

A Communication Consistent Approach to Signal Temporal Logic Task Decomposition in Multi-Agent Systems

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Abstract—We address the problem of decomposing a global task assigned to a multi-agent system, where the task is expressed using a fragment of Signal Temporal Logic (STL) and communication among agents is range-limited. The global task is expressed as the conjunction of local tasks over individual and relative agent's states, thus naturally inducing the definition of a task graph. Due to the limited communication range, inconsistencies among the edges of the communication and task graph prevent the application of previously derived state feedback control laws for the satisfaction of global STL tasks. To resolve this issue, we propose a task decomposition mechanism reassigning tasks between non-communicating agents to communicating ones, thus ensuring consistency between task and communication graphs. By assuming that the STL tasks are defined over concave predicate functions with polytopic super-level sets, the decomposition can be framed as a parameter optimization problem, solvable via decentralized convex optimization. To guarantee the soundness of our approach, we present various conditions under which the tasks defined in the applied STL fragment are unsatisfiable, and we derive sufficient conditions such that our decomposition approach yields satisfiable global tasks after decomposition.

Index Terms—Signal Temporal Logic, Multi-Agent Systems, Optimization.

I. INTRODUCTION

Recently, Signal Temporal Logic (STL) has received increasing attention as a tool to monitor and control the run-time behaviour of multi-agent cyber-physical systems [1]. Notably, an extensive body of literature has been developed dealing with synthesizing admissible plans and feedback solutions that achieve the satisfaction of a given global STL task assigned to a multi-agent system (MAS) [2]–[11]. While the scalability of the proposed approaches varies depending

on the semantic richness of the considered task and the number of agents involved in the task, task decomposition has been recently explored as a solution to mitigate the curse of dimensionality at the control and planning level, borrowing from similar approaches applied for other types of temporal logics [12], [13]. Namely, the authors in [14]–[17] first proposed optimization-based approaches to split complex global STL tasks into simpler ones assigned to teams of agents in the systems, whose satisfaction, when jointly considered, recovers the satisfaction of the original global task. However, the influence of agent-to-agent communication has received only minor attention in these settings [18]–[21]. If, loosely speaking, we denote task dependencies to be consistent when the agents involved in the task are in communication with each other, then such consistency is a key enabler for many of the previous control/planning approaches. From this observation, the contribution of this work is to establish an approach whereby agents can, in a decentralized fashion, reassign task dependencies that are not consistent with the range-limited communication constraints while recovering the original task satisfaction.

We consider the setting in which the global STL task is expressed as a conjunction of local tasks assigned to single agents and pairs of agents in the system. These kinds of specifications are particularly suited for time-varying formation control of multi-agent systems, which find wide applications in robotics in the context of surveillance and reconnaissance tasks, localization and exploration tasks, and coverage control, to mention a few [22]–[25]. Within this framework, task dependencies among pairs of agents are captured by a task graph. At the same time, range-limited communication naturally induces the definition of a communication graph where edges are defined by pairs of communicating agents. We propose to decompose tasks assigned over pairs of non-communicating agents as a conjunction of tasks defined over agents forming a multi-hop communication path between the former non-communicating pair. More specifically, we show that by considering concave predicate functions whose super-level set is defined by a bounded polytope, our decomposition mechanism can be cast as a convex optimization problem. In addition, we show that such optimization can be efficiently decentralized over the edges of the communication graph when its structure is acyclic, thus providing scalability of the pro-

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posed approach. Eventually, we provide sufficient conditions under which the new task derived from the decomposition is satisfiable and its satisfaction implies the original global task. To the best of our knowledge, this work represents the first decentralized solution to an STL task decomposition problem in multi-agent settings.

Previous literature in STL task decomposition includes the work by [17], which first introduces a continuous-time and continuous-space STL task decomposition via convex optimization in a centralized fashion. On the other hand, the authors in [14], [15] employ a Mixed-Integer Linear Program (MILP) formulation to decompose a global task expressed in a variant of the STL formalism, denoted as Capability STL (CaSTL), as local team-wise tasks. Differently from [14], [15], [17], our decomposition approach considers communication constraints at the task decomposition level. It is noted however, that the STL fragment considered in [14], [15] is syntactically richer than the one considered in [17] and our work. The relevance of this work should be considered with respect to previous literature in feedback control solutions for the satisfaction of STL tasks in multi-agent settings, where all-to-all agent communication is explicitly or implicitly assumed [19], [26]–[28]. Namely, we aim to mitigate the complete connectivity requirement (i.e. every agent is directly connected to every other agent) of these previously developed feedback control solutions, while also retaining their adaptive and reactive properties, by decomposing tasks defined over non-communicating agents into conjunctions of tasks over those that do communicate. We list the following contributions:

- i) We provide a task decomposition mechanism to enforce consistency between the task and communication dependencies, enabling previously developed feedback control laws to be leveraged for the satisfaction of a global STL task.
- ii) We prove that by assuming the communication graph to be acyclic, our decomposition can be decentralized over the edges of the communication graph and solved by state-of-the-art decentralized convex optimization algorithms as the ones proposed in [29]–[31].
- iii) We provide a set of conditions under which conjunctions of collaborative tasks over the proposed task graph are unsatisfiable (referenced as conflicting conjunctions hereafter).
- iv) We derive sufficient conditions under which our decomposition approach of the original global task assigned to the system yields a new global task that does not suffer from the aforementioned types of conflicting conjunctions and such that its satisfaction implies the original one.

Compared to our work in [21], the novel contribution of this work is to show that the task decomposition can be achieved in a decentralized manner and over a broader class of predicate types. The paper is organized as follows: Section II reviews polytope operations, STL, and graph theory. The problem statement is provided in Section III. Section IV introduces the concept of conflicting conjunctions of collaborative tasks, while Sections V–VI develop the task decomposition approach. Simulations and conclusions are presented in Sections VII and VIII.

II. NOTATION AND PRELIMINARIES

Bold letters denote vectors, capital letters indicate matrices and sets. Vectors are considered to be columns. Given $\mathbf{a}_i \in \mathbb{R}^{n_i}$, the vector $\mathbf{a} = [\mathbf{a}_i]_{i \in \mathcal{I}} \in \mathbb{R}^{\sum_i n_i}$ is obtained by vertically stacking the vectors $\mathbf{a}_i$ with index $i \in \mathcal{I}$. For a matrix $A \in \mathbb{R}^{n \times n}$ and vector $\mathbf{a} \in \mathbb{R}^n$, the matrix $[A \mid \mathbf{a}] \in \mathbb{R}^{n \times (n+1)}$ is obtained by stacking A and $\mathbf{a}$ horizontally. The notation $\mathbf{x}[k]$ indicates the k^{th} element of $\mathbf{x}$ and $A[i, j]$ indicates the element at the i^{th} row and j^{th} column of the matrix A . Given a vector $\mathbf{x} \in \mathbb{R}^n$, the standard notation for the 2-norm $\|\mathbf{x}\| = \sqrt{\sum_k \mathbf{x}[k]^2}$ applies, while we adopt the notation $\langle \mathbf{x} \rangle = \min_k \{\mathbf{x}[k]\}$ to indicate the element-wise minimum. The notations $|\mathcal{A}|$ and $2^{\mathcal{A}}$ denote cardinality and the power set of $\mathcal{A}$, respectively, while the symbols $\oplus$, $\ominus$ and $\otimes$ indicate the Minkowski sum, Minkowski difference and Cartesian product among sets, respectively. The symbol $\otimes$ indicates the Kronecker product of two matrices/vectors. Let $\text{blk}(A_1, \dots, A_N)$ indicate the block diagonal matrix with blocks $A_1, \dots, A_N$. The notation $\mathbf{1}_{m \times n}$, $\mathbf{1}_n$, $\mathbf{0}_{m \times n}$ and $\mathbf{0}_n$ indicate matrices/vectors of ones and zeros respectively while I_n indicates the n -dimensional identity matrix. The set $\mathbb{R}_+$ denotes the non-negative real numbers.

Let $\mathcal{V} = \{1, \dots, N\}$ be the indices of the agents in a multi-agent system, each following the nonlinear dynamics:

$$\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i, \mathbf{u}_i), \quad (1)$$

where $\mathbf{x}_i \in \mathbb{X}_i \subset \mathbb{R}^n$ and $\mathbf{u}_i \in \mathbb{U}_i \subset \mathbb{R}^{m_i}$ represent the state and control input for agent i with $m_i \geq 1$. Without loss of generality, consider $\mathbb{X}_i$ and $\mathbb{U}_i$ to be compact sets containing the origin and let $f_i : \mathbb{X}_i \times \mathbb{U}_i \rightarrow \mathbb{R}^n$ be locally Lipschitz continuous on $\mathbb{X}_i$ and $\mathbb{U}_i$. Furthermore, let $\mathbf{u}_i : \mathbb{R}_+ \rightarrow \mathbb{U}_i \in \mathcal{U}_i$ and $\mathbf{x}_i : \mathbb{R}_+ \rightarrow \mathbb{X}_i \in \mathcal{X}_i$ represent the input and state *signals* for each agent i where $\mathcal{U}_i$ is the set of Lipschitz continuous input signals and $\mathcal{X}_i$ is the set of absolutely continuous solutions of (1) under the input signals from $\mathcal{U}_i$. The complete state of the MAS is then given by $\mathbf{x} := [\mathbf{x}_i]_{i \in \mathcal{V}} \in \mathbb{X}$, with $\mathbb{X} := \times_{i \in \mathcal{V}} \mathbb{X}_i$, and similarly $\mathbf{x}(t) = [\mathbf{x}_i(t)]_{i \in \mathcal{V}} \in \mathcal{X}$ where $\mathcal{X} := \times_{i \in \mathcal{V}} \mathcal{X}_i$. Let $\mathbf{e}_{ij} := \mathbf{x}_j - \mathbf{x}_i \in \mathbb{X}_{ij}$ represent the relative state vector for each $i, j \in \mathcal{V}$ where $\mathbb{X}_{ij} := \mathbb{X}_j \ominus \mathbb{X}_i$ and let $S \in \mathbb{R}^{n_p \times n}$ be a selection matrix¹ such that $\mathbf{p}_i = S\mathbf{x}_i$, $\forall i \in \mathcal{V}$, represents the position vector of agent i with $n_p \leq n$.

A. Polytopes and their properties

In this subsection, some fundamental properties of polytopes based on [32, Ch. 0-1] and [33, Sec. 3] are revised and adapted to the settings of this work for clarity of presentation. Results not directly available from these references are proven in [34].

Definition 1: ([32, pp. 28]) Given $A \in \mathbb{R}^{m \times n}$, $\mathbf{z} \in \mathbb{R}^m$ and a center $\mathbf{c} \in \mathbb{R}^n$, the set $\mathcal{P}(A, \mathbf{c}, \mathbf{z}) := \{\mathbf{x} \in \mathbb{R}^n \mid A(\mathbf{x} - \mathbf{c}) - \mathbf{z} \leq \mathbf{0}\} \subset \mathbb{R}^n$ is a *polytope* if it is bounded.

The inequality in the definition of $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$ should be interpreted row-wise. Moreover, the centre vector $\mathbf{c}$ is introduced for the convenience of analysis, but the polytope definition

¹A selection matrix applies to select $m \leq n$ unique elements from a vector of dimensions n such that $S \in \mathbb{R}^{m \times n}$ with $S[i, j] \in \{0, 1\}$; $\sum_{j=1}^n S[i, j] = 1$, $\forall i = 1, \dots, m$ and $\sum_{i=1}^m S[i, j] = 1$, $\forall j = 1, \dots, n$.

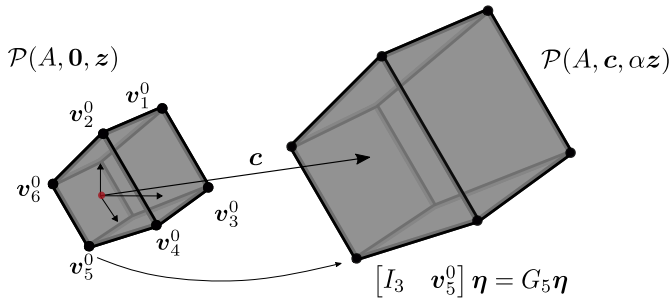


Fig. 1: Graphical representation of similar polytopes. On the left, the polytope $\mathcal{P}(A, \mathbf{0}, \mathbf{z})$ with a red dot representing the origin of a common reference system. On the right, the scaled and shifted polytope $\mathcal{P}(A, \mathbf{c}, \alpha \mathbf{z})$.

$\mathcal{P}(A, \tilde{\mathbf{z}}) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \tilde{\mathbf{z}}\}$ in [32, pp. 28] is equivalent to ours letting $\tilde{\mathbf{z}} = \mathbf{z} + A\mathbf{c}$.

Definition 2: ([32, pp. 4]) For a finite set of points $V = \{\mathbf{v}_k\}_{k=1}^{|V|} \subset \mathbb{R}^n$ with $|V| \geq 1$, the *convex hull* of V is defined as $\text{conv}(V) := \{\sum_{k=1}^{|V|} \lambda_k \mathbf{v}_k | \sum_{k=1}^{|V|} \lambda_k = 1, \lambda_k \geq 0, \forall k = 1, \dots, |V|\}$.

Proposition 1: ([32, Thm. 1.1]) The set $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$ is a polytope if and only if there exists a finite *generator set* V such that $\mathcal{P}(A, \mathbf{c}, \mathbf{z}) = \text{conv}(V) \subset \mathbb{R}^n$. Furthermore, $\mathcal{P}(A, \mathbf{c}, \mathbf{z}) = \{\mathbf{x} \in \mathbb{R}^n | h(\mathbf{x}) \geq 0\}$ where $h(\mathbf{x}) := \langle -(A(\mathbf{x} - \mathbf{c}) - \mathbf{z}) \rangle$.

Proposition 1 establishes three equivalent representations for a polytope $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$. Given $A, \mathbf{z}$ and $\mathbf{c}$, then a generator set V can be obtained algorithmically (see for example [35]), while the functional representation $h(\mathbf{x})$ is obtained at no additional computational cost [32]. While for a given polytope the generator set V is not unique, we define $\nu : \mathcal{P}(A, \mathbf{c}, \mathbf{z}) \mapsto V^*$ as the map between a polytope $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$ and the set of unique generators with minimal cardinality V^* (the set of vertices for $\mathcal{P}$ [32, Proposition 2.2]). We next define when two polytopes are said to be *similar* and what properties can be derived from this similarity relation.

Definition 3: Given a polytope $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$, a scale factor $\alpha > 0$ and a center $\bar{\mathbf{c}} \in \mathbb{R}^n$, then the polytope $\mathcal{P}(A, \mathbf{c} + \bar{\mathbf{c}}, \alpha \mathbf{z})$ is *similar* to $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$.

Thus two polytopes are similar if one is obtained from the other by homogeneous scaling and a translation. Throughout the presentation, the column vector $\boldsymbol{\eta} = [\mathbf{c}, \alpha] \in \mathbb{R}^{n+1}$ compactly represents this similarity transformation. The following proposition relates the generator sets of two similar polytopes.

