Simultaneous Topology Identification and Synchronization of Directed Dynamical Networks

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***Abstract*— We propose an approach for simultaneous topology identification and synchronization of a complex dynamical network with directed interconnections that relies on the edge-agreement framework and on adaptive-control approaches by design of an auxiliary synchronizing network. Our method guarantees the identification of the unknown directed topology without the need for verifying the Linear Independence Conditions normally required by previous works in the literature. Furthermore, it also guarantees that the complex network reaches synchronization as determined by the internal dynamics of the system. Under our identification algorithm we provide strong stability results for the estimation errors in the form of uniform semiglobal practical asymptotic stability of the estimation errors. Finally, we demonstrate the effectiveness of our approach with a numerical example.**

***Index Terms*—Complex networks, topology identification, synchronization, multi-agent systems.**

I. Introduction

Complex dynamical networks are used to represent a great number of real systems including social networks, biological networks, multi-robot systems, power grids, to name a few [1], [2]. The evolution of many of such complex dynamical networks may be modeled by the differential equation [2]

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} c_{ij} h\left(x_j\right), \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state of the sub-system i, $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, $h : \mathbb{R}^n \to \mathbb{R}^n$ is a linear or nonlinear coupling function, and the coefficients c_{ij} describe the topology of the network in which $c_{ij} \neq 0$ if sub-system i directly interacts with sub-system j, and $c_{ij} = 0$ otherwise.

In the last few years, the study of complex networks has mainly focused on the problem of synchronization of systems modeled by (1) [3]–[6], either for control design or analysis. Synchronization means that all the sub-systems ultimately follow the same trajectory, up to a certain "shift". The achievement of synchronization greatly depends of the interaction topology of the network, which is usually assumed either to be known or to have some known properties in terms of connectivity, persistency, etc [2]. However, in many situations the topology of the complex network is not known, therefore the network is not directly accessible to control or analysis.

Aiming to overcome this barrier, many studies have focused on the problem of identifying the network topology. Some of the existing studies use a data-based approach to identify the underlying graph using optimization-based approaches—see e.g. [7]–[10]. Others, rely on synchronization and adaptive control in order to estimate the unknown topology [11]–[18]. In this paper we rely on the latter.

In many existing works in the literature, e.g. [13]–[17], the unknown topology of the network is identified by synchronizing the network of interest with an auxiliary network, or with multiple auxiliary networks [18], with *known* topology, which is described by the differential equation

$$\dot{y}_i = f_i(y_i) - \sum_{j=1}^{N} \hat{c}_{ij} h\left(y_j\right) - d_i \zeta_i, \tag{2}$$

with updating laws $\dot{\hat{c}}_{ij} = -\zeta_i^\top h(y_j)$ and $\dot{d}_i = k_i |\zeta_i|^2$, where $\zeta_i := x_i - y_i$ and k_i is a positive constant. This auxiliary network can be either a virtual system explicitly designed for the identification purpose [12]–[15] or exist as a part of the system within a multi-layer topology [16]. In all of these works the successful identification of the unknown topology relies on the trajectories of the systems satisfying a *Linear Independence Condition*, stated as follows.

The functions $h(x_i)$, $i = 1, \dots, N$, are linearly independent if and only if there do not exist nonzero constants b_i, $i = 1, \dots, N$, such that $b_1 h(x_1) + \cdots + b_N h(x_N) = 0$. When all functions $h(x_i)$ satisfy the *linear independence condition* on the invariant synchronization manifold $\{\zeta_i = 0\}$, then $\hat{c}_{ij} \to c_{ij}$, i.e., the topology of the network of (1) is successfully identified—cf. [13]. Roughly speaking, this condition implies that the trajectories of the system must be "chaotic enough" so that synchronization is avoided and the topology is correctly estimated. Indeed, it is shown in [13] that synchronization poses an obstacle to the topology identification since the information about the topology is lost as the agents synchronize. The Linear Independence Condition is however difficult to verify and reduces the class of systems to which the previous results can be applied.

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In light of this, a new framework for topology identification is developed in [19] where an external input is added to system (1). More precisely, the dynamics of each agent is now described by

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} c_{ij}\Gamma x_j + u_i, \tag{3}$$

where $\Gamma \in \mathbb{R}^{n\times n}$ is a linear coupling and $u_i \in \mathbb{R}^n$ is the external input given by

$$u_i = \dot{y}_i - f_i(y_i) - \sum_{j=1}^{N} \hat{c}_{ij}\Gamma x_j - d_i\zeta_i, \tag{4}$$

with updating laws $\dot{\hat{c}}_{ij} = -\zeta_i^\top \Gamma x_j$ and $\dot{d}_i = k_i|\zeta_i|^2$, k_i is a positive constant, and $\zeta_i := x_i - y_i$ where y_i are the states of *isolated* nodes in an auxiliary network, described by

$$y_i(t) = \nu g_i(t) \tag{5}$$

with $\nu \in \mathbb{R}^n$ satisfying $\Gamma\nu \neq 0$, and $g_i : \mathbb{R} \to \mathbb{R}$ satisfying the following assumption.

Assumption 1: There exists a positive constant, T satisfying $g_k(t) = g_k(t+T)$, $1 \leq k \leq N$. Moreover, there exists a sequence of time points $t_1^{(p)}, \ldots, t_p^{(p)} \in [0, T)$ satisfying

$$\sum_{l=1}^{p} g_k(t_l^{(p)}) = \begin{cases} 0, & 1 \leq k < p \\ \delta_p \neq 0, & k = p \end{cases} \tag{6}$$

for $p = 1, \ldots, N$.

Under Assumption 1, using (4), the authors in [19] show that the topology is successfully recovered by the estimates $\hat{c}_{ij}$, without the need of verifying the Linear Independence Condition. Indeed, the input (4) consists basically in a tracking law making the nodes of the original network x_i track the auxiliary nodes y_i, which are described by the time-varying functions (5) satisfying Assumption 1. This makes the multi-agent system avoid synchronization, effectively overcoming the obstacle raised in [13] and guarantees the successful identification of the unknown topology without the need for verifying the Linear Independence Condition.

Although the approach above successfully estimates the unknown topology, it makes the system behave “erratically” and avoid synchronization, even after the topology has been successfully identified. This may be hindering in most applications of dynamical networks. On the one hand, as it was mentioned above one, knowledge of the network topology is required for controlling the network in order to achieve a desired behavior via, e.g., leader-follower or pinning control. On the other hand, however, synchronization constitutes the basis of coordination protocols and, in many cases, the internal dynamics of complex multi-agent systems and so it is desirable to preserve (or achieve) and keep stable such synchronized behavior. Then, using the approaches mentioned above, controlling a dynamical network with unknown topology is tantamount to accomplishing two conflicting goals. Therefore, in order to overcome this impasse, in this paper we propose a novel approach for simultaneous topology identification and synchronization of dynamical networks.

A. Contributions of this paper

The contributions of this paper are twofold. On one hand, we present a new perspective for the study of the topology identification problem which relies on the edge-agreement framework [20]. On the other hand, we propose a new adaptive-control-based topology-identification algorithm based on the concept of δ-persistency of excitation [21].

More precisely, the study of the identification problem using the edge-agreement framework allow us to write the system in a form usually found in the literature of adaptive control. This makes the control-design process as well as the analysis of the closed-loop system more intuitive as we can directly rely on well-known adaptive-control approaches. Indeed, based on this representation we propose a new algorithm where the dynamics of an auxiliary network are designed to be δ-persistently exciting. The latter, in contrast to [19], allows us to identify the unknown topology while guaranteeing that the synchronization behavior of the system is achieved (preserved) and despite this fact. This makes our approach more suited to synchronization-based applications requiring *simultaneous* identification and control, as would an observer-based feedback-control approach for general dynamical systems be.

Our result distinguishes itself from the the previous works in that, by leveraging the properties and advantages of the edge-agreement representation, we are able to provide strong stability results for the identification errors in terms of *uniform semiglobal practical asymptotic stability*. Such characterization of the error trajectories is not only stronger than the *convergence* results generally proposed in the works of the literature discussed above, but we believe that it is also an important property that may allow to apply the present results to more complex systems and applications.

The rest of this paper is organized as follows. In Section II are presented some preliminaries and the problem formulation. The topology identification methodology is presented in Section III. Finally, some numerical examples are presented in Section IV and concluding remarks are included in Section V.

II. Preliminaries

A. Notations

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is decreasing and $\sigma(s) \to 0$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot) \in \mathcal{L}$ for any $s \in \mathbb{R}_{\geq 0}$. We use $|\cdot|$ for the Euclidean norm of vectors and the induced L_2 norm of matrices. The set $\mathbb{B}(\Delta) \subset \mathbb{R}^n$ is the closed ball of radius Δ centered at the origin, i.e. $\mathbb{B}(\Delta) := \{x \in \mathbb{R}^n \ : \ |x| \leq \Delta\}$. The notation $\mathbb{H}(\delta, \Delta) := \{x \in \mathbb{R}^n \ : \ \delta \leq |x| \leq \Delta\}$. We define $|x|_\delta := \inf_{y \in \mathbb{B}(\delta)} |x - y|$.

