estimate the future states $\mathbf{x}(\kappa_j^{-1}(t))$, we may conclude that the system acts identically to (1) under the control inputs $q_j(\mathbf{x}(t)), j \in \mathcal{J}$, and thus for every $t \geq \kappa_m^{-1}(0)$ the following holds:

$$\frac{d}{dt}(h(x(t)) = \frac{\partial h}{\partial \mathbf{x}}\left(f(\mathbf{x}(t)) + \sum_{j \in \mathcal{J}} g_j(\mathbf{x}(t))q_j(\mathbf{x}(t))\right).$$

By design of the control inputs $q_j(\mathbf{x}(t)), j \in \mathcal{J}$, found as the optimal solution to (3), it holds that:

$$\frac{d}{dt}(h(x(t))) \geq -\alpha(h(\mathbf{x}(t))), \quad \mathbf{x}(t) \in \mathcal{D},\ t \geq \kappa_m^{-1}(0).$$

Note that $\mathbf{x}(\kappa_m^{-1}(0)) \in \mathcal{C}$ by Assumption 8. Hence, by (Khalil, 1996, Ch 4.4), when invoking the Comparison Lemma (Khalil, 1996, Ch. 3.4), it holds that $h(\mathbf{x}(t)) \geq \beta(|h(\mathbf{x}(\kappa_m^{-1}(0)))|, t) \geq 0$, for any $t \geq \kappa_m^{-1}(0)$, where $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function. Thus, $\mathbf{x}(t) \in \mathcal{C}$, for every $t \geq \kappa_m^{-1}(0)$. Considering the latter result in addition to Assumptions 4,7-8, the result follows.

5. NUMERICAL EXAMPLE

Consider the two-dimensional system:

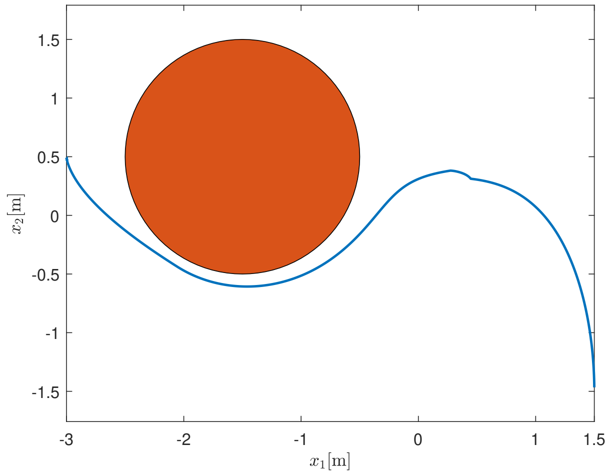
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -x_1(t) - x_2(t) \\ x_1^3(t) \end{bmatrix} + \begin{bmatrix} u_1(\kappa_1(t)) \\ u_2(\kappa_2(t)) \end{bmatrix},$$

where $f(\mathbf{x})$ is obtained by Abel et al. (2020) and $\kappa_j(t), j \in \mathcal{J} = \{1, 2\}$ are time-varying delay functions whose inverse functions are defined as follows:

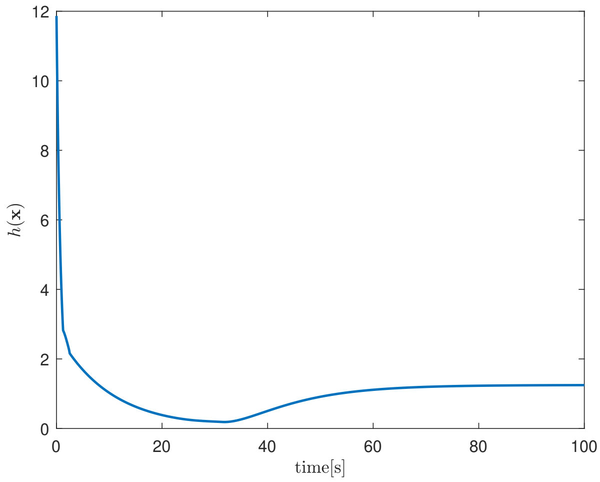
$$\kappa_1^{-1}(t) = t + 1 + \frac{1}{4}\cos t,$$