Proposition 2: ([34]) Let the polytope $\mathcal{P}(A, \mathbf{0}, \mathbf{z})$, the center vector $\mathbf{c} \in \mathbb{R}^n$ and scale $\alpha > 0$ such that $\boldsymbol{\eta} = [\mathbf{c}, \alpha] \in \mathbb{R}^{n+1}$. Moreover, let $V = \nu(\mathcal{P}(A, \mathbf{0}, \mathbf{z}))$ be the generator set, such that $V = \{\mathbf{v}_k\}_{k=1}^{|V|}$, then

$$\nu(\mathcal{P}(A, \mathbf{c}, \alpha \mathbf{z})) = \{G_k \boldsymbol{\eta}\}_{k=1}^{|V|}, \quad (2)$$

where $G_k = [I_n | \mathbf{v}_k] \in \mathbb{R}^{n \times (n+1)}$, $\mathbf{v}_k \in V, \forall k = 1, \dots, |V|$.

In other words, the generators of a polytope $\mathcal{P}(A, \mathbf{c}, \alpha \mathbf{z})$ are directly computed by translating and scaling the generators of the similar polytope $\mathcal{P}(A, \mathbf{0}, \mathbf{z})$ (see Fig. 1). The next proposition extends the result of Proposition 2 to the Minkowski sum of similar polytopes.

Proposition 3: ([34]) Let $\mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}), \forall i \in \mathcal{I}$ be similar

polytopes for some index set $\mathcal{I}$ and $\alpha_i \in \mathbb{R}_+$, then

$$\bigoplus_{i \in \mathcal{I}} \mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}) = \mathcal{P}(A, \sum_{i \in \mathcal{I}} \mathbf{c}_i, \sum_{i \in \mathcal{I}} \alpha_i \mathbf{z}). \quad (3)$$

Moreover, let $\boldsymbol{\eta}_i = [\mathbf{c}_i, \alpha_i]$ and the generator set $V = \nu(\mathcal{P}(A, \mathbf{0}, \mathbf{z})) = \{\mathbf{v}_k\}_{k=1}^{|V|}$, then it holds

$$\nu\left(\bigoplus_{i \in \mathcal{I}} \mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z})\right) = \left\{ \sum_{i \in \mathcal{I}} G_k \boldsymbol{\eta}_i \right\}_{k=1}^{|V|}, \quad (4)$$

with $G_k = [I_n | \mathbf{v}_k] \in \mathbb{R}^{n \times (n+1)}$, $\mathbf{v}_k \in V$.

Next, Proposition 4 provides a set of linear inequalities applicable to check the intersection and inclusion of polytopes.

Proposition 4: ([34]) Let the polytopes $\mathcal{P}(A_1, \mathbf{c}_1, \alpha_1 \mathbf{z}_1)$, $\mathcal{P}(A_2, \mathbf{c}_2, \alpha_2 \mathbf{z}_2)$ such that $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$ and let $\boldsymbol{\eta}_1 = [\mathbf{c}_1, \alpha_1]$ and $\boldsymbol{\eta}_2 = [\mathbf{c}_2, \alpha_2]$. Furthermore, let $V_1 = \nu(\mathcal{P}(A_1, \mathbf{0}, \mathbf{z}_1)) = \{\mathbf{v}_k\}_{k=1}^{|V_1|}$. Then, the inclusion relation $\mathcal{P}(A_1, \mathbf{c}_1, \alpha_1 \mathbf{z}_1) \subseteq \mathcal{P}(A_2, \mathbf{c}_2, \alpha_2 \mathbf{z}_2)$ holds if and only if

$$M_2 \boldsymbol{\eta}_1 - Z_2 \boldsymbol{\eta}_2 \leq \mathbf{0}, \quad (5)$$

where

$$M_2 = \begin{bmatrix} A_2 G_1 \\ \vdots \\ A_2 G_{|V_1|} \end{bmatrix}, \quad Z_2 = \mathbf{1}_{|V_1|} \otimes [A_1 | \mathbf{z}_1] \quad (6)$$

and $G_k = [I_n | \mathbf{v}_k] \in \mathbb{R}^{n \times (n+1)}$, $\mathbf{v}_k \in V_1$. On the other hand, the intersection relation $\mathcal{P}(A_1, \mathbf{c}_1, \alpha_1 \mathbf{z}_1) \cap \mathcal{P}(A_2, \mathbf{c}_2, \alpha_2 \mathbf{z}_2) \neq \emptyset$ holds if and only if

$$A_1 \boldsymbol{\xi} - [A_1 | \mathbf{z}_1] \boldsymbol{\eta}_1 \leq \mathbf{0}, \quad A_2 \boldsymbol{\xi} - [A_2 | \mathbf{z}_2] \boldsymbol{\eta}_2 \leq \mathbf{0}, \quad (7)$$

for some vector $\boldsymbol{\xi} \in \mathbb{R}^n$.

The section is concluded with the following result, whose relevance is clarified in Sec. V-A.

Proposition 5: (From [33, Thm. 3.2]) Let $\mathcal{P}(A, \mathbf{c}, \mathbf{z})$ and the similar polytopes $\mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}), \forall i \in \mathcal{I}$ with $\alpha_i \in \mathbb{R}_+$ and index set $\mathcal{I}$, then

$$\begin{aligned} i) \quad & \bigoplus_{i \in \mathcal{I}} \mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}) = \mathcal{P}(A, \mathbf{c}, \mathbf{z}) \Leftrightarrow \\ & \sum_{i \in \mathcal{I}} \alpha_i = 1 \wedge \bigoplus_{i \in \mathcal{I}} \mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}) \subseteq \mathcal{P}(A, \mathbf{c}, \mathbf{z}), \end{aligned} \quad (8)$$

and

$$ii) \quad \bigoplus_{i \in \mathcal{I}} \mathcal{P}(A, \mathbf{c}_i, \alpha_i \mathbf{z}) \subset \mathcal{P}(A, \mathbf{c}, \mathbf{z}) \Rightarrow \sum_{i \in \mathcal{I}} \alpha_i < 1. \quad (9)$$

Thus Proposition 5 supports the intuition that given a polytope and a Minkowski sum of polytopes similar to the former, then the Minkowski sum is included in the original polytope only if the sum of the scale factors α_i is less than or equal to 1. The relevance of this fact is clarified in Sec. V-A.

B. Signal Temporal Logic

Signal Temporal Logic (STL) is a predicate logic applied to formally define spatio-temporal behaviours (tasks) for state signals $\mathbf{x}(t) \in \mathcal{X}$. Let $h : \mathcal{D} \rightarrow \mathbb{R}$, with $\mathcal{D} \subseteq \mathbb{X}$ be a predicate function and let the boolean-valued predicate $\mu(h(\mathbf{x})) := \begin{cases} \top & \text{if } h(\mathbf{x}) \geq 0 \\ \perp & \text{if } h(\mathbf{x}) < 0 \end{cases}$. Then, the recursively defined STL

grammar applies to construct complex specifications from a set of elementary predicates as

$$\phi := \mu(h(\mathbf{x})) | \neg\phi | \phi_1 \wedge \phi_2 | \phi_1 \vee \phi_2 | F_{[a,b]}\phi | G_{[a,b]}\phi | \phi_1 U_{[a,b]}\phi_2,$$

where G, F and U are the temporal *always*, *eventually* and *until* operators with time interval $[a, b] \subseteq \mathbb{R}_+$, while $\wedge, \vee$ and $\neg$ represent the logical conjunction, disjunction and negation operators. The qualitative semantic rules defining the conditions under which a state signal $\mathbf{x}(t)$ satisfies a task ϕ from time t (written as $(\mathbf{x}(t), t) \models \phi$) were derived in [1, Ch 2.2]. Namely, the semantics relevant for the presentation are :

$$(\mathbf{x}(t), t) \models \mu(h(\mathbf{x}(t))) \Leftrightarrow \mu(h(\mathbf{x}(t))) = \top \quad (10a)$$

$$(\mathbf{x}(t), t) \models \phi_1 \wedge \phi_2 \Leftrightarrow (\mathbf{x}(t), t) \models \phi_1 \wedge (\mathbf{x}(t), t) \models \phi_2 \quad (10b)$$

$$(\mathbf{x}(t), t) \models F_{[a,b]}\phi \Leftrightarrow \exists \tau \in [a, b] \text{ s.t. } (\mathbf{x}, t + \tau) \models \phi \quad (10c)$$

$$(\mathbf{x}(t), t) \models G_{[a,b]}\phi \Leftrightarrow (\mathbf{x}, t + \tau) \models \phi, \forall \tau \in [a, b] \quad (10d)$$

In this work, we deal with a fragment of STL having application in the definition of time-varying missions for multi-agent systems. Namely, let

$$\varphi_i := F_{[a,b]}\mu_i | G_{[a,b]}\mu_i, \quad (11a)$$

$$\varphi_{ij} := F_{[a,b]}\mu_{ij} | G_{[a,b]}\mu_{ij}, \quad (11b)$$

$$\phi_{ij} := \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k, \quad \phi_i := \bigwedge_{k=1}^{K_i} \varphi_i^k, \quad (11c)$$

for some $K_i, K_{ij} \in \mathbb{N}$, such that the boolean-valued predicates μ_i, μ_{ij} take the form

$$\mu_i := \begin{cases} \top & \text{if } h_i(\mathbf{x}_i) \geq 0 \\ \perp & \text{if } h_i(\mathbf{x}_i) < 0, \end{cases} \quad (12a)$$

$$\mu_{ij} := \begin{cases} \top & \text{if } h_{ij}(\mathbf{e}_{ij}) \geq 0 \\ \perp & \text{if } h_{ij}(\mathbf{e}_{ij}) < 0. \end{cases} \quad (12b)$$

where

$$h_i(\mathbf{x}_i) = \langle -(A_i(\mathbf{x}_i - \mathbf{c}_i) - \mathbf{z}_i) \rangle, \quad (13a)$$

$$h_{ij}(\mathbf{e}_{ij}) = \langle -(A_{ij}(\mathbf{e}_{ij} - \mathbf{c}_{ij}) - \mathbf{z}_{ij}) \rangle. \quad (13b)$$

for some centres $\mathbf{c}_i, \mathbf{c}_{ij} \in \mathbb{R}^n$. It is assumed that the sets $\mathcal{B}_i, \mathcal{B}_{ij}$ for $h_i(\mathbf{x}_i), h_{ij}(\mathbf{e}_{ij})$ are bounded, thus taking the form

$$\begin{aligned} \mathcal{B}_i &:= \{\mathbf{x}_i \in \mathbb{X}_i | h_i(\mathbf{x}_i) \geq 0\} = \mathcal{P}(A_i, \mathbf{c}_i, \mathbf{z}_i), \\ \mathcal{B}_{ij} &:= \{\mathbf{e}_{ij} \in \mathbb{X}_{ij} | h_{ij}(\mathbf{e}_{ij}) \geq 0\} = \mathcal{P}(A_{ij}, \mathbf{c}_{ij}, \mathbf{z}_{ij}). \end{aligned} \quad (14)$$

We refer to tasks of type (11a) and (11b) as *independent* tasks and *collaborative* tasks, respectively. Moreover, we refer to $\mathcal{B}_{ij}$ and $\mathcal{B}_i$ as the *truth sets* for tasks φ_{ij} and φ_i respectively since, by definition, these are the sets where $\mu_{ij} = \top$ and $\mu_i = \top$. In other words, independent and collaborative tasks enforce the single state and relative state of a pair of agents to lie within a bounded polytope for a time interval specified by the temporal operator included in the task. Note that recurrent tasks with period T can be encoded in the fragment as conjunctions of the form $\bigwedge_k G_{[a+kT, b+kT]}\mu$. Safety specifications (e.g., collision

avoidance) are not part (11), as we assume safety is enforced by the controller applied to satisfy the tasks.

Concerning the semantics, *quantitative* semantics are applied in this work instead of the qualitative semantics in (10). Namely, let the *robustness* measure $\rho^\phi(\mathbf{x}(t), t) : \mathcal{X} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of task ϕ for a state signal $\mathbf{x}(t)$ be defined as

$$\rho^\mu(\mathbf{x}(t), t) = h(\mathbf{x}(t)), \quad (15a)$$

$$\rho^{F_{[a,b]}\phi}(\mathbf{x}(t), t) = \max_{\tau \in [a,b]} \rho^\phi(\mathbf{x}, t + \tau), \quad (15b)$$

$$\rho^{G_{[a,b]}\phi}(\mathbf{x}(t), t) = \min_{\tau \in [a,b]} \rho^\phi(\mathbf{x}, t + \tau), \quad (15c)$$

$$\rho^{\phi_1 \wedge \phi_2}(\mathbf{x}(t), t) = \min(\rho^{\phi_1}(\mathbf{x}(t), t), \rho^{\phi_2}(\mathbf{x}(t), t)), \quad (15d)$$

then considering, for convenience, the initial time $t = 0$, we have the satisfaction relation $\rho^\phi(\mathbf{x}(t), 0) \geq 0 \Leftrightarrow (\mathbf{x}(t), 0) \models \phi$ (with double implication holding due to absence of the negation operator in fragment (11)) [36, Sec. 2.2] [37].

C. Communication and task graphs

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denote an undirected graph over the agents in $\mathcal{V}$ with undirected edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. Furthermore, let the neighbour set for agent i be defined as $\mathcal{N}(i) = \{j | (i, j) \in \mathcal{E} \wedge i \neq j\}$. In the following, we consider two main graphs: the *communication graph* $\mathcal{G}_c(\mathcal{V}, \mathcal{E}_c)$ and the *task graph* $\mathcal{G}_\psi(\mathcal{V}, \mathcal{E}_\psi)$, such that

$$\mathcal{E}_c = \{(i, j) | \|\mathbf{p}_i - \mathbf{p}_j\| \leq r_c\} \quad (16a)$$

$$\mathcal{E}_\psi = \{(i, j) | \exists \phi_{ij}\} \cup \{(i, i) | \exists \phi_i\}, \quad (16b)$$

where $r_c > 0$ is the communication radius. Note that by (16b) an edge $(i, j) \in \mathcal{E}_\psi$ exists whether a collaborative task ϕ_{ij} as per (11c) exists over the relative state $\mathbf{e}_{ij}$, while self-loops $(i, i) \in \mathcal{E}_\psi$ are induced by independent tasks ϕ_i . Let $\mathcal{N}_c(i)$ and $\mathcal{N}_\psi(i)$ be the neighbour sets for the communication and task graphs respectively. When clear from the context, the shorthand notation $\mathcal{G}_c$ and $\mathcal{G}_\psi$ applies. In general, the communication graph is time-varying according to our definition since the position of the agents changes over time. However, we assume a supervisory controller enforcing the connectivity for each edge $(i, j) \in \mathcal{E}_c$ exists, such that the graph $\mathcal{G}_c$ is considered static (time-invariant) and determined at the initial time [38]. The following definition clarifies when a collaborative task is considered consistent with the communication requirements.

Definition 4: (Communication Consistency) Given the communication graph $\mathcal{G}_c$, then the collaborative task φ_{ij} with associated truth set $\mathcal{B}_{ij}$ as per (11b) is *communication consistent* if $(i, j) \in \mathcal{E}_c$ and it holds

$$\mathcal{B}_{ij} \cap \{\mathbf{e}_{ij} \in \mathbb{X}_{ij} | \|\mathbf{p}_j - \mathbf{p}_i\| \leq r_c\} \neq \emptyset. \quad (17)$$

On the other hand, φ_{ij} is *communication inconsistent* if it is not consistent. Moreover, $\mathcal{G}_\psi$ is communication consistent if all collaborative tasks φ_{ij}^k , $\forall (i, j) \in \mathcal{E}_\psi$, $k = 1, \dots, K_{ij}$, are communication consistent.