We use $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ to denote a weighted graph defined by a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ with cardinality N and corresponding to the agents' states, an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ with cardinality M and characterizing the information exchange between agents, that is, an edge $e_k := (i, j) \in \mathcal{E}$, $k =$

$\{1,\ldots,M\}$, is an ordered pair indicating that agent j has access to information from node i, and a positive diagonal matrix $W\in\mathbb{R}^{M\times M}$, whose diagonal w_k entries represent the weights of the edges. A graph is said to be *undirected* if $(i,j)\in\mathcal{E} \implies (j,i)\in\mathcal{E}$. An *orientation* of an undirected graph is the assignment of directions to its edges. An undirected graph is said to be *connected* if there is an undirected path between every pair of distinct nodes. A *tree* is a subgraph in which every node has exactly one parent except for one node, called the root, which has no parent and which has a path to every other node. A *spanning tree* is a tree subgraph containing all nodes in $\mathcal{V}$. A graph is said to be *complete* if there exists an edge between every pair of agents.

B. Model and problem formulation

We consider a multi-agent system where the agents interact over an *unknown* topology described by a *directed* graph $\mathcal{G}(\mathcal{V},\mathcal{E},W)$ containing a directed spanning tree. Furthermore consider that each agent's dynamics is governed by the law

$$\dot{x}_i = f_i(x_i) - c\sum_{j=1}^{N} w_{ij}\left[x_i - x_j\right] + u_i \quad i\in\mathcal{V}, \tag{7}$$

where $x_i\in\mathbb{R}^1$ is the state of agent i, $f_i:\mathbb{R}\to\mathbb{R}$ is a smooth function, $w_{ij}\equiv w_k$ is the *unknown* weight of the interconnection between agents i and j such that $w_{ij}=0$ if the edge $e_k\notin\mathcal{E}$ and $w_{ij}\neq 0$ if the edge $e_k\in\mathcal{E}$. Also, we assume the following.

Assumption 2: For each $i\le N$, there exists a positive constant L_i such that

$$|f_i(x)-f_i(y)|\le L_i|x-y| \tag{8}$$

for all $x,\, y\in\mathbb{R}$.

Let $E\in\mathbb{R}^{N\times M}$ denote the (unknown) incidence matrix of $\mathcal{G}$. This is a matrix with rows indexed by the nodes and columns indexed by the edges with its (i,k)th entry defined as follows: $[E]_{ik}:=-1$ if i is the terminal node of edge e_k, $[E]_{ik}:=1$ if i is the initial node of edge e_k, and $[E]_{ik}:=0$ otherwise. Similarly, let $E_\odot\in\mathbb{R}^{N\times M}$ denote the (unknown) in-incidence matrix of $\mathcal{G}$, defined as follows: $[E_\odot]_{ik}:=-1$ if i is the terminal node of edge e_k and $[E_\odot]_{ik}:=0$ otherwise. Then, defining $x^\top:=[x_1\,\ldots\,x_N]^\top$, $F(x)^\top:=[f_1(x_1)\,\ldots\,f_N(x_N)]^\top$, and $u^\top:=[u_1\,\ldots\,u_N]^\top$, (1) may be written in compact form as

$$\dot{x}=F(x)-cE_\odot W E^\top x+u. \tag{9}$$

Now, denote by $\bar{E}$ and $\bar{E}_\odot$, respectively, the incidence and in-incidence matrices of a *complete* graph $\mathcal{K}(\mathcal{V},\mathcal{E}_c,\bar{W})$, where $\mathcal{E}\subseteq\mathcal{E}_c$, $|\mathcal{E}_c|=N(N-1)=:\bar{M}$, and let $\bar{W}:=\operatorname{diag}\{\bar{w}_k\}$ where $\bar{w}_k\equiv w_k$ if $\bar{e}_k\in\mathcal{E}$ and $\bar{w}_k=0$ if $\bar{e}_k\in\mathcal{E}_c\backslash\mathcal{E}$. That is, the weight $\bar{w}_k$ is different from 0 if the edge $\bar{e}_k$ of the complete graph exists in the graph $\mathcal{G}$ to be identified. Therefore, akin to (9), we have

$$\dot{x}=F(x)-c\bar{E}_\odot\bar{W}\bar{E}^\top x+u. \tag{10}$$

[1]For simplicity of notation we consider agents in $\mathbb{R}$, however, using the Kronecker product, the results in this paper apply to agents evolving in $\mathbb{R}^n$.

In this paper, we study the networked system using the edge-agreement framework [20] in which we consider the state of the edges of the graph rather than that of the nodes. That is, we study the evolution the edge variable $z:=\bar{E}^\top x$. Then, using (10) we obtain

$$\dot{z}=\bar{E}^\top F(x)-c\bar{E}^\top\bar{E}_\odot\bar{W}z+\bar{E}^\top u. \tag{11}$$

One property the edge-based representation for networked systems is that it is possible to obtain an equivalent reduced system in terms of an arbitrary directed spanning tree. Indeed, using an appropriate labeling of the edges, the incidence matrix of the complete graph $\mathcal{K}$ may be expressed as

$$\bar{E}=\left[\ \bar{E}_\mathcal{T}\quad \bar{E}_\mathcal{C}\ \right] \tag{12}$$

where $\bar{E}_\mathcal{T}\in\mathbb{R}^{N\times(N-1)}$ denotes the full-column-rank incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_\mathcal{T}\subset\mathcal{K}$ and $\bar{E}_\mathcal{C}\in\mathbb{R}^{N\times(\bar{M}-N+1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_\mathcal{T}$. Moreover, defining

$$R:=\left[\ I_{N-1}\quad T\ \right],\quad T:=\left(\bar{E}_\mathcal{T}^\top\bar{E}_\mathcal{T}\right)^{-1}\bar{E}_\mathcal{T}^\top\bar{E}_\mathcal{C}, \tag{13}$$

with I_{N-1} denoting the $N-1$ identity matrix, one obtains an alternative representation of the incidence matrix given by

$$\bar{E}=\bar{E}_\mathcal{T}R. \tag{14}$$

Similarly, the edge state may be divided as

$$z=\left[z_\mathcal{T}^\top\ \ z_\mathcal{C}^\top\right]^\top, \tag{15}$$

where $z_\mathcal{T}\in\mathbb{R}^{(N-1)}$ are the states of the edges corresponding to the spanning tree $\mathcal{G}_\mathcal{T}$ and $z_\mathcal{C}\in\mathbb{R}^{\bar{M}-N+1}$ are the states of the remaining edges. Moreover, using the identity (14) we have that

$$z=R^\top z_\mathcal{T}. \tag{16}$$

Then, from (11), applying (14) and (16), we obtain a reduced-order model

$$\dot{z}_\mathcal{T}=\bar{E}_\mathcal{T}^\top F(x)-c\bar{E}_\mathcal{T}^\top\bar{E}_\odot\bar{W}R^\top z_\mathcal{T}+\bar{E}_\mathcal{T}^\top u. \tag{17}$$

Note that, since $\bar{E}$ and $\bar{E}_\odot$ are the incidence and in-incidence matrices of the complete graph on N nodes, they are *known*. Therefore, using the notation in (17), the topology identification problem reduces to estimating the diagonal entries of matrix $\bar{W}$, i.e. the edge weights $\bar{w}_k$. On the other hand, the synchronization problem is transformed into the stabilization of the origin for the reduced-order system (17). Indeed, from (16) $z_\mathcal{T}\to 0$ implies $z\to 0$, and the latter is equivalent to $x_i-x_j\to 0$ for all $i,j\in\mathcal{V}$.

III. Simultaneous Topology Identification and Synchronization

A. Control design

In this section we design the external input u in order to simultaneously identify the unknown graph topology $\mathcal{G}$ by estimating the edge weights $\bar{w}_k$, $k\le\bar{M}$ and synchronize the dynamical systems (7).

For that purpose, let us define $\bar{w}^\top := [\bar{w}_1 \ \cdots \ \bar{w}_{\bar{M}}] \in \mathbb{R}^{\bar{M}}$ as the vector of unknown weights, $\hat{w}^\top := [\hat{w}_1 \ \cdots \ \hat{w}_{\bar{M}}] \in \mathbb{R}^{\bar{M}}$ as the estimate of $\bar{w}$, and $\hat{W} := \text{diag}\{\hat{w}\}$. Then, the external input is set to

$$u = -c_1(x - \hat{x}(t)) + \dot{\hat{x}}(t) + c\bar{E}_\odot \hat{W}(t)\hat{z}(t) - F(\hat{x}(t)) \quad (18)$$

with the updating law

$$\dot{\hat{w}} = -c\hat{Z}(t)\bar{E}_\odot^\top \tilde{x} - c\hat{Z}(t)\bar{E}_\odot^\top \bar{E}\tilde{z}, \quad (19)$$

where $c_1 > 2L_f$ is a positive constant, $\tilde{z} := z - \hat{z}(t) = \bar{E}^\top(x - \hat{x}(t))$, $\hat{z}(t) := \bar{E}^\top \hat{x}(t)$, $\hat{Z}(t) := \text{diag}\{\hat{z}(t)\}$, and $\hat{x}(t)$ is an auxiliary variable to be designed later.