$$\kappa_2^{-1}(t) = t + 2 + \frac{1}{2}\cos t.$$

The delay functions $\kappa_j(t), j \in \mathcal{J}$ and their inverse functions are shown in Figure 1. The initial conditions of the system are chosen to be $u_j(t) = 0$, for every $t \in [\kappa_j(0), 0), j \in \mathcal{J}$ and $\mathbf{x}(0) = [1.5\ -1.5]^T$. Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:



(a) Evolution of the system's states



(b) Barrier Function Evolution

Fig. 2. Evolution of (5) and $h(\mathbf{x}(t))$ over time with the proposed control approach.

$$h(\mathbf{x}) = \|\mathbf{x} - \mathbf{c}\|_2^2 - 1,$$

where $\mathbf{c} = [-1.5\ 0.5]^T$. Observe that $\frac{\partial h}{\partial \mathbf{x}} = 2(\mathbf{x} - \mathbf{c})^T$ and $g(\mathbf{x}) = I_2$ is of full rank for every $\mathbf{x} \in \mathcal{D}$, thus $\frac{\partial h}{\partial \mathbf{x}} g(\mathbf{x}) = 0$, if and only if $\mathbf{x} = \mathbf{c}$. If $\mathcal{D} \subset \mathbb{R}^2$ is chosen such that $\mathbf{c} \notin \mathcal{D}$, then $h(\mathbf{x})$ is a CBF for the system without delays.

Here, we consider a linear class $\mathcal{K}$ function $\alpha : \xi \mapsto 0.1\xi$ and a nominal controller $\mathbf{u}_{nom} : \mathcal{D} \to \mathbb{R}^2$ that aims at steering the system to $\mathbf{p} = [-3\ 0.5]^T$ and is defined as follows:

$$\mathbf{u}_{nom}(\mathbf{x}) = -f(\mathbf{x}) - \begin{bmatrix} 0.1 & 0 \\ 0 & 0.07 \end{bmatrix}(\mathbf{x} - \mathbf{p}).$$

In Figure 2a the evolution of the system between 0 and 100 sec is shown. In the first $\kappa_1^{-1}(0) = 1.25$ sec, when the system is uncontrolled, the system moves towards violating the safety constraint. Therefore, as shown in Figure 2b, the decrease of $h(\mathbf{x}(t))$ for $t \in [0, 1.25]$ is significantly steep. Between 1.25 and 2.5 sec, where $\kappa_2^{-1}(0) = 2.5$, the control input affecting the x_1 coordinate starts acting at the system, slowing down the evolution of the system towards the unsafe region. The effect of $u_1(t)$ on the system is noticeable in Figure 2b, where the rate of decrease of $h(\mathbf{x}(t))$ with respect to t becomes smaller when $t \in [1.25, 2.5]$. After 2.5 sec, the system becomes fully controlled and starts moving towards the stabilization point while staying away from the unsafe area. As a result, the barrier function $h(\mathbf{x})$ remains positive at all times and $h(\mathbf{x}(t)) \geq 0.1866$, for every $t \in [0, 100]$.

6. CONCLUSIONS

In this work a safety controller is designed for a system with multiple, time-varying input delays. Under a monotonicity assumption on the delays functions, we design a set of state-predictors for the system utilizing the safety controller of the corresponding system without delays. Then, under the assumption of perfect estimation of the future states, we show that the systems remain safe by the time the largest delay acting on the system is compensated for the first time. Future work, will consider ways to relax the monotonicity assumption imposed on the delay functions of the system and extend the current framework to ensure forward invariance of time-varying safety sets.

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