Relation (17) entails the existence of a nonempty set of relative configurations $\mathbf{e}_{ij} \in \mathbb{X}_{ij}$ whereby the predicate μ_{ij} , associated with task φ_{ij} , takes the value $\mu_{ij} = \top$ while agents i and j are within the communication radius. This ensures the

satisfiability of φ_{ij} subject to the limited-range communication constraint. For brevity, we refer to communication consistent tasks simply as *consistent*.

D. Paths and cycles

For a given graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, let the vector $\pi_i^j \in \mathcal{V}^l$ represent a *directed* path of length l defined as a vector of non-repeated indices in $\mathcal{V}$ such that $\pi_i^j[k] \in \mathcal{V}$, $\forall k = 1, \dots, l$; $\pi_i^j[k] \neq \pi_i^j[k+1]$, $\forall k = 1, \dots, l-1$; $(\pi_i^j[k], \pi_i^j[k+1]) \in \mathcal{E}$ and $(\pi_i^j[1], \pi_i^j[l]) = (i, j)$. Similarly, let $\omega \in \mathcal{V}^l$ represent a cycle such that ω is a path of length l with a single repeated index as $\omega[1] = \omega[l]$. A graph is *acyclic* when it does not contain cycles. Moreover, let $\epsilon : \mathcal{V}^l \rightarrow 2^{\mathcal{E}}$ be a set-valued function yielding the edges along the path π_i^j as $\epsilon(\pi_i^j) = \{(\pi_i^j[k], \pi_i^j[k+1]) \mid k = 1, \dots, l-1\}$, with the same definition extending to cycles ω . Thus, we can write

$$e_{ij} = \sum_{(r,s) \in \epsilon(\pi_i^j)} e_{rs}, \quad (18a)$$

$$0 = \sum_{(r,s) \in \epsilon(\omega)} e_{rs}. \quad (18b)$$

The following assumption over $\mathcal{G}_c$ is further considered.

Assumption 1: The communication graph $\mathcal{G}_c$ is connected and acyclic.

Assumption 1 is an initialization assumption imposing acyclicity of the communication graph $\mathcal{G}_c$. Note that when the communication graph is not naturally acyclic based on the communication range, then we assume that the agents can select a subset of the neighbours $\mathcal{N}_c(i)$ that renders $\mathcal{G}_c$ acyclic by electing a leader agent from which an acyclic graph can be constructed using standard graph search algorithms [39, Ch. 5]. Our previous work in [21], dealing with a similar decomposition, considered the milder assumption of connectivity for $\mathcal{G}_c$. The reason why Assumption 1 is considered here instead, relates to the fact that the convexity and decentralization properties of the task decomposition approach proposed in this work may be lost when cycles are present over $\mathcal{G}_c$. Further details justifying this assumption are given in Sec. V-B.

III. PROBLEM FORMULATION

Consider the situation in which a set of tasks in the form (11) is assigned to a multi-agent system such that a task graph $\mathcal{G}_\psi$ can be derived as per (16b). The *global* task ψ assigned to the system is compactly written as

$$\psi := \bigwedge_{i=1}^N \left(\phi_i \wedge \bigwedge_{j \in \mathcal{N}_\psi(i)} \phi_{ij} \right). \quad (19)$$

Global tasks as per (19) are suitable for defining time-varying relative formations from which complex group behaviours for the MAS can be obtained. The problem we approach is formally stated as follows:

Problem 1: Consider the communication graph $\mathcal{G}_c$, task graph $\mathcal{G}_\psi$, the global task ψ as per (19) with the associated collaborative tasks $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$, $\forall (i, j) \in \mathcal{E}_\psi$ as per (11c).

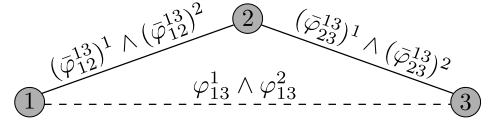


Fig. 2: Decomposition of the inconsistent task $\phi_{13} = \varphi_{13}^1 \wedge \varphi_{13}^2$ over the path $\pi_1^3 = [1, 2, 3]$. Solid lines show the communication graph $\mathcal{G}_c$; the dashed edge (1, 3) is the only one in the task graph $\mathcal{G}_\psi$.

Develop a decentralized algorithm that outputs a global task $\bar{\psi}$ with task graph $\mathcal{G}_{\bar{\psi}}$ in the form of

$$\bar{\psi} := \bigwedge_{i=1}^N \left(\phi_i \wedge \bigwedge_{j \in \mathcal{N}_\psi(i) \cap \mathcal{N}_c(i)} \phi_{ij} \wedge \bigwedge_{j \in \mathcal{N}_\psi(i) \setminus \mathcal{N}_c(i)} \bar{\phi}^{ij} \right), \quad (20)$$

where

$$\bar{\phi}^{ij} := \bigwedge_{k=1}^{K_{ij}} (\bar{\phi}^{ij})^k, \quad (21a)$$

$$(\bar{\phi}^{ij})^k := \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} (\bar{\varphi}_{rs}^{ij})^k, \quad (21b)$$

for all inconsistent edges $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ and with tasks $(\bar{\varphi}_{rs}^{ij})^k$, $\forall (r, s) \in \epsilon(\pi_i^j)$, $\forall k = 1, \dots, K_{ij}$ being communication consistent and as per (11b). Moreover, $\bar{\psi}$ should respect

$$(x(t), t) \models \bar{\psi} \Rightarrow (x(t), t) \models \psi. \quad (22)$$

In other words, by recalling that each inconsistent task is defined as $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$, then the objective in Problem 1 is to find a conjunction of task $(\bar{\phi}^{ij})^k$, as per (21b), replacing the task φ_{ij}^k for all $k = 1, \dots, K_{ij}$. Note that each $(\bar{\phi}^{ij})^k$ is defined as a conjunction of new collaborative tasks $(\bar{\varphi}_{rs}^{ij})^k$ over the edges $(r, s) \in \epsilon(\pi_i^j) \subseteq \mathcal{E}_c$, where π_i^j is a path over $\mathcal{G}_c$ connecting i to j . The following example provides further intuition on the meaning of the proposed decomposition.

Example 1: Consider Figure 2 where two communication edges $(1, 2), (2, 3) \in \mathcal{E}_c$ (solid edges) are applied to decompose the inconsistent task $\phi_{13} = \varphi_{13}^1 \wedge \varphi_{13}^2$ (dashed line). The decomposition of ϕ_{13} yields the task $\bar{\phi}^{13} = (\bar{\phi}^{13})^1 \wedge (\bar{\phi}^{13})^2$ where $(\bar{\phi}^{13})^1 = (\bar{\varphi}_{12}^{13})^1 \wedge (\bar{\varphi}_{23}^{13})^1$ and $(\bar{\phi}^{13})^2 = (\bar{\varphi}_{12}^{13})^2 \wedge (\bar{\varphi}_{23}^{13})^2$, as per (21b). Hence, the conjunctions $(\bar{\phi}^{13})^1$ and $(\bar{\phi}^{13})^2$ replace φ_{13}^1 and φ_{13}^2 , respectively.

Section V-A details the design of tasks $\bar{\phi}^{ij}$. Before that, the next section presents key results to explain how these tasks can be introduced into the task graph $\mathcal{G}_\psi$ without inducing conflicts with other tasks in $\mathcal{G}_\psi$.

IV. CONFLICTING CONJUNCTIONS

The concept of conflicting conjunction is introduced first.

Definition 5: (Conflicting conjunction) The conjunction of collaborative tasks $\bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$ is a *conflicting conjunction* if there does not exist a continuous state signal $x(t)$ for the multi-agent system such that $(x(t), t) \models \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$.

It is desired that any formula $\bar{\psi}$ as per (20), resulting from the decomposition approach, should not induce conflicting conjunctions as this fact would result directly in $\bar{\psi}$ being unsatisfiable. We here present five types of conflicting conjunctions according to Def. 5 which can arise when collaborative tasks are set in conjunction. Due to space limitations,

proofs of the results in this section can be found in [34]. Intuitively, conflicting conjunctions of collaborative tasks in the fragment (11) occur when two or more collaborative tasks feature an intersection in the temporal domain (in terms of the intersection of their time intervals) which does not correspond to an intersection in the spatial domain (in terms of their truth sets). We here propose a more general definition of conflicting conjunction compared to the one in [21]. We conjecture that these are the only five types of conflicting conjunctions over collaborative tasks according to fragment (11). We acknowledge that conflicting conjunctions can also arise among conjunctions of independent and collaborative tasks, but we limit the presentation to conflicts over collaborative tasks. Adding conditions for conflicting conjunctions of independent and collaborative tasks is the subject of future work. In the presentation of the different types of conflicting conjunctions, we consider the general conjunction of tasks $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$ over an edge (i, j) as per (11c), with truth sets $\mathcal{B}_{ij}^k, \forall k = 1, \dots, K_{ij}$ as per (14). Moreover, let the set of indices $\mathcal{I}_G, \mathcal{I}_F$ such that $\mathcal{I}_G \cup \mathcal{I}_F = \{1, \dots, K_{ij}\}$ with

$$\begin{aligned} \mathcal{I}_G &= \{k = 1 \dots K_{ij} \mid \varphi_{ij}^k = G_{[a^k, b^k]} \mu_{ij}^k\}, \\ \mathcal{I}_F &= \{k = 1 \dots K_{ij} \mid \varphi_{ij}^k = F_{[a^k, b^k]} \mu_{ij}^k\}. \end{aligned} \quad (23)$$

While we present Facts 1-3 consecutively, Example 2 provides intuition on their meanings and implications.

Fact 1: (Conflict of type 1) Consider the conjunction of tasks $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$ and let $\mathcal{L}_{ij}$ be a set of subsets of indices in $\mathcal{I}_G$ such that

$$\mathcal{L}_{ij} := \{L \in 2^{\mathcal{I}_G} \setminus \{\emptyset\} \mid \bigcap_{l \in L} [a^l, b^l] \neq \emptyset\}. \quad (24)$$

If there exists a set of indices $L \in \mathcal{L}_{ij}$ such that $\bigcap_{l \in L} [a^l, b^l] \neq \emptyset$ and $\bigcap_{l \in L} \mathcal{B}_{ij}^l = \emptyset$ then ϕ_{ij} is a conflicting conjunction.

Intuitively, conflicting conjunctions as per Fact 1 occur any time a set of collaborative tasks, featuring an always operator, also feature a non-empty time interval intersection. Then, for any time in this time interval intersection, the relative state e_{ij} should lie inside the truth set $\mathcal{B}_{ij}$ of all the considered tasks, which is possible only if the truth sets are intersecting as well. The next two facts involve conflicting conjunctions of tasks with mixed temporal operators.

Fact 2: (Conflict of type 2) Consider the conjunction of tasks $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$. For a given index $d \in \mathcal{I}_F$, let $\mathcal{C}_{ij}(d)$ be the set of subsets of indices in $\mathcal{I}_G$ such that

$$\mathcal{C}_{ij}(d) = \{C \in 2^{\mathcal{I}_G} \setminus \{\emptyset\} \mid [a^d, b^d] \subseteq \bigcup_{l \in C} [a^l, b^l] \wedge [a^l, b^l] \cap [a^d, b^d] \neq \emptyset, \forall l \in C\}. \quad (25)$$

Then, if there exists $d \in \mathcal{I}_F$ and a set of indices $C \in \mathcal{C}_{ij}(d)$ such that $\mathcal{B}_{ij}^d \cap \bigcap_{l \in C} \mathcal{B}_{ij}^l = \emptyset, \forall l \in C$ then ϕ_{ij} is a conflicting conjunction.

Fact 3: (Conflict of type 3) Consider the conjunction of tasks $\phi_{ij} = \bigwedge_{k=1}^{K_{ij}} \varphi_{ij}^k$. For a given index $d \in \mathcal{I}_F$, let $\mathcal{D}_{ij}(d)$ be a set of subsets of indices in $\mathcal{I}_G$ such that

$$\mathcal{D}_{ij}(d) = \{D \in 2^{\mathcal{I}_G} \setminus \{\emptyset\} \mid [a^d, b^d] \subseteq \bigcap_{l \in D} [a^l, b^l]\}. \quad (26)$$

Then, if there exists $d \in \mathcal{I}_F$ and a set of indices $D \in \mathcal{D}_{ij}(d)$ such that $\mathcal{B}_{ij}^d \cap \bigcap_{l \in D} \mathcal{B}_{ij}^l = \emptyset$, then ϕ_{ij} is a conflicting conjunction.

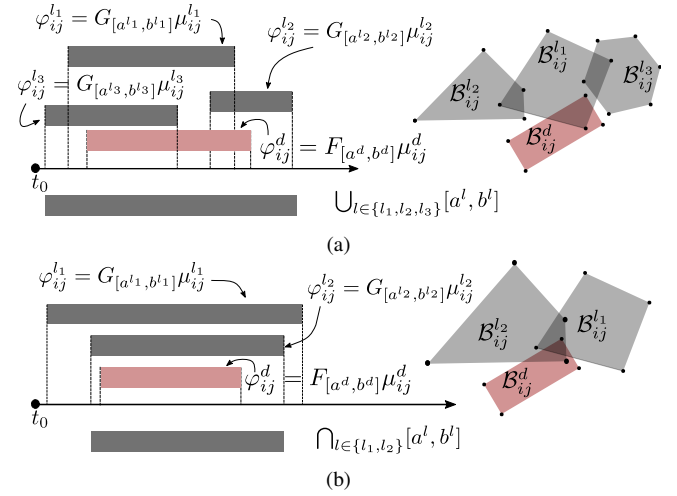


Fig. 3: Example of two conjunctions of tasks that do not induce conflicting tasks as per Fact 2. Detailed explanation in Example 2.

We provide some intuitions over the results of Facts 2 and 3. Consider a collaborative task $\varphi_{ij}^d = F_{[a^d, b^d]} \mu_{ij}^d$, with $d \in \mathcal{I}_F$, such that its satisfaction can potentially occur at any time in the interval $[a^d, b^d]$ as per (10c). Conflicts as per Fact 2 can then occur if there exists a set of tasks $\varphi_{ij}^l = G_{[a^l, b^l]} \mu_{ij}^l$ such that the union $\bigcup_l [a^l, b^l]$ is covering the interval $[a^d, b^d]$ in the sense that $\bigcup_l [a^l, b^l] \supseteq [a^d, b^d]$. In this case, at least one of the tasks φ_{ij}^l must necessarily be satisfied together with φ_{ij}^d . On the other hand, conflicts as per Fact 3 occur when the time interval intersection of the tasks φ_{ij}^l is covering $[a^d, b^d]$ in the sense that $\bigcap_l [a^l, b^l] \supseteq [a^d, b^d]$. In this case, there exists at least one time instant $\tau \in [a^d, b^d]$ such that all tasks $\varphi_{ij}^l = G_{[a^l, b^l]} \mu_{ij}^l$ must necessarily be satisfied together with $\varphi_{ij}^d = F_{[a^d, b^d]} \mu_{ij}^d$. An example is provided to support these proposed intuitions.