Now, defining $\tilde{w} := \bar{w} - \hat{w}$, from (10), (11), (18), and (19), and using the identities (14) and (16) we have that the closed-loop system is given by

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}}_\mathcal{T} \\ \dot{\tilde{w}} \end{bmatrix} = \begin{bmatrix} -c_1 I - c\bar{L} & 0 & -c\bar{E}_\odot \hat{Z}(t) \\ 0 & -c_1 I - c\bar{L}_e & -c\bar{E}_\mathcal{T}^\top \bar{E}_\odot \hat{Z}(t) \\ c\hat{Z}(t)\bar{E}_\odot^\top & c\hat{Z}(t)\bar{E}_\odot^\top \bar{E}_\mathcal{T} RR^\top & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{z}_\mathcal{T} \\ \tilde{w} \end{bmatrix} + \begin{bmatrix} \tilde{F}(x, \hat{x}) \\ \bar{E}_\mathcal{T}^\top \tilde{F}(x, \hat{x}) \\ 0 \end{bmatrix}, \quad (20)$$

where $\bar{L} := \bar{E}_\odot \bar{W} \bar{E}^\top$, $\bar{L}_e := \bar{E}_\mathcal{T}^\top \bar{E}_\odot \bar{W} R^\top$, and $\tilde{F}(x, \hat{x}) := F(x) - F(\hat{x}(t))$.

Note that in view of Assumption 2, the last term on the right-hand side of (20) satisfies

$$|\tilde{F}(x, \hat{x})| = |[F(x) - F(\hat{x}(t))]| \le L_f |\tilde{x}|, \quad (21)$$

where $L_f := \max_{i \in \mathcal{V}} \{L_i\}$.

Before presenting the result on simultaneous synchronization and identification in Sections III-C and III-D, we state the following intermediate result which relies on the signal $\hat{z}(t)$ being *persistently exciting*—see Appendix I for a definition.

B. Identification without synchronization

Proposition 1: Assume that the signal $\hat{z}(t)$ is bounded, globally Lipschitz and persistently exciting. Then, the topology of network (7) is identified by the estimate $\hat{w}$, that is, $\bar{w} = \lim_{t \to \infty} \hat{w}(t)$, applying input (18) and update law (19). □

Proof: Eq. (20) can be seen as a perturbed version of

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}}_\mathcal{T} \\ \dot{\tilde{w}} \end{bmatrix} = \begin{bmatrix} -c_1 I - c\bar{L} & 0 & -c\bar{E}_\odot \hat{Z}(t) \\ 0 & -c_1 I - c\bar{L}_e & -c\bar{E}_\mathcal{T}^\top \bar{E}_\odot \hat{Z}(t) \\ c\hat{Z}(t)\bar{E}_\odot^\top & c\hat{Z}(t)\bar{E}_\odot^\top \bar{E}_\mathcal{T} RR^\top & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{z}_\mathcal{T} \\ \tilde{w} \end{bmatrix}. \quad (22)$$

Systems in the form of (22), with $\hat{Z}(t)$ persistently exciting, have been studied extensively in adaptive control. First note that $\bar{L}$ is the Laplacian of a directed graph containing a spanning tree and therefore all of its eigenvalues have non-negative real parts. Similarly, $\bar{L}_e$ denotes the *essential edge Laplacian* [22], whose eigenvalues have positive real parts. From the latter $-c_1 I - c\bar{L}$ and $-c_1 I - c\bar{L}_e$ are Hurwitz. Now, consider the integral

$$\int_t^{t+T} \hat{Z}(\tau)\bar{E}_\odot^\top \bar{E}_\odot \hat{Z}(\tau) + \hat{Z}(\tau)\bar{E}_\odot^\top \bar{E}_\mathcal{T} RR^\top \bar{E}_\mathcal{T}^\top \bar{E}_\odot \hat{Z}(\tau) d\tau. \quad (23)$$

Since $\hat{Z}$ is a diagonal matrix and $\hat{z}$ denotes an edge representation of the auxiliary variables $\hat{x}$, akin to (15), the integral (23) can be equivalently expressed as

$$\int_t^{t+T} \hat{z}_\mathcal{T}(\tau)^\top \left[RR^\top + R\bar{E}_\odot^\top \bar{E}_\mathcal{T} RR^\top \bar{E}_\mathcal{T}^\top \bar{E}_\odot R^\top \right] \hat{z}_\mathcal{T}(\tau) d\tau \ge \int_t^{t+T} \hat{z}_\mathcal{T}(\tau)^\top \hat{z}_\mathcal{T}(\tau) d\tau \ge \mu > 0, \quad (24)$$

where we have used the facts that $RR^\top$ is symmetric positive definite with its smallest eigenvalue equal to 1, the smallest eigenvalue of $\bar{E}_\mathcal{T}^\top \bar{E}_\odot R^\top$, $\lambda_{min}(\bar{E}_\mathcal{T}^\top \bar{E}_\odot R^\top)$, has positive real part [22], and $\hat{z}(t)$ is persistently exciting by assumption. *Global exponential stability* of the origin for (22) follows from (24) and from standard results of adaptive control—see e.g. [23], [24].

Denote $\xi^\top := [\tilde{x}^\top \ \tilde{z}_\mathcal{T}^\top \ \tilde{w}^\top] \in \mathbb{R}^{\bar{M} + (2N-1)}$. From the global exponential stability of (22) and from converse Lyapunov theorems, there exists a Lyapunov function $V : \mathbb{R}^{\bar{M} + (2N-1)} \to \mathbb{R}_{\ge 0}$ such that

$$\beta_1 |\xi|^2 \le V(\xi) \le \beta_2 |\xi|^2 \quad (25)$$

$$\left| \frac{\partial V}{\partial \xi} \right| \le \beta_3 |\xi|, \quad (26)$$

with $\beta_1, \beta_2, \beta_3 > 0$, and whose derivative along the trajectories of (22) satisfies

$$\dot{V}(\xi) \le -\beta_4 |\xi|^2, \quad \beta_4 > 0. \quad (27)$$

Then, along the trajectories of (20), in view of (21), we have

$$\begin{aligned} \dot{V}(\xi) \le & -\beta_4 |\xi|^2 + \frac{\partial V}{\partial \xi}^\top \tilde{F}(x, \hat{x}) \\ \le & -\beta_4 |\xi|^2 + \beta_3 L_f |\xi|^2 \\ \le & -\beta_4' |\xi|^2, \end{aligned} \quad (28)$$

where $\beta_4' := \beta_4 - \beta_3 L_f$. Then, by properly choosing V such that $\beta_4 > \beta_3 L_f$, from (28) the origin of (20) is globally exponentially stable and the result follows. ■

Remark 1: The results stated in Proposition 1 are comparable to those in [19]. Indeed, as mentioned in the Introduction, the control law (4) consists basically in a tracking law making the nodes of the original network x_i track the auxiliary nodes y_i, which are described by the time-varying functions (5), which, if taken as the family of functions defined in [19, Theorem 2] that satisfy Assumption 1, are also persistently exciting. However, the functions g_k in [19] are restricted to the family of periodic functions. In contrast, in Proposition 1 the assumption only requires that $\hat{z}(t)$ be bounded, globally Lipschitz and persistently exciting, not periodic, therefore extending the class of functions that can be considered. Moreover, the persistency-of-excitation property is easier to verify than Assumption 1. Note also that in the proof of Proposition 1, the origin $\tilde{w} = 0$ is shown to be globally exponentially stable, which is a stronger property than the convergence property usually established in the literature. •

Remark 2: Although the result in Proposition 1 allows us to identify the unknown topology relying on the presistently-exciting signal $\hat{z}(t)$, similarly to [19] synchronization of the multi-agent system is prevented even when the topology has been correctly estimated. For this reason, in what follows we design an update law for $\hat{z}(t)$ so that it is *uniformly δ-persistently exciting* (uδ-PE)—see Appendix I for a definition—and synchronization is achieved as the topology is identified. •

C. Simultaneous identification and synchronization

In order to achieve simultaneous topology identification and synchronization we set $\hat{z}(t)$ to be the state of an auxiliary dynamical system instead of just a persistently exciting signal. For that purpose, let $\phi(t,\xi) : \mathbb{R}_{\geq 0} \times \mathbb{R}^{\bar{M}+2N-1} \to \mathbb{R}^N$, $(t,\xi) \mapsto \phi(t,\xi)$. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous non-decreasing function. Assume that for all $\xi \in \mathbb{R}^{\bar{M}+2N-1}$ and almost all $t \in \mathbb{R}_{\geq 0}$

$$\max\left\{|\phi(\cdot)|, \left|\frac{\partial \phi(\cdot)}{\partial t}\right|, \left|\frac{\partial \phi(\cdot)}{\partial \xi}\right|\right\} \leq \rho(|\xi|). \tag{29}$$

Then, we set the following update law for the auxiliary system:

$$\dot{\hat{x}} = F(\hat{x}) - c_2\hat{x} + \phi(t,\xi), \tag{30}$$

or equivalently, in the edge coordinates:

$$\dot{\hat{z}} = \bar{E}^\top F(\hat{x}) - c_2\hat{z} + \bar{E}^\top \phi(t,\xi) \tag{31}$$

where $c_2 > L_f$ is a positive constant. Now, we state the following proposition.

Proposition 2: Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{\bar{M}+2N-1} \to \mathbb{R}^N$, $(t,\xi) \mapsto \phi(t,\xi)$ satisfying (29). Then, if ϕ is uδ-PE in the sense of Lemma 2 in Appendix I, the origin of the closed-loop system (20) with $\hat{z}(t)$ given by the update law (31) is *uniformly globally asymptotically stable* and the topology of network (11) can be identified by the estimate $\hat{w}$, that is, $\bar{w} = \lim_{t\to\infty} \hat{w}(t)$ with the update law (19). Moreover, the system synchronizes, i.e. $z_k(t) \to 0$ as $t \to \infty$ for all $k \leq \bar{M}$ or, equivalently, $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{V}$. □

Proof: The total derivative of the Lyapunov function

$$V_1(\xi) = \frac{1}{2}\left[|\tilde{x}|^2 + |\tilde{z}_\mathcal{T}|^2 + |\tilde{w}|^2\right] \tag{32}$$

along the system's trajectories (20) yields

$$\dot{V}_1(\xi) \leq -c_1'|\tilde{x}|^2 - c_1|\tilde{z}_\mathcal{T}|^2 \leq 0, \quad c_1' := c_1 - 2L_f > 0. \tag{33}$$

From (33) we conclude that the system (20) is uniformly globally stable. Therefore, the solutions $\xi(t,t_0,\xi_0)$ are uniformly globally bounded.

Now, for the auxiliary system (30) consider the Lyapunov function $\hat{V}(\hat{x}) := 0.5|\hat{x}|^2$ whose total derivative satisfies

$$\begin{aligned}\dot{\hat{V}}(\hat{x}) \leq& -c_2|\hat{x}|^2 + L_f|\hat{x}|^2 + |\hat{x}||\bar{E}||\phi(t,\xi)| \\ \leq& -c_2'|\hat{x}|^2 + |\rho(|\xi|)|^2 \\ \leq& -c_2'|\hat{x}|^2 + \sigma,\end{aligned} \tag{34}$$

where we have used (29) and the last inequality follows from the fact that the solutions $\xi(t,t_0,\xi_0)$ are uniformly globally bounded, therefore there exists a positive constant σ such that $|\rho(|\xi(t,t_0,\xi_0)|)|^2 \leq \sigma$ for all t. Similarly, from (34) we conclude that the solutions $\hat{x}(t,t_0,\hat{x}_0)$, and hence the solutions $\hat{z}(t,t_0,\hat{z}_0)$, are uniformly globally bounded. Therefore, since all the assumptions in Lemma 4 in Appendix I are satisfied, $\hat{z}(t)$ is uδ-PE with respect to ξ.

Now, note that from (29), (32), and (33) Assumptions 6 and 7, in Appendix I hold for system (20). Then, invoking Lemma 5 in Appendix I we conclude that the origin of (20) is uniformly globally asymptotically stable. Therefore, $\lim_{t\to\infty} \hat{w} = \bar{w}$. Moreover, since $\lim_{t\to\infty} \xi(t) = 0$ we have that, from (34), $\lim_{t\to\infty} \hat{x}(t) = 0$, implying that $\lim_{t\to\infty} \hat{z}(t) = 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$, meaning that the system asymptotically reaches synchronization. ■

Remark 3: The proof of Proposition 2 relies on the fact that ϕ in (31) is uδ-PE with respect to ξ. As a matter of fact, Proposition 2 still holds even if ϕ is uδ-PE "only" with respect to $\tilde{w}$—cf. [21]. However, for this purpose, one would have to design ϕ dependent on $\tilde{w}$, hence, on $\bar{w}$ which are the *unknown* weights to estimate. Therefore, in the following, we study the stability properties of (20) and (31) when ϕ is uδ-PE "only" with respect to $\tilde{z}_\mathcal{T}$. •

Proposition 3: Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{N-1} \times \mathbb{R}^{n_\theta} \to \mathbb{R}^N$, $(t,\tilde{z}_\mathcal{T},\theta) \mapsto \phi(t,\tilde{z}_\mathcal{T},\theta)$, be parameterized by the free constants $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$. Moreover, let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous non-decreasing function and denote $\phi_\theta(t,\tilde{z}_\mathcal{T}) := \phi(t,\tilde{z}_\mathcal{T},\theta)$. Assume that

$$\max\left\{|\phi_\theta(\cdot)|, \left|\frac{\partial \phi_\theta(\cdot)}{\partial t}\right|, \left|\frac{\partial \phi_\theta(\cdot)}{\partial \tilde{z}_\mathcal{T}}\right|\right\} \leq \rho(|\tilde{z}_\mathcal{T}|), \quad \forall \theta \in \Theta. \tag{35}$$

Then, if ϕ_θ is uδ-PE with respect to $\tilde{z}_\mathcal{T}$, the origin of the closed-loop system (20) with $\hat{z}(t)$ given by the update law

$$\dot{\hat{z}} = \bar{E}^\top F(\hat{x}) - c_3\hat{z} + \bar{E}^\top \phi_\theta(t,\tilde{z}_\mathcal{T}) \tag{36}$$

is *uniformly semiglobally practically asymptotically stable* on Θ—cf. Appendix II. Moreover, the system synchronizes, i.e. $z_k(t) \to 0$ as $t \to \infty$ for all $k \leq \bar{M}$ or, equivalently, $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{V}$. □

Proof: First, we introduce the following result.

Consider the parameterized nonlinear time-varying system

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A(t,x_1) + B\phi_\theta(t,x_1)^\top x_2 \\ -\phi_\theta(t,x_1)B^\top x_1\end{bmatrix} \tag{37}$$

where $x^\top := \begin{bmatrix}x_1^\top & x_2^\top\end{bmatrix}$, $\phi_\theta(t,x_1) := \phi(t,x_1,\theta)$, $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}^n$ is piece-wise continuous in t and continuous in x_1, with $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ a constant free parameter. Furthermore, assume the following:

Assumption 3: The function A is locally Lipschitz in x uniformly in t. Moreover, there exists a continuous nondecreasing function $\rho_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\rho_1(0) = 0$ and for all $(t,x_1) \in \mathbb{R} \times \mathbb{R}^{n_1}$, $|A(t,x_1)| \leq \rho_1(|x_1|)$.

Assumption 4: There exists a locally Lipschitz function $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and positive constants α_1, α_2, α_3 such that

$$\alpha_1|x|^2 \leq V_1(t,x) \leq \alpha_2|x|^2 \tag{38}$$

and satisfies

$$\dot{V}_1(t,x) \le -\alpha_3|x_1|^2 \tag{39}$$

along the trajectories of (37).

Lemma 1: For system (37) suppose there exists a continuous non-decreasing function $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$\max\left\{|\phi_\theta(\cdot)|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial t}\right|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial x_1}\right|\right\} \le \rho(|x_1|). \tag{40}$$

Then, under Assumptions 3 and 4, if $\phi_\theta(t,x_1)$ is uδ-PE with respect to x_1, the origin of (37) is *uniformly semiglobally practically asymptotically stable* on Θ. □

Proof: Let us define the functions

$$V_2(t,x) := -x_1^\top B\phi_\theta(t,x_1)^\top x_2 \tag{41}$$

$$V_3(t,x) := -\int_t^\infty e^{(t-\tau)}\left|B\phi_\theta(\tau,x_1)^\top x_2\right|^2 d\tau \tag{42}$$

$$V_4(t,x) := V_2(t,x) + V_3(t,x) \tag{43}$$

Note that for any $\Delta > 0$, in view of the δ-persistently excitation of ϕ_θ, (40), and Lemma 3 in Appendix I, (43) satisfies, for all $(t,x) \in \mathbb{R}\times\mathbb{B}(\Delta)$

$$V_4(t,x) \le b|x_1|\rho(|x_1|)|x_2| - \gamma_\theta(|x_1|)|x_2|^2 \tag{44}$$

where $b := |B|$ and $\gamma_\theta(|x_1|) := e^{\vartheta_\Delta(|x_1|)}\gamma_\Delta(|x_1|,\theta)$.