Example 2: Consider Figure 3a where a conjunction of four tasks $\phi_{ij} = \varphi_{ij}^{l_1} \wedge \varphi_{ij}^{l_2} \wedge \varphi_{ij}^{l_3} \wedge \varphi_{ij}^d$ is represented with $\varphi_{ij}^l = G_{[a^l, b^l]} \mu_{ij}^l, \forall l \in \{l_1, l_2, l_3\}$ and $\varphi_{ij}^d = F_{[a^d, b^d]} \mu_{ij}^d$. The left of Figure 3a shows the time interval intersection for the tasks, while the truth sets intersection is given on the right. This conjunction does not suffer from conflicting conjunctions as per Fact 1, 2, or 3. Indeed, concerning conflicts of type 1, we have that $\mathcal{L}_{ij} = \{\{l_1, l_2\}, \{l_1, l_3\}, \{l_2\}, \{l_3\}\}$. First we note that that $[a^l, b^l] \neq \emptyset$ and $\mathcal{B}_{ij}^l \neq \emptyset$ for all $l \in \{l_1, l_2, l_3\}$, thus avoiding conflicts as per type 1 over the singleton sets in $\mathcal{L}_{ij}$. Moreover, $[a^{l_1}, b^{l_1}] \cap [a^{l_2}, b^{l_2}] \neq \emptyset$ and $[a^{l_1}, b^{l_1}] \cap [a^{l_3}, b^{l_3}] \neq \emptyset$, while at the same time $\mathcal{B}_{ij}^{l_1} \cap \mathcal{B}_{ij}^{l_2} \neq \emptyset$ and $\mathcal{B}_{ij}^{l_1} \cap \mathcal{B}_{ij}^{l_3} \neq \emptyset$ (Figure 3a on the right). Thus conflicts of type 1 also do not arise for $\{l_1, l_2\}, \{l_1, l_3\} \in \mathcal{L}_{ij}$. Passing to conflicts of type 2 we have $\mathcal{C}_{ij}(d) = \{\{l_1, l_2, l_3\}\}$ as per (25). From Figure 3a it is shown that $\mathcal{B}_{ij}^{l_1} \cap \mathcal{B}_{ij}^{l_2} \cap \mathcal{B}_{ij}^{l_3} \neq \emptyset$ thus avoiding the conflict of type 2. Finally, turning to conflicts of type 3, we have that $\mathcal{D}_{ij} = \emptyset$ such that no conflicts of this type case arise. Consider now Figure 3b where conjunction of three tasks $\phi_{ij} = \varphi_{ij}^{l_1} \wedge \varphi_{ij}^{l_2} \wedge \varphi_{ij}^d$ is represented in the same fashion as Figure 3a. Again $\{l_1, l_2\} = \mathcal{I}_G$ while $\{d\} \in \mathcal{I}_F$. In this case, we focus on conflict of type 3, while the absence of conflicts of type 1, 2 can be verified as in the previous example. Then we have $\mathcal{D}_{ij}(d) = \{\{l_1\}, \{l_2\}, \{l_1, l_2\}\}$, as per (26), since

the time intervals of the tasks $\varphi_{ij}^{l_1}$ and $\varphi_{ij}^{l_2}$, as well as their intersection, contain the time interval of the task φ_{ij}^d . Since the intersection $\mathcal{B}_{ij}^{l_1} \cap \mathcal{B}_{ij}^{l_2} \cap \mathcal{B}_{ij}^d$ is not empty (on the right in 3b), then conflicting conjunctions of type 3 do not arise.

The next two conflicting conjunctions are defined over cycles of tasks in $\mathcal{G}_\psi$ rather than on a single edge. Namely, consider a cycle ω over the task $\mathcal{G}_\psi$ and assume, for simplicity, that for each edge $(r, s) \in \epsilon(\omega)$ a single collaborative task $\varphi_{rs} = \varphi_{rs}$ is defined. Then, the global task ψ contains the conjunction of tasks $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$ and conflicting conjunction can arise if the cycle closure relation (18b) can not be satisfied together with the conjunction of tasks $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$. This intuition is formalised in the following two facts.

Fact 4: (Conflict of type 4) Consider a cycle ω over $\mathcal{G}_\psi$ and the conjunction of tasks $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$ defined over ω such that, with a slight abuse of notation, $\varphi_{rs} = G_{[a_{rs}, b_{rs}]} \mu_{rs}$, $\forall (r, s) \in \epsilon(\omega)$ with respective truth sets $\mathcal{B}_{rs}$, $\forall (r, s) \in \epsilon(\omega)$ as per (14). If $\bigcap_{(r,s) \in \epsilon(\omega)} [a_{rs}, b_{rs}] \neq \emptyset$ and

$$0 \notin \bigoplus_{(r,s) \in \epsilon(\omega)} \mathcal{B}_{rs}, \quad (27)$$

then $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$ is a conflicting conjunction.

Fact 5: (Conflict of type 5) Consider a cycle ω over $\mathcal{G}_\psi$ and the conjunction of tasks $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$ defined over ω with respective truth sets $\mathcal{B}_{rs}$, $\forall (r, s) \in \epsilon(\omega)$ as per (14). Furthermore, consider the single edge path $\pi_i^p = [i, p]$ and the path π_p^i such that $\omega = [\pi_i^p, \pi_p^i]$. Let, without the loss of generality, $\varphi_{ip} = F_{[a_{ip}, b_{ip}]} \mu_{ip}$ and $\varphi_{rs} = G_{[a_{rs}, b_{rs}]} \mu_{rs}$, $\forall (r, s) \in \epsilon(\pi_p^i)$. If it holds that $\bigcap_{(r,s) \in \epsilon(\pi_p^i)} [a_{rs}, b_{rs}] \supseteq [a_{ip}, b_{ip}]$ and

$$0 \notin \mathcal{B}_{ip} \oplus \bigoplus_{(r,s) \in \epsilon(\pi_p^i)} \mathcal{B}_{rs} \quad (28)$$

then $\bigwedge_{(r,s) \in \epsilon(\omega)} \varphi_{rs}$ is a conflicting conjunction.

While we provided the statement for conflicting conjunction of type 4 and 5, we show in Section V-B that these can not arise for the proposed decomposition approach due to the acyclicity assumption on $\mathcal{G}_c$. We choose to report them as they represent an independent contribution to future work in this direction. The following assumption is considered further

Assumption 2: The original task graph $\mathcal{G}_\psi$ does not suffer from any conflicting conjunction as per Facts 1-5.

This is a reasonable assumption over $\mathcal{G}_\psi$ as otherwise the task ψ would be unsatisfiable even before the decomposition. Section V-B details how the proposed decomposition approach preserves the conflict-free property passing from $\mathcal{G}_\psi$ to $\mathcal{G}_\psi$.

V. TASK DECOMPOSITION

In this section, the decomposition approach to obtain tasks $\bar{\varphi}^{ij}$ from the inconsistent tasks ϕ_{ij} defined over the edges $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ is presented as per Problem 1. For the sake of clarity and to reduce the burden of notation, we present our results assuming that all the inconsistent tasks ϕ_{ij} do not contain conjunctions such that simply $\phi_{ij} = \varphi_{ij}$. Hence, by dropping the index k in (21) and replacing (21b) in (21a), we get that $\phi_{ij} = \varphi_{ij}$ is decomposed into the task $\bar{\varphi}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} \bar{\varphi}_{rs}^{ij}$. Note that this simplifying assumption is not required from our approach but is nonetheless introduced for

the sake of clarity. Indeed, the same approach developed to decompose a single collaborative task φ_{ij} into an appropriate task $\bar{\varphi}^{ij}$ as per (21b), can be repeated for any index $k \in \{1, \dots, K_{ij}\}$ in the conjunction. With these considerations, in Sec. V-A the first main decomposition result is provided in the form of Lemma 1, where a procedure to define the task $\bar{\varphi}^{ij}$ from an inconsistent task φ_{ij} is derived. The task $\bar{\varphi}^{ij}$ resulting from this decomposition is a function of a set of parameters that are later shown to be computable via convex optimization. Moreover, Sec. V-B and V-C detail how convex constraints over the former parameters are enforced to avoid conflicting conjunctions and guarantee communication consistency.

A. Parametric tasks for decomposition

Consider the communication inconsistent task $\phi_{ij} = \varphi_{ij}$ with $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$, where $h_{ij} = \langle -(A_{ij}(e_{ij} - c_{ij}) - z_{ij}) \rangle$ is the predicate function associated with φ_{ij} and $\mathcal{B}_{ij} = \mathcal{P}(A_{ij}, c_{ij}, z_{ij})$ is the associated truth set as in (13)-(14). Furthermore, consider the family of parametric predicate functions

$$\bar{h}_{rs}^{ij}(e_{rs}, \eta_{rs}^{ij}) := \langle -(A_{ij}(e_{rs} - c_{rs}^{ij}) - \alpha_{rs}^{ij} z_{ij}) \rangle, \quad (29a)$$

$$\bar{\mathcal{B}}_{rs}^{ij}(\eta_{rs}^{ij}) := \{e_{rs} \in \mathbb{X}_{rs} \mid \bar{h}_{rs}^{ij}(e_{rs}, \eta_{rs}^{ij}) \geq 0\}, \quad (29b)$$

such that

$$\bar{\mu}_{rs}^{ij} := \begin{cases} \top & \text{if } \bar{h}_{rs}^{ij}(e_{rs}, \eta_{rs}^{ij}) \geq 0 \\ \perp & \text{if } \bar{h}_{rs}^{ij}(e_{rs}, \eta_{rs}^{ij}) < 0, \end{cases} \quad (30)$$

where $\eta_{rs}^{ij} = [c_{rs}^{ij}, \alpha_{rs}^{ij}]$ is a free parameter vector that needs to be optimised. The parameter vector η_{rs}^{ij} intuitively defines the position and the scale of the polytope $\mathcal{P}(A_{ij}, c_{rs}^{ij}, \alpha_{rs}^{ij} z_{ij}) = \bar{\mathcal{B}}_{rs}^{ij}(\eta_{rs}^{ij})$ which is similar to $\mathcal{P}(A_{ij}, c_{ij}, z_{ij})$.

From the family of parametric predicate functions in (29) a decomposition of the inconsistent task φ_{ij} is obtained by a two-step procedure: 1) define a path of agents π_i^j from i to j through the communication graph $\mathcal{G}_c$; 2) define a set of *parametric* tasks $\bar{\varphi}_{rs}^{ij} \forall (r, s) \in \epsilon(\pi_i^j)$ with corresponding predicate functions $\bar{h}_{rs}^{ij}$ as per (29) such that $(x(t), 0) \models \bar{\varphi}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} \bar{\varphi}_{rs}^{ij} \Rightarrow (x(t), 0) \models \varphi_{ij}$. The solution to 1) is well studied in the literature so that we assume a path π_i^j between any pair $i, j \in \mathcal{V}$ can be found efficiently [39, Ch. 2]. On the other hand, a solution to step 2) is provided by Lemma 1.

Lemma 1: Consider a collaborative task φ_{ij} defined over the inconsistent edge $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$, with associated truth set $\mathcal{B}_{ij} = \mathcal{P}(A_{ij}, c_{ij}, z_{ij})$ as per (14). Moreover, consider the path π_i^j through the communication graph $\mathcal{G}_c$ and $\bar{\varphi}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} \bar{\varphi}_{rs}^{ij}$ as per (21) such that

$$\bar{\varphi}_{rs}^{ij} = \begin{cases} F_{[\bar{t}, \bar{t}]} \bar{\mu}_{rs}^{ij} & \text{if } \varphi_{ij} = F_{[a,b]} \mu_{ij}, \text{ s.t } \bar{t} \in [a, b] \\ G_{[a,b]} \bar{\mu}_{rs}^{ij} & \text{if } \varphi_{ij} = G_{[a,b]} \mu_{ij}, \end{cases} \quad (31)$$

where $\bar{\mu}_{rs}^{ij}$, $\bar{h}_{rs}^{ij}(e_{rs}, \eta_{rs}^{ij})$, $\bar{\mathcal{B}}_{rs}^{ij}(\eta_{rs}^{ij})$ are as per (29)-(30). If for each $(r, s) \in \epsilon(\pi_i^j)$ the tasks $\bar{\varphi}_{rs}^{ij}$ are defined according to (31) and there exists parameters η_{rs}^{ij} , $\forall (r, s) \in \epsilon(\pi_i^j)$ such that

$$\bigoplus_{(r,s) \in \epsilon(\pi_i^j)} \bar{\mathcal{B}}_{rs}^{ij}(\eta_{rs}^{ij}) \subseteq \mathcal{B}_{ij}, \quad (32)$$

then $(\mathbf{x}(t), 0) \models \bar{\phi}^{ij} \Rightarrow (\mathbf{x}(t), 0) \models \varphi_{ij}$.

Proof: We prove the lemma for $\varphi_{ij} := F_{[a,b]}\mu_{ij}$ while the case of $\varphi_{ij} := G_{[a,b]}\mu_{ij}$ follows a similar reasoning. We omit the dependency of $\bar{h}_{rs}^{ij}, \bar{B}_{rs}^{ij}$ with respect to η_{rs}^{ij} to reduce the notation. Given the path π_i^j over $\mathcal{G}_c$, then $\bar{\phi}^{ij}$ is defined according to (31) as

$$\bar{\phi}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} \bar{\varphi}_{rs}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} F_{[\bar{t}, \bar{t}]} \bar{\mu}_{rs}^{ij},$$

where $[\bar{t}, \bar{t}] \subseteq [a, b]$. By (15d) it holds $\rho^{\bar{\phi}^{ij}}(\mathbf{x}(t), 0) = \min_{(r,s) \in \epsilon(\pi_i^j)} \{\rho^{\bar{\varphi}_{rs}^{ij}}(\mathbf{x}(t), 0)\} \geq 0 \Leftrightarrow (\mathbf{x}(t), 0) \models \bar{\phi}^{ij}$. Since, $\bar{\varphi}_{rs}^{ij} = F_{[\bar{t}, \bar{t}]} \bar{\mu}_{rs}^{ij}$ for all $(r, s) \in \epsilon(\pi_i^j)$, then by (15b) it holds, for each $(r, s) \in \epsilon(\pi_i^j)$, that $\rho^{F_{[\bar{t}, \bar{t}]} \bar{\mu}_{rs}^{ij}}(\mathbf{x}(t), 0) \geq 0 \Leftrightarrow \max_{t \in [\bar{t}, \bar{t}]} \bar{h}_{rs}^{ij}(e_{rs}(t)) \geq 0 \Rightarrow \bar{h}_{rs}^{ij}(e_{rs}(\bar{t})) \geq 0$. Thus, by (29b), $e_{rs}(\bar{t}) \in \bar{B}_{rs}^{ij}$, $\forall (r, s) \in \epsilon(\pi_i^j)$. On the other hand, from (18a) and (32) it holds at time $\bar{t}$ that

$$e_{ij}(\bar{t}) \stackrel{(18a)}{=} \sum_{(r,s) \in \epsilon(\pi_i^j)} e_{rs}(\bar{t}) \in \bigoplus_{(r,s) \in \epsilon(\pi_i^j)} \bar{B}_{rs}^{ij} \stackrel{(32)}{\subseteq} \mathcal{B}_{ij}.$$

Thus, for every satisfying trajectory $(\mathbf{x}(t), 0) \models \bar{\phi}^{ij}$ it must hold $e_{ij}(\bar{t}) \in \mathcal{B}_{ij} \Rightarrow h_{ij}(e_{ij}(\bar{t})) \geq 0$. Since $\bar{t} \in [a, b]$, then by (15b) it holds $\rho^{\varphi_{ij}}(\mathbf{x}(t), 0) = \max_{t \in [a, b]} (h_{ij}(e_{ij}(t))) \geq h_{ij}(e_{ij}(\bar{t})) \geq 0 \Rightarrow (\mathbf{x}(t), 0) \models \varphi_{ij}$, concluding the proof. ■

The following clarifying example is provided to build a graphical intuition of the result in Lemma 1.