We proceed now to evaluate the total derivative of $V_4(t,x)$ along the trajectories of the system. First, we have

$$\begin{aligned}\dot{V}_2(t,x) &= x_1^\top B\phi_\theta(t,x_1)^\top\phi_\theta(t,x_1)B^\top x_1\\ &\quad - x_2^\top\phi_\theta(t,x_1)^\top B^\top B\phi_\theta(t,x_1)x_2\\ &\quad - x_2^\top\phi_\theta(t,x_1)^\top B^\top A(t,x_1) - x_2^\top\overbrace{\phi_\theta(t,x_1)}^{\cdot}B^\top x_1\\ &= V_2(t,x) - |B\phi_\theta(t,x_1)x_2|^2 + \left|\phi_\theta(t,x_1)B^\top x_1\right|^2\\ &\quad - x_2^\top\phi_\theta(t,x_1)^\top B^\top(A(t,x_1)-x_1)\\ &\quad - x_2^\top\overbrace{\phi_\theta(t,x_1)}^{\cdot}B^\top x_1.\end{aligned} \tag{45}$$

Next, we write

$$\frac{\partial V_3}{\partial x_1} = -\int_t^\infty 2e^{(t-\tau)}x_2^\top\phi_\theta(\tau,x_1)B^\top B\left[\frac{\partial\phi_\theta(\tau,x_1)}{x_1}^\top x_2\right]d\tau$$

$$\frac{\partial V_3}{\partial x_2} = -\int_t^\infty 2e^{(t-\tau)}\phi_\theta(\tau,x_1)B^\top B\phi_\theta(\tau,x_1)^\top x_2 d\tau$$

$$\frac{\partial V_3}{\partial t} = \left|B\phi_\theta(\tau,x_1)^\top x_2\right|^2 - \int_t^\infty\frac{\partial}{\partial t}\left[e^{(t-\tau)}\left|B\phi_\theta(\tau,x_1)^\top x_2\right|^2\right]d\tau.$$

Finally, from Assumption 3, (40), and (44) we can obtain a bound for the derivative of (43). Define $b_\rho := b\rho(\Delta)$ and $\bar\rho(r,s) := b_\rho\left[4rs + \rho_1(r)s + b_\rho r^2 + b_\rho s^2\right]$, then, for $(t,x) \in \mathbb{R}\times\mathbb{B}(\Delta)$,

$$\dot{V}_4(t,x) \le \bar\rho(|x_1|,|x_2|) - \gamma_\theta(|x_1|)|x_2|^2. \tag{46}$$

Now, consider the candidate Lyapunov function $V(t,x) := V_1(t,x) + \varepsilon V_4(t,x)$, where $V_1(t,x)$ is given in Assumption 4 and ε is a small positive constant to be defined. Notice that in view of (44), $\varepsilon V_4(t,x)$ satisfies on $\mathbb{R}\times\mathbb{H}(\delta,\Delta)$

$$\begin{aligned}-\varepsilon\gamma_\theta(\Delta)|x_2|^2 - \varepsilon b_\rho|x_1||x_2| \le \varepsilon V_4(t,x) \le &\varepsilon b_\rho|x_1||x_2|\\ &- \varepsilon\gamma_\theta(\delta)|x_2|^2.\end{aligned} \tag{47}$$

So, from (38) and (47), for any $\Delta > \delta > 0$ and for a sufficiently small ε, there exist $\underline{\alpha}_{\delta,\Delta} > 0$ and $\overline{\alpha}_{\delta,\Delta} > 0$ such that for all $(t,x) \in \mathbb{R}\times\mathbb{H}(\delta,\Delta)$

$$\underline{\alpha}_{\delta,\Delta}|x|^2 \le V(t,x) \le \overline{\alpha}_{\delta,\Delta}|x|^2. \tag{48}$$

Using (39) and (46) the total derivative of $V(t,x)$ satisfies, for all $(t,x) \in \mathbb{R}\times\mathbb{H}(\delta,\Delta)$,

$$\begin{aligned}\dot{V}(t,x) \le &-\alpha_3|x_1|^2 - 5\varepsilon b_\rho|x_1||x_2| + \varepsilon b_\rho^2|x_1|^2 + \varepsilon b_\rho^2|x_2|^2\\ &- \varepsilon\gamma_\theta(\delta)|x_2|^2.\end{aligned}$$

Choosing $\theta^*(\delta,\Delta)$ such that $\gamma_\theta(\delta) \ge 2b_\rho^2$ we have

$$\begin{aligned}\dot{V}(t,x) &\le -\left(\alpha_3 - \varepsilon b_\rho^2\right)|x_1|^2 - 5\varepsilon b_\rho|x_1||x_2| - \varepsilon b_\rho^2|x_2|^2\\ &\le -\left(\alpha_3 - \left(b_\rho^2 + \frac{25}{2}\right)\varepsilon\right)|x_1|^2 - \frac{\varepsilon b_\rho^2}{2}|x_2|^2.\end{aligned} \tag{49}$$

Choosing ε sufficiently small such that $\alpha_3' := \left(\alpha_3 - \left(b_\rho^2 + \frac{25}{2}\right)\varepsilon\right) > 0$ and letting $c := \min\{\alpha_3', \varepsilon b_\rho^2/2\}$, we have

$$\dot{V}(t,x) \le -c|x|^2. \tag{50}$$

Therefore, from (48), (50), and Theorem 1 in Appendix II the result follows. ■

Following the same analysis as in the proof of Proposition 2 we can conclude that $\hat{z}(t)$, given by the update law (36), is uδ-PE with respect to $\tilde{z}_\mathcal{T}$. Therefore, uniform semiglobal practical asymptotical stability follows from Lemma 1 replacing x_1, x_2, $A(t,x_1)$, B, and ϕ_θ by $\left[\tilde{x}^\top\ \ \tilde{z}_\mathcal{T}^\top\right]^\top$, $\tilde{w}$,

$$\begin{bmatrix}\tilde{F}(x,\hat{x}) - \left(c_1 I + \bar{E}_\odot\bar{W}\bar{E}\right)\tilde{x}\\ \bar{E}_\mathcal{T}^\top\tilde{F}(x,\hat{x}) - \left(c_1 I + \bar{E}_\mathcal{T}^\top\bar{E}_\odot\bar{W}R^\top\right)\tilde{z}_\mathcal{T}\end{bmatrix},$$

$-\left[\bar{E}_\odot^\top\ \ \bar{E}_\odot^\top\bar{E}_\mathcal{T}\right]^\top$, and $\hat{Z}$, respectively.

Finally, to show that the system synchronizes, note that the Lyapunov function V_1 in (32) satisfies

$$\frac{1}{2}|\xi|_\mathcal{A}^2 \le V_1(\xi) \le \frac{1}{2}|\xi|^2 \tag{51}$$

$$\dot{V}_1(\xi) \le -c_1'|\xi|_\mathcal{A}^2 \tag{52}$$

where $\mathcal{A} := \{0\}^{2N-1}\times\mathbb{R}^{\bar{M}}$. Therefore, the set $\mathcal{A}$ is uniformly globally asymptotically stable—cf. [25, Theorem 2.8]. The latter implies that $\tilde{z}_\mathcal{T}(t) \to 0$ and, akin to the proof of Proposition 2, from (34) (replacing ξ by $\tilde{z}_\mathcal{T}$), that $\lim_{t\to\infty}\hat{x}(t) = 0$, implying that $\lim_{t\to\infty}\hat{z}(t) = 0$. Therefore, $\lim_{t\to\infty}z(t) = 0$, i.e., the system asymptotically reaches synchronization. ■

Remark 4: An example of the parameterized δ-persistently-exciting function ϕ_θ in (36) can be given as

$$\phi_\theta(t,\tilde{z}_\mathcal{T}) = \tanh\left(\kappa\bar{E}_\odot R^\top\tilde{z}_\mathcal{T}\right)\sin(t), \tag{53}$$

where $\kappa \equiv \theta^*(\delta,\Delta)$ is the design parameter that can be chosen large enough so that the ultimate bound of the estimation error is as small as desired. Note that (53) is δ-persistently exciting with respect to $\tilde{z}_\mathcal{T}$ as per Definition 2 in Appendix I. •

D. Another design for the auxiliary system

Under the strategy presented in Section III-C we successfully identify the topology of the network and synchronize the dynamical systems. As a consequence, note that under the control law (18), the system (11) (in edge representation) converges to the auxiliary system with dynamics given by (36). Although synchronization is achieved, note that using the dynamics (36) is equivalent to driving the states x_i of all the systems to the origin ($x_i \to 0$). To see this, note that the auxiliary system (30) has $\{\hat{x} = 0\}$ as its equilibrium, and therefore, as $x_i \to \hat{x}_i$ and $\hat{x}_i \to 0$ we have that $x_i \to 0$. By the latter we lose the effect of the interaction represented by the (identified) graph topology. Indeed, without the control action (18), the systems (7) would converge to a consensus value, generally different from zero, determined by the topology. Therefore, in this subsection we modify slightly the auxiliary system in order to recover the influence of the topology in the synchronized state.

Let the update law for the auxiliary system be given by:

$$\dot{\hat{x}} = F(\hat{x}) - c\bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x} + \phi_\theta(t, \tilde{z}_{\mathcal{T}}). \tag{54}$$

Equivalently, in the edge coordinates, we have:

$$\dot{\hat{z}} = \bar{E}^{\top}F(\hat{x}) - c\bar{E}^{\top}\bar{E}_{\odot}\hat{W}\hat{z} + \bar{E}^{\top}\phi_\theta(t, \tilde{z}_{\mathcal{T}}) \tag{55}$$

where $\hat{W} := \mathrm{diag}\{\hat{w}\}$ with $\hat{w}$ being the estimated weights, and ϕ_θ is parameterized by the free constants $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ and satisfies (35). Moreover we assume the following.

Assumption 5: All the autonomous auxiliary systems, i.e.,

$$\dot{\hat{x}}_i = f_i(\hat{x}_i) + u_i, \quad i \in \mathcal{V} \tag{56}$$

are strictly semi-passive [26] with respect to the input u_i and output y_i with continuously differentiable and radially unbounded storage functions $\hat{V}_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. That is, there exist positive definite and radially unbounded storage functions $\hat{V}_i$, positive constants ϱ_i, continuous functions H_i and positive continuous functions χ_i such that

$$\dot{\hat{V}}_i(\hat{x}_i) \leq u_i^{\top}y_i - H_i(\hat{x}_i) \tag{57}$$

and $H_i(\hat{x}_i) \geq \chi(|\hat{x}_i|)$ for all $|\hat{x}_i| \geq \varrho_i$.

Remark 5: Semi-passivity means that for large values of the state the system behaves as a passive system, which is a reasonable assumption in practice. Indeed, many dynamical systems, especially physical or biological systems, can be expected to be semi-passive [26]. Furthermore, semi-passivity is a more general property than passivity as it encompasses a much broader class of dynamical behaviors such as, e.g., oscillatory or chaotic behavior. •

Proposition 4: If ϕ_θ is uδ-PE with respect to $\tilde{z}$, the origin of the closed-loop system (20) with $\hat{z}(t)$ given by the update law (55) is *uniformly semiglobally practically asymptotically stable* on Θ and the topology of network (11) can be identified by the estimate $\hat{w}$, using the update law (19). Moreover, the system achieves *practical* synchronization, i.e. $|z(t)| \to \epsilon$ as $t \to \infty$ for all $k \leq \bar{M}$, with ϵ a small constant. □

Proof: We first show that $\hat{z}$ is uδ-PE with respect to $\tilde{z}$. For that purpose consider the autonomous part of the heterogeneous auxiliary system with output y

$$\dot{\hat{x}} = F(\hat{x}) + u \tag{58}$$

$$y = \bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x}. \tag{59}$$

Then, defining the storage function $V_\Sigma = \sum_{i\in\mathcal{V}} \hat{V}_i$, from Assumption 5, we have

$$\dot{V}_\Sigma(\hat{x}) \leq -\sum_{i\in\mathcal{V}} H_i(\hat{x}_i) + u^{\top}y. \tag{60}$$

Replacing the output (59) and the input $u = -c\bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x} + \phi_\theta(t, \tilde{z})$ into (60), we obtain

$$\dot{V}_\Sigma(\hat{x}) \leq -\sum_{i\in\mathcal{V}} H_i(\hat{x}_i) - c|\bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x}|^2 + |\bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x}||\phi_\theta(t, \tilde{z})|. \tag{61}$$

By definition the δ-PE function ϕ_θ is bounded, therefore there exists a constant $\bar{\phi}$ such that $|\phi_\theta(t, \tilde{z})| \leq \bar{\phi}$ for all t. Then, there exist positive constants c' and λ such that applying Young's inequality to the last term on the right-hand side of (61) we have

$$\begin{aligned}\dot{V}_\Sigma(\hat{x}) &\leq -\sum_{i\in\mathcal{V}} H_i(\hat{x}_i) - c'|\bar{E}_{\odot}\hat{W}\bar{E}^{\top}\hat{x}|^2 + \lambda\bar{\phi} \\ &\leq -\sum_{i\in\mathcal{V}} H_i(\hat{x}_i) + \lambda\bar{\phi}.\end{aligned} \tag{62}$$

Next, let $\bar{\varrho} := \max_{i\leq N}\{\varrho_i\}$ and consider the function $\bar{\chi} : [\bar{\varrho}, +\infty) \to \mathbb{R}_{\geq 0}$ given by $\bar{\chi}(s) = \min_{i\leq N}\{\chi_i(s)\}$. Furthermore, note that, by assumption, $H_i(\hat{x}_i) \geq \chi(|\hat{x}_i|)$ for all $|\hat{x}_i| \geq \varrho_i$, and for any $\hat{x} \in \mathbb{R}^N$ such that $|\hat{x}| \geq N\bar{\varrho}$, there exists $l \in \mathcal{V}$ such that $|\hat{x}_l| \geq \frac{1}{N}|\hat{x}| \geq \bar{\varrho}$. Thus, for all $|\hat{x}| \geq N\bar{\varrho}$ and any $l \leq N$,

$$\sum_{i\in\mathcal{V}} H_i(\hat{x}_i) \geq H_l(\hat{x}_l) \geq \bar{\chi}(|\hat{x}_l|) \geq \bar{\chi}\left(\frac{1}{N}|\hat{x}|\right).$$

In view of the latter and (62), we obtain

$$\dot{V}_\Sigma(\hat{x}) \leq -\bar{\chi}\left(\frac{1}{N}|\hat{x}|\right) + \lambda\bar{\phi}, \tag{63}$$

which implies that the trajectories $\hat{x}(t, t_0, \hat{x}_0)$ are uniformly globally bounded. Furthermore, since $\hat{z} = \bar{E}^{\top}\hat{x}$, we have that the trajectories $\hat{z}(t, t_0, \hat{z}_0)$ are also uniformly globally bounded. Therefore, all the conditions of Lemma 4 in Appendix I are satisfied and we conclude that $\hat{z}$, given by the update law (55) is uδ-PE with respect to $\tilde{z}$.

Next, following the same arguments as in the proof of Proposition 3 it can be concluded that the origin of the closed-loop system (20) is uniformly semiglobally practically asymptotically stable on Θ.

Finally, we show that the system reaches practical synchronization. First, note that (51) and (52) imply that the set $\mathcal{A}$ is uniformly globally asymptotically stable, hence, $\tilde{z}_{\mathcal{T}}(t) \to 0$. Now, the reduced-order model (17) in closed-loop with the control input (18) and the update law (54) yields

$$\dot{z}_{\mathcal{T}} = \bar{E}_{\mathcal{T}}^{\top}F(x) - c\bar{E}_{\mathcal{T}}^{\top}\bar{E}_{\odot}\bar{W}R^{\top}z_{\mathcal{T}} + \zeta(\tilde{z}_{\mathcal{T}}), \tag{64}$$

where $\zeta(\tilde{z}_{\mathcal{T}}) := -c_1\tilde{z}_{\mathcal{T}} + \bar{E}_{\mathcal{T}}^{\top}\phi_\theta(t, \tilde{z}_{\mathcal{T}})$. Moreover, the kth component of the first term on the right-hand side of (64) can be written as

$$\begin{aligned}\left[\bar{E}_{\mathcal{T}}^{\top}F(x)\right]_k =& f_i(x_i)-f_j(x_j)\\ =&\left(f_i(x_i)-f_i(x_j)\right)+\left(f_i(x_j)-f_j(x_j)\right).\end{aligned}$$

Therefore, we have

$$\dot{z}_{\mathcal{T}} = F_z(z_{\mathcal{T}}) - c\bar{E}_{\mathcal{T}}^{\top}\bar{E}_{\odot}\bar{W}R^{\top}z_{\mathcal{T}} + \Pi F(x) + \zeta(\tilde{z}_{\mathcal{T}}), \quad (65)$$

where $[F_z(z_{\mathcal{T}})]_k = f_i(x_i) - f_i(x_j)$ and $[\Pi F(x)]_k = f_i(x_j) - f_j(x_j)$. Furthermore, from the uniform global boundedness of $\tilde{x}$ and $\hat{x}$, x is also uniformly globally bounded and we have

$$|F_z(z_{\mathcal{T}})| \le |z_{\mathcal{T}}| \quad (66)$$

$$|\Pi F(x)| \le \Delta_f, \quad (67)$$

where $\Delta_f := \max_{|x|\le B_x}\max_{i,j\in\mathcal{V}}\{|f_i(x_j) - f_j(x_j)|\}$.

Now, the derivative of the candidate Lyapunov function $V_z = 0.5z_{\mathcal{T}}^{\top}Pz_{\mathcal{T}}$ along (65) yields

$$\begin{aligned}\dot{V}_z =& z_{\mathcal{T}}^{\top}PF_z(z_{\mathcal{T}}) - cz_{\mathcal{T}}^{\top}P\bar{E}_{\mathcal{T}}^{\top}\bar{E}_{\odot}\bar{W}R^{\top}z_{\mathcal{T}}\\ &+ z_{\mathcal{T}}^{\top}P\Pi F(x) + z_{\mathcal{T}}^{\top}P\zeta(\tilde{z}_{\mathcal{T}}).\end{aligned} \quad (68)$$

By assumption the directed graph contains a directed spanning tree. Therefore, for any symmetric positive definite $Q \in \mathbb{R}^{(N-1)\times(N-1)}$, there exists a symmetric positive definite $P \in \mathbb{R}^{(N-1)\times(N-1)}$ which is a solution to the Lyapunov equation $P\bar{E}_{\mathcal{T}}^{\top}\bar{E}_{\odot}\bar{W}R^{\top} + R\bar{W}\bar{E}_{\odot}^{\top}\bar{E}_{\mathcal{T}}P = Q$—cf. [22]. Then, using (66) and (67), we obtain

$$\dot{V}_z \le -c'|z_{\mathcal{T}}|^2 + \psi(|\tilde{z}_{\mathcal{T}}|) + \Delta_f^2, \quad (69)$$

where ψ is a $\mathcal{K}_{\infty}$ function such that $\psi(|\tilde{z}_{\mathcal{T}}|) \ge |\zeta(\tilde{z}_{\mathcal{T}})|^2$. Eq. (69) implies that the origin $z_{\mathcal{T}} = 0$ is input-to-state practically stable—cf. [27]. Therefore, since $\tilde{z}_{\mathcal{T}}(t) \to 0$, from (69) we conclude that $|z_{\mathcal{T}}(t)| \to \epsilon$ and the system achieves practical synchronization. ■