Example 3: Figure 4a shows a system of 6 drones with states $\mathbf{x}_i = \mathbf{p}_i$ being the positions of each drone, while on the left in Figure 4b the task graph $\mathcal{G}_\psi$ (dashed lines) and a communication graph $\mathcal{G}_c$ (solid lines) assigned to the agents are shown. The edge $(1, 4) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ is an inconsistent edge that requires decomposition through $\pi_1^4 = [1, 2, 3, 4]$ such that a new task $\bar{\phi}^{14} = \bar{\varphi}_{12}^{14} \wedge \bar{\varphi}_{23}^{14} \wedge \bar{\varphi}_{34}^{14}$ is introduced as per Lemma 1. The new task graph resulting from the decomposition is then $\mathcal{G}_{\bar{\psi}}$ (on the right in Figure 4b). Figure 4a also shows the similar truth sets $\bar{B}_{rs}^{14}$, $\forall (r, s) \in \epsilon(\pi_1^4) = \{(1, 2), (2, 3), (3, 4)\}$ (gray rectangles) and $\mathcal{B}_{14}$ (red rectangle). The inclusions relation (32) is graphically understood by letting the Minkowski sum $\bigoplus_{(r,s) \in \epsilon(\pi_1^4)} \bar{B}_{rs}^{14}$ to be a subset of $\mathcal{B}_{14}$.

Note that when an inconsistent task φ_{ij} featuring the eventually operator F has to be decomposed, then the tasks $\bar{\varphi}_{rs}^{ij}$, $\forall (r, s) \in \epsilon(\pi_i^j)$ are necessarily synchronized over a common time instant $\bar{t}$ as specified by (31). This synchronization is needed for the satisfaction of the original task φ_{ij} , although $\bar{t}$ can be chosen arbitrarily within $[a, b]$. Concerning the inclusions relation (32), it is known from Prop. 3 and Prop. 4 that (32) can be efficiently verified by a set of linear inequalities. Namely, consider again the truth set $\mathcal{B}_{ij} = \mathcal{P}(A_{ij}, \mathbf{c}_{ij}, \mathbf{z}_{ij})$ for the inconsistent task φ_{ij} and let the decomposition path π_i^j . Furthermore, let the similar polytopes $\bar{B}_{rs}^{ij} = \mathcal{P}(A_{ij}, \mathbf{c}_{rs}^{ij}, \alpha_{rs}^{ij} \mathbf{z}_{ij})$, $\forall (r, s) \in \epsilon(\pi_i^j)$ derived from the parametric task $\bar{\varphi}_{rs}^{ij}$ as per Lemma 1 with parameters $\eta_{rs}^{ij} = [\mathbf{c}_{rs}^{ij}, \alpha_{rs}^{ij}]$. By recalling the results of Prop. 3 and Prop. 4, and by letting the generator set $V_{ij} = \nu(\mathcal{P}(A_{ij}, \mathbf{0}, \mathbf{z}_{ij}))$, then the inclusion relation (32) is verified if it holds

$$\sum_{(r,s) \in \epsilon(\pi_i^j)} \left(M_{ij} \eta_{rs}^{ij} - \frac{Z_{ij}}{|\epsilon(\pi_i^j)|} \begin{bmatrix} \mathbf{c}_{ij} \\ 1 \end{bmatrix} \right) \leq \mathbf{0}, \quad (33)$$

where

$$M_{ij} = \begin{bmatrix} A_{ij} G_1 \\ \vdots \\ A_{ij} G_{|V_{ij}|} \end{bmatrix}, \quad Z_{ij} = \mathbf{1}_{|V_{ij}|} \otimes [A_{ij} \mid \mathbf{z}_{ij}], \quad (34)$$

and $G_k = [I_n \mid \mathbf{v}_k] \in \mathbb{R}^{n \times (n+1)}$, $\mathbf{v}_k \in V_{ij}$, $\forall k = 1, \dots, |V_{ij}|$. From Prop. 5, if (32) holds, then also

$$\sum_{(r,s) \in \epsilon(\pi_i^j)} \alpha_{rs}^{ij} \leq 1, \quad (35)$$

where (35) holds with equality if (32) does. We refer to the left-hand side of (35) as the decomposition accuracy of the decomposed task φ_{ij} . We show in Section VI that maximising the decomposition accuracy on the right-hand side of (35) is the main objective of our decomposition approach

Hereafter, it is assumed that all the inconsistent tasks have been decomposed using a conjunction of parametric tasks as per Lemma 1. Next, in Section V-B and V-C, we provide a set of constraints that are imposed over the parameters η_{rs}^{ij} such that the respective tasks $\bar{\varphi}_{rs}^{ij}$ are communication consistent and such that the tasks $\bar{\varphi}_{rs}^{ij}$ do not cause conflicts as per Facts 1-5.

B. Handling conflicting conjunctions

In this section, it is clarified how conflicting conjunctions as per Facts 1-5 might arise during decomposition and how to resolve them by appropriately constraining the parameters η_{rs}^{ij} associated with each $\bar{\varphi}_{rs}^{ij}$. Namely, we analyze how the consistent and conflict-free collaborative tasks ϕ_{rs} for all $(r, s) \in \mathcal{E}_c \cap \mathcal{E}_\psi$, that were in place *before* the decomposition, are modified by the introduction of the new tasks $\bar{\varphi}_{rs}^{ij}$, $\forall (i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ resulting from the decomposition, and how the consistency and conflict-free properties for these can be preserved. As in the previous sections, it is assumed that inconsistent tasks are without conjunctions as $\phi_{ij} = \varphi_{ij}$. A

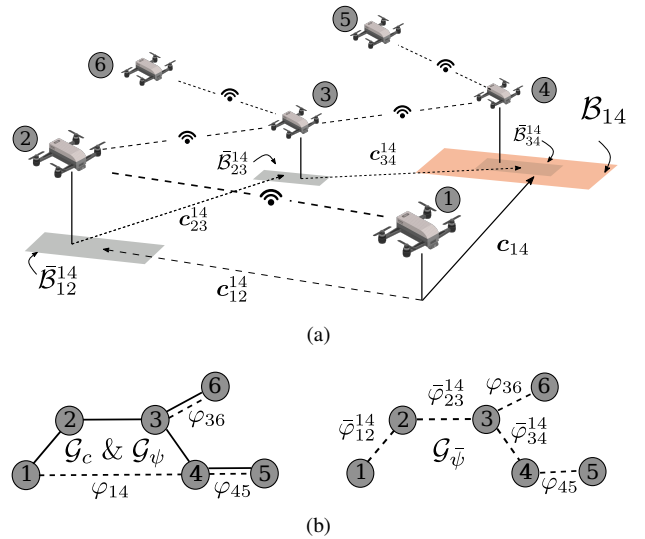


Fig. 4: (a) A team of 6 drones with respective communication graph. (b)(Left) Initial task graph $\mathcal{G}_\psi$ (dashed line) and communication graph $\mathcal{G}_c$ (solid line) of the system. (Right) final task graph $\mathcal{G}_{\bar{\psi}}$ after applying the proposed decomposition.

change of notation is required at this point of the presentation and applies hereafter. Namely, let the sets

$$\mathcal{E}_\pi := \bigcup_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} \epsilon(\pi_i^j), \quad (36a)$$

$$\Pi_{rs} := \{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c \mid (r,s) \in \epsilon(\pi_i^j)\}, \quad (36b)$$

where the set $\mathcal{E}_\pi \subseteq \mathcal{E}_c$ contains the edges in $\mathcal{G}_c$ that are used for the decomposition of all the inconsistent tasks ϕ_{ij} with $(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$, while $\Pi_{rs} \subseteq \mathcal{E}_\psi \setminus \mathcal{E}_c$ contains the inconsistent edges $(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ such that the decomposition path π_i^j is passing through $(r,s) \in \mathcal{E}_\pi \subseteq \mathcal{E}_c$. Moreover, let the collaborative task $\phi_{rs} := \bigwedge_{k=1}^{K_{rs}} \varphi_{rs}^k$, as per (11c), be defined over $(r,s) \in \mathcal{E}_\pi$ **before** the decomposition. Then, **after** the decomposition of the inconsistent tasks φ_{ij} , $\forall (i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$, an additional conjunction of parametric tasks given by $\bigwedge_{(i,j) \in \Pi_{rs}} \bar{\varphi}_{rs}^{ij}$ is included over the edge $(r,s) \in \mathcal{E}_\pi$. Let then

$$\bar{\mathcal{K}}_{rs} = \{K_{rs} + 1, \dots, K_{rs} + |\Pi_{rs}|\}, \quad (37)$$

and let the one-to-one index mapping

$$y_{\Pi_{rs}} : \Pi_{rs} \rightarrow \bar{\mathcal{K}}_{rs}, \quad (38)$$

that maps each unique $(i,j) \in \Pi_{rs}$ (an inconsistent edge whose decomposition path π_i^j passes through (r,s)) to a unique index in $\bar{\mathcal{K}}_{rs}$ so that the following notation equivalence is established

$$\bigwedge_{(i,j) \in \Pi_{rs}} \bar{\varphi}_{rs}^{ij} \equiv \bigwedge_{k \in \bar{\mathcal{K}}_{rs}} \varphi_{rs}^k, \quad (39)$$

where we dropped the bar notation over the parametric tasks to ease the notation. Hence, **after** decomposition, the tasks ϕ_{rs} for each $(r,s) \in \mathcal{E}_\pi$ take the form

$$\phi_{rs} := \underbrace{\bigwedge_{k=1}^{K_{rs}} \varphi_{rs}^k}_{\text{before decomposition}} \wedge \underbrace{\bigwedge_{k \in \bar{\mathcal{K}}_{rs}} \varphi_{rs}^k}_{\text{parametric tasks}} = \bigwedge_{k \in \mathcal{K}_{rs}} \varphi_{rs}^k, \quad (40)$$

where $\mathcal{K}_{rs} = \{1, \dots, K_{rs}\} \cup \bar{\mathcal{K}}_{rs}$. An example is provided to clarify the new notation.

Example 4: Consider Figure 5. The left panel represents the communication graph $\mathcal{G}_c$ (solid edges) together with the task graph $\mathcal{G}_\psi$ (dashed edges) such that tasks φ_{24} and φ_{25} are inconsistent. The consistent edge $(1,5) \in \mathcal{E}_c$ is on the decomposition paths $\pi_2^4 = [2, 1, 5, 4]$ and $\pi_2^5 = [2, 1, 5]$ for both inconsistent tasks φ_{24} and φ_{25} , while the task φ_{15} is assigned **before** the decomposition. From the index mapping $y_{\Pi_{15}}$, with $y_{\Pi_{15}}((2,4)) = 2$ and $y_{\Pi_{15}}((2,5)) = 3$ the notational equivalence

$$\varphi_{15} \wedge \bar{\varphi}_{15}^{24} \wedge \bar{\varphi}_{15}^{25} \equiv \varphi_{15}^1 \wedge \varphi_{15}^2 \wedge \varphi_{15}^3,$$

$y_{\Pi_{15}}(\cdot)$

is established. The final task graph and communication graphs are then shown on the right panel in Figure 5.

With this new notation, this section is concerned with stating sufficient conditions on the parameters η_{rs}^k , $\forall k \in \bar{\mathcal{K}}_{rs}$ associated with the parametric tasks φ_{rs}^k , $\forall k \in \bar{\mathcal{K}}_{rs}$ such that the task ϕ_{rs} in (40) is not a conflicting conjunction. Indeed, we highlight again that after the introduction of the index map $y_{\Pi_{rs}}$, the tasks φ_{rs}^k , $\forall k \in \bar{\mathcal{K}}_{rs}$ are parametric as per (31), with

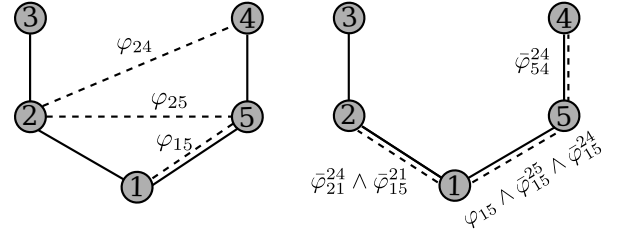


Fig. 5: Decomposition of formulas φ_{25} , φ_{15} , φ_{24} . (Left) Initial task graph (dashed) and communication graph (solid). (Right) Final task graph (dashed) and communication graph (solid).

corresponding truth set given by $\mathcal{B}_{rs}^k = \mathcal{P}(A_{rs}^k, c_{rs}^k, \alpha_{rs}^k z_{rs}^k)$ as per (29b) and such that $\eta_{rs}^k := [c_{rs}^k, \alpha_{rs}^k]$ is a parameter to be optimised. On the other hand, the tasks φ_{rs}^k , $k \in \{1, \dots, K_{rs}\}$ are *not* parametric, as per (11b), and such that their truth set is a fixed polytope in the same general form $\mathcal{B}_{rs}^k = \mathcal{P}(A_{rs}^k, c_{rs}^k, \alpha_{rs}^k z_{rs}^k)$, but with $\alpha_{rs}^k = 1$ and $\eta_{rs}^k = [c_{rs}^k, 1]$ having fixed value as per (14). Note that, in general, the polytopes $\mathcal{B}_{rs}^k$, $\forall k \in \mathcal{K}_{rs}$ are *not* similar (cf. Def 3). We are now ready to present the next main results. Following the presentation of the different types of conflicting conjunctions, we often refer to the set of indices $\mathcal{I}_G$ and $\mathcal{I}_F$ as per (23) where $\mathcal{I}_G \cup \mathcal{I}_F = \mathcal{K}_{rs}$, $\varphi_{rs}^k = G_{[a^k, b^k]} \mu_{rs}^k$, $\forall k \in \mathcal{I}_G$ and $\varphi_{rs}^k = F_{[a^k, b^k]} \mu_{rs}^k$, $\forall k \in \mathcal{I}_F$.

Proposition 6: Consider $\phi_{rs} = \bigwedge_{k \in \mathcal{K}_{rs}} \varphi_{rs}^k$ as per (40) and the set of subset of indices $\mathcal{L}_{rs}$ as per (24) over the edge (r,s) . Furthermore, let $\mathcal{L}_{rs}^M \subseteq \mathcal{L}_{rs}$ be defined as $\mathcal{L}_{rs}^M = \{L \in \mathcal{L}_{rs} \mid L \not\subseteq L', \forall L' \in \mathcal{L}_{rs}\}$. Then if for each set of indices $L \in \mathcal{L}_{rs}^M$ there exists $\xi \in \mathbb{R}^n$, such that

$$A_{rs}^l \xi - [A_{rs}^l \mid z_{rs}^l] \eta_{rs}^l \leq 0, \forall l \in L \quad (41)$$

then ϕ_{rs} is not a conflicting conjunction as per Fact 1.

Proof: By Prop. 4 we know that the set of inequalities (41) guarantees that for every $L \in \mathcal{L}_{rs}^M$ it holds $\bigcap_{l \in L} \mathcal{B}_{rs}^l \supseteq \{\xi\} \neq \emptyset$, thus disproving the conditions for conflicting conjunction over the sets of indices $L_{rs} \in \mathcal{L}_{rs}^M$ as per Fact 1. Consider now a set L' such that $L' \in \mathcal{L}_{rs} \setminus \mathcal{L}_{rs}^M$. Then necessarily, by definition of $\mathcal{L}_{rs}^M$, there must be an $L \in \mathcal{L}_{rs}^M$ such that $L' \subset L$. We can then write $\bigcap_{l \in L'} \mathcal{B}_{rs}^l \supseteq \bigcap_{l \in L} \mathcal{B}_{rs}^l \supseteq \{\xi\} \neq \emptyset$. Since L' was chosen arbitrarily from $\mathcal{L}_{rs}$, then the conditions for conflicting conjunction as per Facts 1 do not hold for any $L' \in \mathcal{L}_{rs}$ concluding the proof. ■

The set $\mathcal{L}_{rs}^M$ represents the set of subsets in $\mathcal{L}_{rs}$ that are “maximal”, meaning that any set of indices $L \in \mathcal{L}_{rs}^M$ is not contained in any other set $L' \in \mathcal{L}_{rs}$. Therefore the result of Prop. 6 suggests that checking for conflicting conjunctions over the subset of indices in $\mathcal{L}_{rs}^M$ is sufficient to avoid conflicting conjunctions of type 1, instead of checking all subsets in $\mathcal{L}_{rs}$ where $|\mathcal{L}_{rs}^M| \leq |\mathcal{L}_{rs}|$. Second, once the power set of indices $2^{\mathcal{I}_G}$ is computed, then extracting the set $\mathcal{L}_{rs} \subseteq 2^{\mathcal{I}_G}$ (and thus $\mathcal{L}_{rs}^M$) is computationally efficient as it requires only to verify the condition $\max_{l \in L} a^l \leq \min_{l \in L} b^l$ for each set $L \in 2^{\mathcal{I}_G}$, with $[a^l, b^l]$ being the time interval of the respective tasks φ_{rs}^l . Third, only the sets of indices $L \in \mathcal{L}_{rs}^M$ such that $L \cap \bar{\mathcal{K}}_{rs} \neq \emptyset$ have to be checked for conflicting conjunctions since, by Assumption 2, the conjunction $\bigwedge_{l \in \{1, \dots, K_{rs}\}} \varphi_{rs}^l$ as per (40) is not a conflict of type 1.

Proposition 7: Consider $\phi_{rs} = \bigwedge_{k \in \mathcal{K}_{rs}} \varphi_{rs}^k$ as per (40) and the set of subset of indices $\mathfrak{C}_{rs}(d)$ as per (25). Furthermore let $\mathfrak{C}_{rs}^m(d) \subseteq \mathfrak{C}_{rs}(d)$ be defined as $\mathfrak{C}_{rs}^m(d) = \{C \in \mathfrak{C}_{rs}(d) \mid C \not\supseteq C', \forall C' \in \mathfrak{C}_{rs}(d)\}$. If for every $d \in \mathcal{I}_F$ and for every $C \in \mathfrak{C}_{rs}^m(d)$ there exists $l \in C$ such that

$$\begin{aligned} A_{rs}^l \xi - [A_{rs}^l \mid z_{rs}^l] \eta_{rs}^l &\leq 0, \\ A_{rs}^d \xi - [A_{rs}^d \mid z_{rs}^d] \eta_{rs}^d &\leq 0, \end{aligned} \quad (42)$$

for some vector $\xi \in \mathbb{R}^n$, then ϕ_{rs} is not a conflicting conjunction as per Fact 2.

Proof: For a given $d \in \mathcal{I}_F$ consider the set $C \in \mathfrak{C}_{rs}^m(d) \subset \mathfrak{C}_{rs}(d)$ such that, by definition of $\mathfrak{C}_{rs}(d)$ in (25), we have $[a^d, b^d] \subseteq \bigcup_{l \in C} [a^l, b^l]$. Then by equation Prop. 4 it is known that (42) ensures the existence of at least one index $l \in C$ such that $\mathcal{B}_{rs}^d \cap \mathcal{B}_{rs}^l \supseteq \{\xi\} \neq \emptyset$. Therefore, a conflict as per Fact 2 is avoided for the index set C . Now consider another index set $C' \neq C$ such that $C' \in \mathfrak{C}_{rs}(d) \setminus \mathfrak{C}_{rs}^m(d)$. By definition of $\mathfrak{C}_{rs}^m(d)$, it also holds $C \subset C'$. It is now sufficient to note that the satisfaction (42) for one index $l \in C \subset C'$ disproves again the condition for conflicting conjunction as per Fact 2 over C' . Since the result is valid for any $d \in \mathcal{I}_F$, any $C \in \mathfrak{C}_{rs}^m(d)$ and any $C' \in \mathfrak{C}_{rs}(d) \setminus \mathfrak{C}_{rs}^m(d)$, we have proved that conflicts as per type 2 do not arise over ϕ_{rs} . ■

Similarly to Prop. 6, the set $\mathfrak{C}_{rs}^m(d)$ represents the set of subsets in $\mathfrak{C}_{rs}(d)$ for a given $d \in \mathcal{I}_F$, that is “minimal”, in the sense that any set of indices $C \in \mathfrak{C}_{rs}^m(d)$ does not contain any set $C' \in \mathfrak{C}_{rs}(d)$ thus reducing the number of subsets of indices of tasks to be checked for conflicting conjunctions. The same remarks highlighted for Prop. 6 are also valid for Prop. 7. The last result of this section is presented next.

Proposition 8: Consider $\phi_{rs} = \bigwedge_{k \in \mathcal{K}_{rs}} \varphi_{rs}^k$ as per (40) and the set of subset of indices $\mathfrak{D}_{rs}(d)$ as per (26). Furthermore, let $\mathfrak{D}_{rs}^M(d) \subseteq \mathfrak{D}_{rs}(d)$ be defined as $\mathfrak{D}_{rs}^M(d) = \{D \in \mathfrak{D}_{rs}(d) \mid D \not\supseteq D', \forall D' \in \mathfrak{D}_{rs}(d)\}$. If for every $d \in \mathcal{I}_F$ and for every $D \in \mathfrak{D}_{rs}^M(d)$ it holds

$$\begin{aligned} A_{rs}^l \xi - [A_{rs}^l \mid z_{rs}^l] \eta_{rs}^l &\leq 0, \forall l \in D, \\ A_{rs}^d \xi - [A_{rs}^d \mid z_{rs}^d] \eta_{rs}^d &\leq 0, \end{aligned} \quad (43)$$

for some vector $\xi \in \mathbb{R}^n$, then ϕ_{rs} is not a conflicting conjunction as per Fact 3.

Proof: For a given $d \in \mathcal{I}_F$ consider the set $D \in \mathfrak{D}_{rs}^M(d) \subset \mathfrak{D}_{rs}(d)$ such that, by definition of $\mathfrak{D}_{rs}(d)$ in (26), we have $[a^d, b^d] \subseteq \bigcap_{l \in D} [a^l, b^l]$. Then by equation Prop. 4 it is known that (43) ensures that $\mathcal{B}_{rs}^d \cap \bigcap_{l \in D} \mathcal{B}_{rs}^l \supseteq \{\xi\}$. Therefore, a conflict as per Fact 3 is avoided over D . Now consider another set of indices $D' \neq D$ such that $D' \in \mathfrak{D}_{rs}(d) \setminus \mathfrak{D}_{rs}^M(d)$ and, by definition, it holds $D \supset D'$. Then the condition (43) directly disproves the condition for conflicting conjunction as per Fact 3 since, it holds $\mathcal{B}_{rs}^d \cap \bigcap_{l \in D'} \mathcal{B}_{rs}^l \supseteq \mathcal{B}_{rs}^d \cap \bigcap_{l \in D} \mathcal{B}_{rs}^l \supseteq \{\xi\}$. Since the argument is valid for any $d \in \mathcal{I}_F$, any $D \in \mathfrak{D}_{rs}^M(d)$ and any $D' \in \mathfrak{D}_{rs}(d) \setminus \mathfrak{D}_{rs}^M(d)$, we have proved that conflicts as per type 3 do not arise over ϕ_{rs} . ■

One last definition provides a compact and uniform way to verify the absence of any type of conflicting conjunctions over $\phi_{rs} = \bigwedge_{k \in \mathcal{K}_{rs}} \varphi_{rs}^k$ as per Prop. 6, 7 and 8. Namely, the union of all the subset of indices that can cause conflicting

conjunctions of tasks is defined as

$$\begin{aligned} \Omega_{rs} = & \underbrace{\left(\bigcup_{d \in \mathcal{I}_F} \{ \{d\} \cup \{l\} \mid l \in C, \forall C \in \mathfrak{C}_{rs}^m(d) \} \right)}_{\text{sets of indices related to conflicts of type 2}} \cup \\ & \underbrace{\left(\bigcup_{d \in \mathcal{I}_F} \{ \{d\} \cup D \mid \forall D \in \mathfrak{D}_{rs}^M(d) \} \right)}_{\text{sets of indices related to conflicts of type 3}} \cup \mathfrak{L}_{rs}^M, \end{aligned} \quad (44)$$

and let

$$\Xi_{rs} = \{\xi_{rs}^q, \forall q = 1, \dots, |\Omega_{rs}|\} \quad (45a)$$

$$y_{\Omega_{rs}} : \Omega_{rs} \mapsto \{1, \dots, |\Omega_{rs}|\}, \quad (45b)$$

where Ξ_{rs} is the set of auxiliary variables applied to impose the conditions (41), (42) and (43), while $y_{\Omega_{rs}}$ is the index mapping relating each set of indices $Q \in \Omega_{rs}$ with an auxiliary variable ξ_{rs}^q such that $q = y_{\Omega_{rs}}(Q)$. With these new definitions, the conjunctions $\bigwedge_{l \in Q} \varphi_{rs}^l$ for all $Q \in \Omega_{rs}$ represent potential conflicting conjunctions over the edge $(r, s) \in \mathcal{E}_\pi$. To avoid conflicts, the satisfaction of (41), (42) and (43) over $(r, s) \in \mathcal{E}_\pi$ can be compactly written as

$$A_{rs}^l \xi_{rs}^q - [A_{rs}^l \mid z_{rs}^l] \eta_{rs}^l \leq 0, \forall Q \in \Omega_{rs}, \forall l \in Q, \quad (46)$$

such that $q = y_{\Omega_{rs}}(Q)$. It now remains to clarify how conflicting conjunctions as per Fact 4-5 can be avoided. In this respect, Assumption. 1 ensures that cycles of tasks as per Fact 4-5 can not arise during decomposition of $\mathcal{G}_\psi$ into $\mathcal{G}_{\bar{\psi}}$. Indeed, since only edges in the communication graph $\mathcal{G}_c$ are applied for the decomposition and since $\mathcal{G}_c$ is acyclic, then, by construction, the new task graph $\mathcal{G}_{\bar{\psi}}$ must necessarily be acyclic, for which conflicts of types 4 and 5 do not arise. The reason why conditions (27)-(28) are excluded in our decomposition approach is due to the fact that their enforcement leads to non-convex constraints over the parameter vectors η_{rs}^l . Moreover, the possibility of decentralizing the proposed decomposition, as presented in Section VI-B, is lost if this last type of constraint is introduced. When the acyclicity assumption on $\mathcal{G}_c$ can not be enforced, a solution to avoid conflicting conjunctions of type 4 and 5 was provided in our previous work [21] by restricting the sets $\mathcal{B}_{rs}$ to be hyper-rectangles instead of general polytopes.

C. Handling communication consistency

Concerning communication consistency of the parametric tasks $\varphi_{rs}^l, \forall l \in \mathcal{K}_{rs}$ over the edges $(r, s) \in \mathcal{E}_\pi$, the following result is provided

Proposition 9: Consider an edge $(r, s) \in \mathcal{E}_\pi$ and the parametric task φ_{rs}^l with $l \in \mathcal{K}_{rs}$ as specified in (40). Furthermore, consider the truth set $\mathcal{B}_{rs}^l = \mathcal{P}(A_{rs}^l, c_{rs}^l, \alpha_{rs}^l z_{rs}^l)$ and parameter vector η_{rs}^l associated with φ_{rs}^l . Let the generator set $V_{rs}^l = \nu(\mathcal{P}(A_{rs}^l, \mathbf{0}, z_{rs}^l))$ as per Prop. 1 and the selection matrix S such that $p_i = Sx_i \forall i \in \mathcal{V}$. Then φ_{rs}^l is communication consistent if

$$(\eta_{rs}^l)^T N_k^l \eta_{rs}^l \leq r_c^2, \forall k = 1, \dots, |V_{rs}^l|, \quad (47)$$

where

$$N_k^l = (G_k^l)^T S^T S G_k^l, G_k^l = [I_n \mid v_k], v_k \in V_{rs}^l. \quad (48)$$

Proof: We prove that (47) implies (17). First let the polytope $\mathcal{L}_{rs} = \{Se_{rs} = \mathbf{p}_s - \mathbf{p}_r \mid e_{rs} \in \mathcal{B}_{rs}^l\}$ represent the projection of $\mathcal{B}_{rs}^l$ over the space of relative positions of agent r and s . Moreover, let the distance function $f(Se_{rs}) = \|Se_{rs}\|^2 - r_c^2, \forall Se_{rs} \in \mathcal{L}_{rs}$. By Def. 2 and (4) the generators of $\mathcal{L}_{rs}$ are given by the vectors $\{SG_k^l \boldsymbol{\eta}_{rs}^l\}_{k=1}^{|V_{rs}^l|}$, where $\{G_k^l \boldsymbol{\eta}_{rs}^l\}_{k=1}^{|V_{rs}^l|} = \nu(\bar{\mathcal{B}}_{rs}^l) = \nu(\mathcal{P}(A_{rs}^l, \mathbf{c}_{rs}^l, \alpha_{rs}^l \mathbf{z}_{rs}^l))$. Thus $Se_{rs} = \sum_k \lambda_k SG_k^l \boldsymbol{\eta}_{rs}^l, \forall Se_{rs} \in \mathcal{L}_{rs}$ with $\lambda_k > 0, \sum_k \lambda_k = 1$. From, Jensen's inequality [33, Thm. 3.4] we also have that $f(SG_k^l \boldsymbol{\eta}_{rs}^l) \leq 0, \forall k = 1, \dots, |V_{rs}^l| \Rightarrow f(Se_{rs}) = \|Se_{rs}\|^2 - r_c^2 = \|\mathbf{p}_r - \mathbf{p}_s\|^2 - r_c^2 \leq 0, \forall Se_{rs} \in \mathcal{L}_{rs}$. It is then sufficient to note that $f(SG_k^l \boldsymbol{\eta}_{rs}^l) \leq 0, \forall k = 1, \dots, |V_{rs}^l|$ is equivalent to (47) such that $\mathcal{B}_{rs}^l \cap \{e_{rs} \in \mathbb{X}_{rs} \mid \|Se_{rs}\| \leq r_c\} = \mathcal{L}_{rs}$, which implies (17). ■

Note that N_k^l are positive semi-definite matrices, and (47) is thus convex in $\boldsymbol{\eta}_{rs}^l$.