Remark 6: Note that, from (69), the ultimate bound ϵ for the synchronization error $z_{\mathcal{T}}$ is a function of Δ_f, i.e. $\epsilon(\Delta_f)$, which is a measure of the heterogeneity of the system. Hence, if the system is homogeneous, i.e., $f_i(x_j) = f_j(x_j)$ for all $i, j \in \mathcal{V}$, then $\Delta_f \equiv 0$ and the system synchronizes, that is, $z_{\mathcal{T}} \to 0$ as $t \to \infty$. •

IV. Numerical example

We consider a network of $N = 10$ harmonic oscillators modeled by the differential equation (7) with $x_i = [x_{1,i}\ \ x_{2,i}]^{\top} \in \mathbb{R}^2$ and $f(x_i) := [x_{2,i}\ \ -x_{1,i}]^{\top}$. The interaction among the agents is determined by the *unknown* directed topology $\mathcal{G}(\mathcal{V},\mathcal{E},W)$, represented in Fig. 1. The objective is to identify the directed topology and its weights w_{ij} by estimating the *unknown* weights $\bar{w}_{ij}$ of the complete graph. For this purpose, the external input u is given by (18) with (19) and (55), where

$$\begin{aligned}\phi_\theta(t,\tilde{z}) =& \tanh\left(\kappa\bar{E}_{\odot}\tilde{z}\right)p_e(t)\\ p_e(t) =& 2\sin(\pi t) + 0.3\cos(6\pi t) - 0.5\sin(8\pi t)\\ &- 0.1\cos(10\pi t),\end{aligned}$$

and the control gains are taken as $c_1 = 10$, $c_3 = 10$, $\kappa = 400$.

The simulation results are presented in Figs. 2-5 (due to limited space only the second coordinate is presented for Figs.

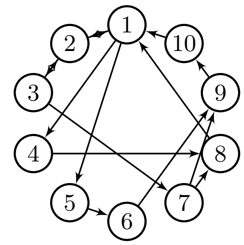

Fig. 1: Connected undirected graph representing the network topology with the leaders in gray.

4 and 5, that is $x_{2,i}$ and $z_{2,k}$). In Figs. 2 and 3 are presented, respectively, the weight estimation errors and the identified weights. As predicted by the theory the estimation errors do not converge exactly to the origin, but to a neighborhood of the origin which is determined by the parameterized δ-PE function ϕ_θ. Moreover, it is clear Fig. 5 that the system reaches practical synchronization, where the steady-state error only depends on the degree of heterogeneity of the system. Note further that since the estimation error of the weights does not converge exactly to the origin, exact identification of the topology is not achieved. However, we obtain a good estimation of which edges are present in the real graph and the neighborhood of convergence can be made smaller by properly tuning the parameter κ and the parameterized δ-PE function ϕ_θ. To see the effect of the parameter κ, in Fig. 6 are presented the estimation errors of the weights setting $\kappa = 40$ instead of $\kappa = 400$. Comparing Fig. 6 with Fig. 2 it is evident that the steady state errors are larger for a smaller value of κ.

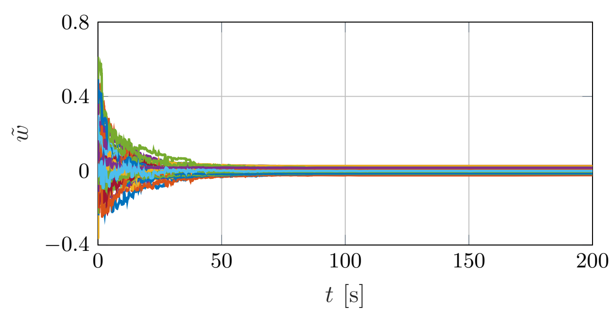

Fig. 2: Estimation errors of the interconnection weights.

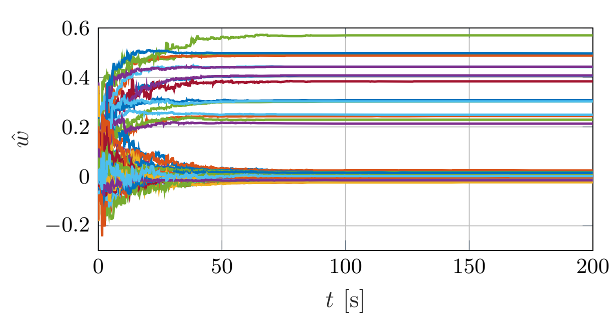

Fig. 3: Estimated values of the topology weights.

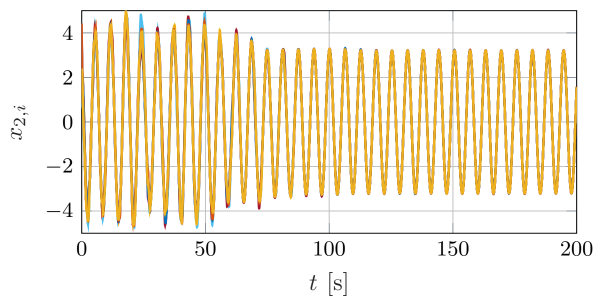

Fig. 4: Evolution of the state of each agent.

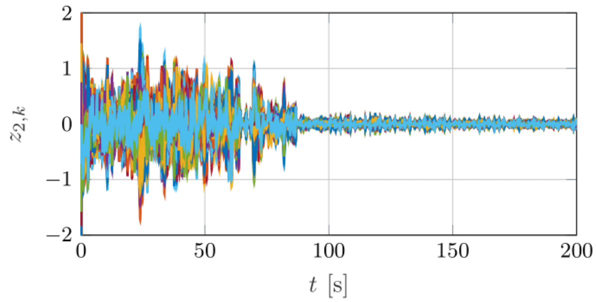

Fig. 5: Evolution of the synchronization errors.

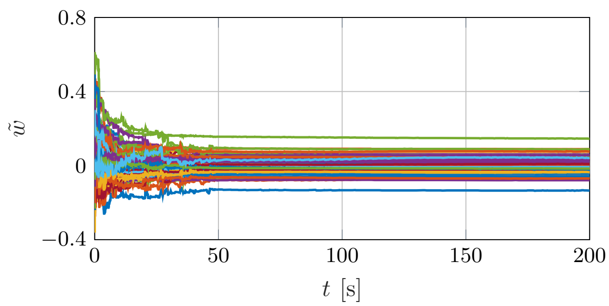

Fig. 6: Estimation errors of the graph weights with $\kappa = 40$.

V. Conclusions

In this paper we presented a new methodology for simultaneously synchronizing a complex dynamical network and identifying its directed topology. The novel control algorithm is based on the edge-agreement framework and on the design of a known auxiliary synchronizing network which is designed to be uniformly δ-persistently exciting. The latter allows us to successfully (up to a small tunable error) identify the directed topology without requiring the verification of the Linear Independence Condition and despite the achievement and preservation of the synchronized behavior of the system. It is important to emphasize that, beyond mere convergence properties as usually established in the literature, we guarantee that the estimation error is uniformly semiglobally practically asymptotically stable. We believe that these properties may be useful to extend the current approach to more complex synchronization-based control applications, and to provide stronger results such as finite- or fixed-time estimation and synchronization. These are topics of further research.

Appendix I
On δ-Persistency of Excitation

Definition 1 (Persistency of excitation): A function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is said to be persistently exciting if there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$

$$\int_t^{t+T} |\phi(\tau)| d\tau \geq \mu. \tag{70}$$

Let $x \in \mathbb{R}^n$ be partitioned as $x^\top := \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Define the set $\mathcal{D}_1 := (\mathbb{R}^{n_1} \backslash \{0\}) \times \mathbb{R}^{n_2}$ and the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ where $t \mapsto \phi(t, x)$ is locally integrable.

Lemma 2: ([28, Lemma 1]) If $x \mapsto \phi(t, x)$ is continuous uniformly in t, then $\phi(\cdot, \cdot)$ is uniformly δ-persistently exciting with respect to x_1 if and only if for each $x \in \mathcal{D}_1$ there exist $T > 0$ and $\mu > 0$ such that,

$$\int_t^{t+T} |\phi(\tau, x)| d\tau \geq \mu, \tag{71}$$

for all $t \in \mathbb{R}$. □

If $\phi(\cdot, \cdot)$ is uniformly δ-persistently exciting (δ-PE) with respect to the whole state x then $\phi(\cdot, \cdot)$ is called uniformly δ-PE.

The following characterization of δ-PE, presented in [28], is a technical tool used in the proof of convergence.

Lemma 3: For each $\Delta > 0$ there exist $\gamma_\Delta \in \mathcal{K}$ and $\vartheta_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ continuous and strictly decreasing such that,

$$\{|x_1|, |x_2| \in [0, \Delta] \backslash \{x_1 = 0\}\}$$
$$\implies \int_t^{t+\vartheta_\Delta(|x_1|)} |\phi(\tau, x)| d\tau \geq \gamma_\Delta(|x_1|),$$

for all $t \in \mathbb{R}$. □

The next Lemma establishes that the output of a strictly proper stable filter driven by a δ-PE input conserves such property. A reminiscent version of the lemma was originally presented in [21]. For completeness and to better fit the contents of this paper we rewrite it here with a different formulation and proof.