VI. PARAMETERS OPTIMIZATION

In this section, the convex optimization to obtain optimal parameters $\boldsymbol{\eta}_{rs}^l, \forall (r, s) \in \mathcal{E}_\pi, \forall l \in \bar{\mathcal{K}}_{rs}$ is presented, with the constraints required for ensuring conflict-free conjunctions of tasks over the communication consistent edges $\mathcal{E}_\pi$ as well as communication consistency.

A. Centralized parameters optimization

1) *Variables:* The variables of the optimization program are the parameters $\boldsymbol{\eta}_{rs}^l, \forall (r, s) \in \mathcal{E}_\pi$ of the tasks φ_{rs}^l introduced for the decomposition for all $l \in \bar{\mathcal{K}}_{rs}$ and all $(r, s) \in \mathcal{E}_\pi$, together with all the auxiliary variables $\boldsymbol{\xi}_{rs}^q \in \Xi_{rs}, \forall (r, s) \in \mathcal{E}_\pi, \forall q = 1, \dots, |\Xi_{rs}|$ (as per (45a)) to avoid conflicts as explained in Sec. V-B. We define the set of parameters $\mathcal{H}_{rs}$ for each edge (r, s) as $\mathcal{H}_{rs} = \{\boldsymbol{\eta}_{rs}^l \mid l \in \bar{\mathcal{K}}_{rs}\}$. We recall that the mapping $y_{\Pi_{rs}}$ as per (38) assigns a unique index $l \in \bar{\mathcal{K}}_{rs}$ to each inconsistent edge $(i, j) \in \Pi_{rs}$ passing through (r, s) . Indeed, when an inconsistent task φ_{ij} with decomposition path π_i^j passes through (r, s) , then a collaborative task $\bar{\varphi}_{rs}^{ij} \equiv \varphi_{rs}^l$, with $l \in \bar{\mathcal{K}}_{rs}$, is introduced over (r, s) .

2) *Cost Function:* Let a single inconsistent task $\phi_{ij} = \varphi_{ij}$ with truth set $\mathcal{B}_{ij}$ be decomposed over the path π_i^j . By Lemma 1, the inclusion relation (32) must hold for a valid decomposition. By Prop. 5 we also know that (32) implies the relation (35) $\sum_{(r,s) \in \epsilon(\pi_i^j)} \alpha_{rs}^{ij} \leq 1$ (where the notation in place before the definition of the index mapping $y_{\Pi_{rs}}$ is used here and replaced in (49)), which holds with equality iff (32) does. Intuitively, the optimal decomposition is obtained when (32) holds with equality, as this is equivalent to recovering the truth set $\mathcal{B}_{ij}$ in its full extent, after the decomposition (cf. Prop. 5). Since this occurs only if $\sum_{(r,s) \in \epsilon(\pi_i^j)} \alpha_{rs}^{ij} = 1$, then the decomposition accuracy in (35) is considered an appropriate metric to define "optimality" for our proposed decomposition. The objective function of our decomposition becomes

$$\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} \sum_{(r,s) \in \epsilon(\pi_i^j)} \alpha_{rs}^{ij} = \sum_{(r,s) \in \mathcal{E}_\pi} \sum_{(i,j) \in \Pi_{rs}} \alpha_{rs}^{ij} \quad (49)$$

$$\equiv \sum_{y_{\Pi_{rs}}} \sum_{(r,s) \in \mathcal{E}_\pi} \sum_{l \in \bar{\mathcal{K}}_{rs}} \alpha_{rs}^l,$$

where the first equality is obtained by swapping an iteration over the inconsistent edges $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$, with an iteration over the decomposition edges $(r, s) \in \mathcal{E}_\pi$. On the other hand, the last equivalence is only notational though $y_{\Pi_{rs}}$ as per (38).

3) *Constraints:* The constraints for our optimization program are of three main types: communication consistency, conflicting conjunctions and decomposition constraints. Consider first the communication constraints over an edge $(r, s) \in \mathcal{E}_\pi$. For each parametric task $\varphi_{rs}^l, l \in \bar{\mathcal{K}}_{rs}$ there is a parametric truth set $\mathcal{B}_{rs}^l = \mathcal{P}(A_{rs}^l, \mathbf{c}_{rs}^l, \alpha_{rs}^l \mathbf{z}_{rs}^l)$ with generators set $V_{rs}^l = \nu(\mathcal{P}(A_{rs}^l, \mathbf{0}, \mathbf{z}_{rs}^l))$. Then, by Prop. 9, communication consistency is enforced by (47). We can repeat the same argument for each $l \in \bar{\mathcal{K}}_{rs}$ to obtain:

$$(\boldsymbol{\eta}_{rs}^l)^T N_k^l \boldsymbol{\eta}_{rs}^l \leq r_c^2, \forall l \in \bar{\mathcal{K}}_{rs}, \forall k = 1, \dots, |V_{rs}^l|$$

where the matrices $N_k^l, \forall k = 1, \dots, |V_{rs}^l|$ are as per (48). The second type of constraint guarantees conflict-free conjunctions over the tasks ϕ_{rs} as per (40). In this respect, consider, for each edge $(r, s) \in \mathcal{E}_\pi$, the set Ω_{rs} as per (44). The set Ω_{rs} is the set of subsets of indices in $\bar{\mathcal{K}}_{rs}$ such that, for each $Q \in \Omega_{rs}$, a conjunction of tasks $\bigwedge_{l \in Q} \varphi_{rs}^l$ over $(r, s) \in \mathcal{E}_\pi$ could cause a conflict. Then, conflicting conjunctions are avoided by imposing (46) over each edge $(r, s) \in \mathcal{E}_\pi$. The third, and last, type of constraint is the one required to obtain a valid decomposition of each inconsistent task $\phi_{ij} = \varphi_{ij}, \forall (i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ by the satisfaction of (33). This is achieved by the satisfaction of (33) and (34). Differently from the previous two types of constraints, this constraint is *shared* among multiple edges $(r, s) \in \epsilon(\pi_i^j)$ in the sense that its satisfaction involves parameters vectors spanning all the edges $(r, s) \in \epsilon(\pi_i^j)$ along the path π_i^j . With the intent of uniforming the notation via the index mapping $y_{\Pi_{rs}}$, we let $l_{rs} = y_{\Pi_{rs}}((i, j))$ for each $(r, s) \in \epsilon(\pi_i^j)$ to be the unique index associated with the inconsistent edge (i, j) over the edge $(r, s) \in \epsilon(\pi_i^j)$, such that the constraint (33) can be rewritten as

$$\sum_{(r,s) \in \epsilon(\pi_i^j)} \left(M_{ij} \boldsymbol{\eta}_{rs}^{l_{rs}} - \frac{Z_{ij}}{|\epsilon(\pi_i^j)|} \begin{bmatrix} \mathbf{c}_{ij} \\ 1 \end{bmatrix} \right) \leq \mathbf{0},$$

for all $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$. Recalling that the truth set of the inconsistent task φ_{ij} is given by $\mathcal{B}_{ij} = \mathcal{P}(A_{ij}, \mathbf{c}_{ij}, \mathbf{z}_{ij})$, the definition of M_{ij} and Z_{ij} is given in (34).

4) *Optimization program:* By letting $\mathcal{H} = \bigcup_{(r,s) \in \epsilon(\omega)} \mathcal{H}_{rs}$ and $\Xi = \bigcup_{(r,s) \in \epsilon(\omega)} \Xi_{rs}$ the centralised optimization program applied to solve the task decomposition problem is given by:

$$\max_{\mathcal{H}, \Xi} \sum_{(r,s) \in \mathcal{E}_\pi} \sum_{l \in \bar{\mathcal{K}}_{rs}} \alpha_{rs}^l \quad (50a)$$

$$(\boldsymbol{\eta}_{rs}^l)^T N_k^l \boldsymbol{\eta}_{rs}^l \leq r_c^2 \quad \forall (r, s) \in \mathcal{E}_\pi, \forall l \in \bar{\mathcal{K}}_{rs}, \quad \forall k = 1, \dots, |V_{rs}^l|, \quad (50b)$$

$$A_{rs}^l \boldsymbol{\xi}_{rs}^q + [A_{rs}^l \mid \mathbf{z}_{rs}^l] \boldsymbol{\eta}_{rs}^l \leq \mathbf{0}, \quad \forall (r, s) \in \mathcal{E}_\pi, \forall Q \in \Omega_{rs}, \quad \text{with } q = y_{\Omega_{rs}}(Q) \quad \forall l \in Q, \quad (50c)$$

$$\sum_{(r,s) \in \epsilon(\pi_i^j)} \left(M_{ij} \boldsymbol{\eta}_{rs}^{l_{rs}} - \frac{Z_{ij}}{|\epsilon(\pi_i^j)|} \begin{bmatrix} \mathbf{c}_{ij} \\ 1 \end{bmatrix} \right) \geq \mathbf{0}, \quad (50d)$$

with $l_{rs} = y_{\Pi_{rs}}((i, j)), \quad \forall (i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c,$

$$\begin{aligned} \|\xi_{rs}^q\| &\leq \xi_{max} \quad \forall q = 1, \dots, |\Omega_{rs}|, \\ \|\eta_{rs}^l\| &\leq \eta_{max} \quad \forall l \in \bar{\mathcal{K}}_{rs}, \end{aligned} \quad \forall (r, s) \in \mathcal{E}_\pi \quad (50e)$$

where the cost function (50a) represents the sum of all the decomposition accuracies over all the inconsistent tasks, constraint (50b) enforces communication consistency over all parametric tasks introduced for the decomposition, constraint (50c) enforces all the necessary inclusions to avoid conflicting conjunctions over the decomposition edges in $\mathcal{E}_\pi$ and constraint (50d) represents all the inclusion relations needed to satisfy (32) for all the inconsistent tasks, with M_{ij} and Z_{ij} defined as per (34) for each $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$. Constraint (50e) with constants $\xi_{max} > 0$ and $\eta_{max} > 0$, is a norm constraint introduced to ensure the domain of the convex program (50) is compact. The upper bounds ξ_{max} , η_{max} should be chosen to be sufficiently large as they only apply to bound the domain of the optimal parameters search.

Remark 1: The proposed task decomposition approach is agnostic to the chosen dynamics in (1). Indeed, the decomposition derived as the solution to (50) depends only on the geometry of the predicate level sets of the tasks and their logical relation (e.g., potential conflicting conjunctions). Future work will address the interaction between dynamics, communication, and STL constraints to enhance the generality of the proposed decomposition framework.

B. A decentralized solution

While problem (50) can be solved in a centralised fashion, the next proposition confirms the intuition that (50) can be written as a set of constraint-coupled programs defined over the edges $(r, s) \in \mathcal{E}_\pi$, which are coupled via the shared constraint (50d).

Proposition 10: For each $(r, s) \in \mathcal{E}_\pi$, let the stacked vector of parameters and auxiliary variables $\chi_{rs} \in \mathbb{R}^{(n+1) \cdot |\mathcal{H}_{rs}| + n \cdot |\Omega_{rs}|}$ defined as

$$\chi_{rs} = [[\eta_{rs}^l]_{l \in \bar{\mathcal{K}}_{rs}}, [\xi_{rs}^q]_{q=1, \dots, |\Omega_{rs}|}]. \quad (51)$$

Then the optimization program (50) can be rewritten as

$$\min_{\chi_{rs} \forall (r,s) \in \mathcal{E}_\pi} \sum_{(r,s) \in \mathcal{E}_\pi} f_{rs}(\chi_{rs}) \quad (52a)$$

$$\text{s.t. : } \chi_{rs} \in \mathcal{C}_{rs} \quad \forall (r, s) \in \mathcal{E}_\pi, \quad (52b)$$

$$\sum_{(r,s) \in \mathcal{E}_\pi} (T_{rs} \chi_{rs} - \mathbf{t}_{rs}) \leq \mathbf{0}, \quad (52c)$$

where the functions $f_{rs}(\chi_{rs}) = \sum_{l \in \bar{\mathcal{K}}_{rs}} \alpha_{rs}^l$, $\forall (r, s) \in \mathcal{E}_\pi$, $\forall (r, s) \in \mathcal{E}_\pi$ are convex and the sets $\mathcal{C}_{rs}$, $\forall (r, s) \in \mathcal{E}_\pi$ are compact and convex sets. Moreover, $T_{rs} \in \mathbb{R}^{(\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} m_{ij}) \times ((n+1) \cdot |\mathcal{H}_{rs}| + n \cdot |\Omega_{rs}|)}$ and $\mathbf{t}_{rs} \in \mathbb{R}^{(\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} m_{ij})}$ with m_{ij} being the number of rows of each matrices M_{ij} , $\forall (i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ as per (34).

Proof: First note that $f_{rs}(\chi_{rs})$, $\forall (r, s) \in \mathcal{E}_\pi$, are linear (and thus convex) functions since they are obtained as sums of elements of the vectors χ_{rs} . Next, we prove 1) the equivalence between (50a) and (52a), 2) the equivalence between (50b)-(50c)-(50e) and (52b), 3) the equivalence between (50d) and (52c). The equivalence 1) of the cost function (52a) and (50a) is evident since $f_{rs}(\chi_{rs}) = \sum_{l \in \bar{\mathcal{K}}_{rs}} \alpha_{rs}^l$. The equivalence