Lemma 4 (Filtration property): Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ and consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} f(t, x, \omega) \\ f_1(t, \omega) + f_2(t, x)\omega + \phi(t, x) \end{bmatrix} \tag{72}$$

with $f_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz in ω uniformly in t and measurable in t and satisfies $|f_1(\cdot)| \leq |\omega|$ for all t; $f_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$ is locally Lipschitz in x uniformly in t and measurable in t. Assume that $\phi(t, x)$ is uniformly δ-persistently exciting with respect to x. If ϕ is locally Lipschitz and there exists a non-decreasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that, for almost all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$:

$$\max\left\{|\phi(\cdot)|, |f(\cdot)|, |f_2(\cdot)|, \left|\frac{\partial \phi(\cdot)}{\partial t}\right|, \left|\frac{\partial \phi(\cdot)}{\partial x}\right|\right\} \leq \alpha(|x|). \tag{73}$$

Assume further that all solutions $t \mapsto x_\phi$, with $x_\phi^\top = \begin{bmatrix} x^\top & \omega^\top \end{bmatrix}$, are defined in $[t_0, \infty)$ and satisfy

$$|x_\phi(t, t_0, x_{\phi 0})| \leq r \quad \forall t \geq t_0, \tag{74}$$

for a positive constant r, then ω is uniformly δ-persistently exciting with respect to x. □

Proof: Defining $\rho = -\omega^\top \phi$ we have that

$$\begin{aligned}\dot{\rho} =& -|\phi|^2 - \phi^\top f_1 - \omega^\top \left[f_2\phi + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f \right] \\ \leq& -|\phi|^2 + 2|\omega| \left[\alpha^2(r) + \alpha(r)\right] =: -|\phi|^2 + c(r)|\omega|. \quad (75)\end{aligned}$$

Inverting the sign of the inequality (75) and integrating both sides from t to $t+T_f$, with $T_f := (k+1)T$, we obtain that

$$\begin{aligned}\omega(t)^\top \phi(t,x) - &\omega(t+T_f)^\top \phi(t+T_f, x) \\ &\geq \int_t^{t+T_f} |\phi(\tau,x)|^2 d\tau - \int_t^{t+T_f} c(r)|\omega(\tau)| d\tau.\end{aligned} \quad (76)$$

Using the bounds in (73) and (74) on the left-hand side of inequality (76), the latter is equivalent to

$$2\alpha(r)r \geq \int_t^{t+T_f} |\phi(\tau,x)|^2 d\tau - \int_t^{t+T_f} c(r)|\omega(\tau)| d\tau.$$

Since $\phi(t,x)$ is uδ-PE, there exists μ such that

$$\int_t^{t+(k+1)T} |\phi(\tau,x)|^2 d\tau \geq (k+1)\mu^2.$$

Then we obtain

$$\int_t^{t+T_f} |\omega(\tau)| d\tau \geq \frac{\left((k+1)\mu^2 - 2\alpha(r)r\right)}{c(r)} =: \mu_r.$$

Finally, choosing k large enough so that $\mu_r > 0$ we obtain that $\omega(t)$ is uδ-PE with respect to x. ■

The following lemma presented originally in [21] is included here for completeness. Consider the nonlinear time-varying system

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} H(t,x) + B(t,x)^\top x_2 \\ D(t,x)\end{bmatrix} \quad (77)$$

where $x^\top := \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix} \in \mathbb{R}^{n_1+n_2}$ and $H : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$, $D : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$, $B : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1 \times n_2}$ are continuous. Furthermore, assume that the system (77) satisfies the following hypotheses.

Assumption 6: There exists a locally Lipschitz function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions α_1, α_2, and a continuous positive definite function α_3 such that

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \quad (78)$$

$$\dot{V}(t,x) \leq -\alpha_3(|x_1|). \quad (79)$$

Assumption 7: Each element of $B(\cdot,\cdot)$ is locally Lipschitz and there exist nondecreasing functions $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 1,2,3$, such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$

$$\max\{|H(t,x)|, |D(t,x)|\} \leq \rho_1(|x|)|x_1| \quad (80)$$

$$|B(t,x)| \leq \rho_2(|x|) \quad (81)$$

$$\max\left\{ \left|\frac{\partial B(t,x)}{\partial (x_j)_i}\right|, \left|\frac{\partial B(t,x)}{\partial t}\right| \right\} \leq \rho_3(|x|), \quad (82)$$

$$j \in \{1,2\}, \quad i \in \{1,\dots,n_j\}.$$

Lemma 5: For the system (77), if Assumptions 6 and 7 hold and B is uδ-PE with respect to x_2, then the origin of the system (77) is uniformly globally asymptotically stable. □

APPENDIX II
ON UNIFORM SEMIGLOBAL PRACTICAL ASYMPTOTIC STABILITY

The following definitions and Theorem on set stability are taken from [29] and included here for completeness.

For nonlinear time-varying systems of the form

$$\dot{x} = f(t,x), \quad (83)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ piecewise continuous in t and locally Lipschitz in x, we present the following definitions in which $\Delta > \delta$ are non-negative numbers.

Definition 2 (uniform stability of a ball): The ball $\mathbb{B}(\delta)$ is said to be *uniformly stable on* $\mathbb{B}(\Delta)$ for system (83) if there exists a class $\mathcal{K}_\infty$ function η such that the solutions of (83) from any initial state $x_0 \in \mathbb{B}(\Delta)$ and any initial time $t_0 \in \mathbb{R}_{\geq 0}$ satisfy $|x(t,t_0,x_0)|_\delta \leq \eta(|x_0|)$ for all $t \geq t_0$.

Definition 3 (uniform attractivity of a ball): The ball $\mathbb{B}(\delta)$ is said to be *uniformly attractive on* $\mathbb{B}(\Delta)$ for system (83) if there exists a class $\mathcal{L}$ function σ such that the solutions of (83) from any initial state $x_0 \in \mathbb{B}(\Delta)$ and any initial time $t_0 \in \mathbb{R}_{\geq 0}$ satisfy $|x(t,t_0,x_0)|_\delta \leq \sigma(t-t_0)$ for all $t \geq t_0$.

Definition 4 (uniform asymptotic stability of a ball): The ball $\mathbb{B}(\delta)$ is said to be *uniformly asymptotically stable on* $\mathbb{B}(\Delta)$ for system (83) if it is both uniformly stable and uniformly attractive on $\mathbb{B}(\Delta)$.

Consider parameterized nonlinear time-varying systems

$$\dot{x} = f(t,x,\theta), \quad (84)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}^{n_\theta}$ is a constant parameter, and $f(t,x,\theta)$ is piecewise continuous in t and locally Lipschitz in x for all θ.

Definition 5: Let $\Theta \in \mathbb{R}_{n_\theta}$ be a set of parameters. System (84) is said to be *uniformly semiglobally practically asymptotically stable* on Θ if, given any $\Delta > \delta > 0$, there exists $\theta^*(\delta,\Delta) \in \Theta$ such that $\mathbb{B}(\delta)$ is uniformly asymptotically stable on $\mathbb{B}(\Delta)$ for the system $\dot{x} = f(t,x,\theta^*)$.

The following statement introduced in [29] gives a sufficient condition for a system (84) to be uniformly semiglobally practically asymptotically stable in terms of a Lyapunov function.

Theorem 1: Suppose that given any $\Delta > \delta > 0$, there exist a parameter $\theta^*(\delta,\Delta) \in \Theta$, a continuously differentiable function $V_{\delta,\Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\delta,\Delta}$, $\overline{\alpha}_{\delta,\Delta}$, $\alpha_{\delta,\Delta}$ such that, for all $x \in \mathbb{H}(\delta,\Delta)$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_{\delta,\Delta}(|x|_\delta) \leq V_{\delta,\Delta}(t,x) \leq \overline{\alpha}_{\delta,\Delta}(|x|), \quad (85)$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t,x) + \frac{\partial V_{\delta,\Delta}}{\partial x}(t,x) f(t,x,\theta^*) \leq -\alpha_{\delta,\Delta}(|x|). \quad (86)$$

Assume further that, given any $\Delta^* > \delta^* > 0$, there exist $\Delta > \delta > 0$ such that

$$\underline{\alpha}_{\delta,\Delta}^{-1} \circ \overline{\alpha}_{\delta,\Delta}(\delta) \leq \delta^*, \quad \overline{\alpha}_{\delta,\Delta}^{-1} \circ \underline{\alpha}_{\delta,\Delta}(\Delta) \geq \Delta^*.$$

Then, system (84) is uniformly semiglobally practically asymptotically stable on the parameter set Θ. □

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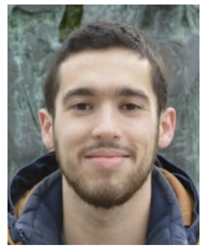

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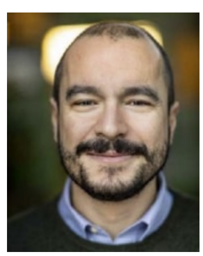

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