2) follows by considering, for each $(r, s) \in \mathcal{E}_\pi$, the sets $\mathcal{C}_{rs}^1 = \{\chi_{rs} \mid (\eta_{rs}^l)^T N_k^l (\eta_{rs}^l) \leq r_c^2, \forall l \in \bar{\mathcal{K}}_{rs} \ k = 1, \dots, |\mathcal{V}_{rs}^l|\}$, $\mathcal{C}_{rs}^2 = \{\chi_{rs} \mid A_{rs}^l \xi_{rs}^q + [A_{rs}^l \mid \mathbf{z}_{rs}^l] \eta_{rs}^l \leq \mathbf{0}, \forall l \in \bar{\mathcal{K}}_{rs}, \forall q \in \Omega_{rs}, \text{ s.t. } q = y_{\Omega_{rs}}(Q)\}$, $\mathcal{C}_{rs}^3 = \{\chi_{rs} \mid \|\xi_{rs}^q\| \leq \xi_{max} \ \forall q = 1, \dots, |\Omega_{rs}| \wedge \|\eta_{rs}^l\| \leq \eta_{max} \ \forall l \in \bar{\mathcal{K}}_{rs}\}$, which are, by definition, equivalent to (50b)-(50c)-(50e) when considered for a single edge $(r, s) \in \mathcal{E}_\pi$. Since $\mathcal{C}_{rs}^1$, $\mathcal{C}_{rs}^2$ and $\mathcal{C}_{rs}^3$ are convex, and $\mathcal{C}_{rs}^3$ is bounded, then $\mathcal{C}_{rs} = \mathcal{C}_{rs}^1 \cap \mathcal{C}_{rs}^2 \cap \mathcal{C}_{rs}^3$ is also convex and bounded, thus proving the equivalence. The last equivalence between (50d) and (52c) is proved as follows. Consider, for each $(r, s) \in \mathcal{E}_\pi$, the matrix $W_{rs} \in \mathbb{R}^{(\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} m_{ij}) \times (n+1) \cdot |\Pi_{rs}|}$, where we recall that Π_{rs} contains all the inconsistent edges $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ whose decomposition path is passing by $(r, s) \in \mathcal{E}_\pi$ as per (36b) and, by definition, $|\bar{\mathcal{K}}_{rs}| = |\Pi_{rs}| = |\mathcal{H}_{rs}|$. Moreover m_{ij} is the number of rows of the matrix M_{ij} associated with each inconsistent edge $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$. We define the matrices W_{rs} by blocks as $W_{rs}[p, w]$ with row-blocks $p = 1, \dots, |\mathcal{E}_\psi \setminus \mathcal{E}_c|$ and $w = 1, \dots, |\Pi_{rs}|$ column-blocks such that all the matrices W_{rs} have the same number of row-blocks, but only a number of column-blocks equal to the number of edges $|\Pi_{rs}|$ that are decomposed over the edge $(r, s) \in \mathcal{E}_\pi$. We define the one-to-one index mapping $y_{\mathcal{E}_\psi \setminus \mathcal{E}_c} : \mathcal{E}_\psi \setminus \mathcal{E}_c \rightarrow \{1, \dots, |\mathcal{E}_\psi \setminus \mathcal{E}_c|\}$ associating each inconsistent edges $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ with a unique row-block index. Then each matrix W_{rs} has non-zero blocks only at those (p, w) index pairs such that $(i, j) = y_{\mathcal{E}_\psi \setminus \mathcal{E}_c}^{-1}(p)$ is an inconsistent edge and the mapping $y_{\Pi_{rs}}^{-1}(w + K_{rs}) = (i, j)$, as per (38), is well defined, meaning that the decomposition path $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ passes through (r, s) . Thus, W_{rs} is defined by blocks as

$$W_{rs}[p, w] = \begin{cases} M_{ij} & \text{if } y_{\mathcal{E}_\psi \setminus \mathcal{E}_c}^{-1}(p) = y_{\Pi_{rs}}^{-1}(w + K_{rs}) = (i, j) \\ \mathbf{0}_{m_{ij} \times (n+1)} & \text{else,} \end{cases}$$

At this point we define the selection matrix S_{rs} of appropriate dimensions such that $[\eta_{rs}^l]_{l \in \bar{\mathcal{K}}_{rs}} = S_{rs} \chi_{rs}$ and finally $T_{rs} = W_{rs} S_{rs}$. Concerning to vector $\mathbf{t}_{rs}$, this can also be defined by a number of $|\mathcal{E}_\psi \setminus \mathcal{E}_c|$ row-blocks of dimension $\mathbb{R}^{m_{ij}}$ as

$$\mathbf{t}_{rs}[p] = \begin{cases} \frac{Z_{ij}}{|\epsilon(\pi_i^j)|} \begin{bmatrix} c_{ij} \\ 1 \end{bmatrix} & \text{if } y_{\mathcal{E}_\psi \setminus \mathcal{E}_c}^{-1}(p) = (i, j) \in \Pi_{rs} \\ \mathbf{0}_{m_{ij}} & \text{else,} \end{cases}$$

With these definitions, if we let $\mathbf{o}_{rs} = T_{rs} \chi_{rs} - \mathbf{t}_{rs}$, then $\mathbf{o}_{rs}$ is a vector defined by $|\mathcal{E}_\psi \setminus \mathcal{E}_c|$ row-blocks of dimension $\mathbb{R}^{m_{ij}}$ such that

$$\mathbf{o}_{rs}[p] = \begin{cases} M_{ij} \eta_{rs}^l - \frac{Z_{ij}}{|\epsilon(\pi_i^j)|} \begin{bmatrix} c_{ij} \\ 1 \end{bmatrix} & \text{if } y_{\mathcal{E}_\psi \setminus \mathcal{E}_c}^{-1}(p) = (i, j) \in \Pi_{rs} \\ \text{s.t } l_{rs} = y_{\Pi_{rs}}((i, j)) & \\ \mathbf{0}_{m_{ij}} & \text{else,} \end{cases} \quad (53)$$

where we recall that $l_{rs} = y_{\Pi_{rs}}((i, j))$ is the index assigned to the inconsistent edge (i, j) over the edge $(r, s) \in \epsilon(\pi_i^j)$ applied for the decomposition, which is different from the index $l_{r's'} = y_{\Pi_{r's'}}((i, j))$ assigned to the same edge (i, j) by another edge $(r', s') \in \epsilon(\pi_i^j)$ applied for the decomposition.

Eventually, noting that $\sum_{(r,s) \in \mathcal{E}_\pi} \mathbf{o}_{rs}$ is equivalent to (50d) concludes the proof. ■

Given the optimization problem (50), a solution can be found in a decentralized fashion by leveraging the fact that (50) has the edge-wise decentralized structure in (52). We propose to employ the algorithm in [29] to solve (52) in a decentralized fashion as explained next. First, we wish to define an edge-computing graph denoted as $\mathcal{G}_\Theta$ such that each node in $\mathcal{G}_\Theta$ corresponds to an edge $(r, s) \in \mathcal{E}_\pi$. Hence, the set of nodes for $\mathcal{G}_\Theta$ is given by $\mathcal{V}_\Theta = \mathcal{E}_\pi$. On the other hand, the set of edges for $\mathcal{G}_\Theta$ is given by

$$\mathcal{E}_\Theta = \{((r, s), (r', s')) \in \mathcal{V}_\Theta \times \mathcal{V}_\Theta \mid \{r, s\} \cap \{r', s'\} \neq \emptyset\}. \quad (54)$$

In other words, a tuple $((r, s), (r', s'))$ is an edge in $\mathcal{E}_\Theta$ if $(r, s) \in \mathcal{E}_\pi$ shares a node with $(r', s') \in \mathcal{E}_\pi$. Then the edge-computing graph is defined as $\mathcal{G}_\Theta(\mathcal{V}_\Theta, \mathcal{E}_\Theta)$ and the neighbour set is defined as $\mathcal{N}_\Theta((r, s)) = \{(r', s') \in \mathcal{V}_\Theta \mid ((r, s), (r', s')) \in \mathcal{E}_\Theta\}$. The edge-computing graph $\mathcal{G}_\Theta$ can then be applied to find a solution for (52) in a decentralized fashion where each node $(r, s) \in \mathcal{V}_\Theta$ (an edge in $\mathcal{E}_\pi$) solves iteratively the following local problem iteratively

$$\min_{\mathbf{x}_{rs}, \rho_{rs}} f_{rs}(\mathbf{x}_{rs}) + c_\rho \rho_{rs}, \quad (55a)$$

$$\text{s.t. : } \mathbf{x}_{rs} \in \mathcal{C}_{rs}, \rho_{rs} \geq 0, \quad (55b)$$

$$T_{rs}\mathbf{x}_{rs} - \mathbf{t}_{rs} + \sum_{(r', s') \in \mathcal{N}_\Theta((r, s))} ({}^t\lambda_{rs, r's'} - {}^t\lambda_{r's', rs}) \leq \mathbf{1}\rho_{rs}. \quad (55c)$$

In (55), the superscript t indicates the current iteration and is set on the upper-left to avoid confusion with the previous notation. The term $\rho_{rs} \in \mathbb{R}$ is a penalty variable constrained to be non-negative as per constraint (55b). The coefficient $c_\rho > 0$ penalizes non-zero values of ρ_{rs} . At the same time, by (55c), each node (r, s) holds the value at iteration t of the additional consensus vectors ${}^t\lambda_{rs, r's'} \in \mathbb{R}^{\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} m_{ij}}$, $\forall (r', s') \in \mathcal{N}_\Theta((r, s))$ and receives the consensus vectors ${}^t\lambda_{r's', rs}$ from the each neighbour $(r', s') \in \mathcal{N}_\Theta((r, s))$. The introduction of a consensus term like the one introduced in (55) is common in distributed optimization of a global optimization program with a separable structure as the one in (52) [29], [30]. Note that summing the left-hand side of constraint (55c) for each $(r, s) \in \mathcal{E}_\pi = \mathcal{V}_\Theta$ yields the original left-hand side in (52c). At this point Algorithm 1 borrowed from [29] applies to converge to an optimal solution of the vectors $\mathbf{x}_{rs}$, $\forall (r, s) \in \mathcal{E}_\pi$, after sufficiently many iterations. While referring the reader to [29] for a detailed analysis of Alg. 1, we here seek to provide an intuitive explanation of the algorithm. Namely, at each iteration t , the edge $(r, s) \in \mathcal{V}_\Theta$ solves an instance of (55), where the penalty variable $\rho_{rs} \in \mathbb{R}$ relaxes the satisfaction of the shared constraint (55c). If we let ${}^t\rho_{rs}$ to be the optimal value of ρ_{rs} at iteration t and if we let ${}^t\mu_{rs} \in \mathbb{R}^{\sum_{(i,j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c} m_{ij}}$ to be the Lagrangian multiplier vector associated with (55c) at each iteration t , then the consensus vectors $\lambda_{rs, r's'}$ are updated by node (r, s) , at iteration t , as per line 7 in Alg. 1. Note that this update only requires the local knowledge of the Lagrangian multiplier vectors μ_{rs} , $\mu_{r's'}$, $\forall (r', s') \in \mathcal{N}_\Theta((r, s))$. The update rate ${}^t\gamma > 0$ must be known to all

agents and such that $\lim_{t \rightarrow \infty} {}^t\gamma = \infty$ and $\lim_{t \rightarrow \infty} ({}^t\gamma)^2 < \infty$ to guarantee convergence [29]. If we let $[\mathbf{x}_{rs}^*]_{(r,s) \in \mathcal{E}_\pi}$ to be a non-unique optimal (feasible) solution of (52), then, by [29, Thm II.6], it is known that the iterates of optimal solutions $\{[{}^t\mathbf{x}_{rs}]_{(r,s) \in \mathcal{E}_\pi}\}$ found by Alg. 1 converge asymptotically to one such solution $[\mathbf{x}_{rs}^*]_{(r,s) \in \mathcal{E}_\pi}$. This, in turns, entails that $\lim_{t \rightarrow \infty} {}^t\rho_{rs} = 0$, $\forall (r, s) \in \mathcal{V}_\Theta$. Once Alg. 1 terminates for all the agents, a set of optimal parameters $(\eta_{rs}^*)^*$, $\forall l \in \bar{\mathcal{K}}_{rs}$, is obtained from each edge $(r, s) \in \mathcal{E}_\pi$ such that the parametric tasks $\bar{\varphi}_{rs}^{ij} \equiv \varphi_{rs}^l$, $\forall l \in \bar{\mathcal{K}}_{rs}$ are fully determined as well as the new global task $\bar{\psi}$ as per (20), thus solving Problem 1. With these considerations, our final main result is presented next.

Theorem 1: Let $\mathcal{G}_c$ be acyclic as per Assumption 1 and let Assumption 2 hold over $\mathcal{G}_\psi$. Moreover, let the optimization program (50) be feasible and solved in a decentralized fashion by the edge-computing graph $\mathcal{G}_\Theta$ as per Alg. 1 such that the final global task $\bar{\psi}$ is obtained, together with the task graph $\mathcal{G}_{\bar{\psi}}$, as per (20). Then the task graph $\mathcal{G}_{\bar{\psi}}$ is communication consistent and $\bar{\psi}$ does not suffer from conflicting conjunctions as per Fact 1-5. Moreover, for all state signals $\mathbf{x}(t) \in \mathcal{X}$ such that $(\mathbf{x}(t), 0) \models \bar{\psi}$ we have $(\mathbf{x}(t), 0) \models \psi$.

Proof: When (50) is feasible, then the satisfaction of the constraints (50b)-(50c) implies, by Prop. 6, 7, 8 and 9, that the resulting tasks ϕ_{ij} , $\forall (i, j) \in \mathcal{E}_{\bar{\psi}}$ are communication consistent and are not conflicting conjunctions. Moreover, by the conditions of Lemma 1, which are enforced via constraint (50d), we know that for each inconsistent task ϕ_{ij} with $(i, j) \in \mathcal{E}_\psi \setminus \mathcal{E}_c$ in $\mathcal{G}_\psi$, the conjunction of tasks $\bar{\phi}^{ij} = \bigwedge_{(r,s) \in \epsilon(\pi_i^j)} \bar{\varphi}_{rs}^{ij}$ as per (21) is such that $(\mathbf{x}(t), 0) \models \bar{\phi}^{ij} \Rightarrow (\mathbf{x}(t), 0) \models \phi_{ij}$. From these considerations, the result follows. ■

C. Computational aspects

The optimization problem (50) has a compact convex domain and a convex objective such that a bounded set of solutions to (50) exists if the domain set is not empty. The infeasibility of (50) occurs when the intersection between the feasible set of (50c) and (50d) is empty. This intuitively happens when an edge $(r, s) \in \mathcal{E}_\pi$ is overloaded with an excessive amount of tasks including conflicting conjunctions that can not be resolved. When (50) is distributed among the agents as per (52), the convergence to the solution of the centralized problem is sublinear [29] and the dimensionality of (52) (in terms of number of variables) scales as $(n+1)|\Omega_{rs}| + n|\mathcal{K}_{rs}|$, where $|\mathcal{K}_{rs}|$ is the number of tasks decomposed over the edge

Algorithm 1 For single edge $(r, s) \in \mathcal{E}_\pi = \mathcal{E}_\Theta$ [29]

- 1: **allocate** $\mathbf{x}_{rs}, \rho_{rs}, \mu_{rs}, \lambda_{rs, r's'}, \forall (r', s') \in \mathcal{N}_\Theta((r, s))$
 - 2: **initialize** ${}^0\lambda_{rs, r's'}, \forall (r', s') \in \mathcal{N}_\Theta((r, s))$ and $c_\rho >> 0$
 - 3: **while** not converged **do**
 - 4: **gather** ${}^t\lambda_{rs, r's'}$ from $(r', s') \in \mathcal{N}_\Theta((r, s))$
 - 5: **get** $(({}^{t+1}\mathbf{x}_{rs}, {}^{t+1}\rho_{rs}), {}^{t+1}\mu_{rs})$ solution of (55)
 - 6: **for** $(r', s') \in \mathcal{N}_\Theta((r, s))$ **do**
 - 7: **gather** ${}^{t+1}\mu_{r's'}$ and **update** ${}^{t+1}\lambda_{rs, r's'}$ as
 ${}^{t+1}\lambda_{rs, r's'} = {}^t\lambda_{rs, r's'} - {}^t\gamma ({}^{t+1}\mu_{rs} - {}^{t+1}\mu_{r's'})$
 - 8: $t = t + 1$
-

(r, s) and $|\mathcal{Q}_{rs}|$ is the number of conflicting conjunctions over the edge. Intuitively, the complexity of (52) grows when edges are overloaded with many tasks that are potentially in conflict. On the other hand, if we let $\bar{m}$ be the maximum number of rows among the matrices M_{ij} in (50d), then the number of exchanged Lagrangian multiplier variables (see Alg. 1, line 4) has a worst case scaling of $\bar{m} \cdot |\mathcal{K}_{rs}|$. Thus, the more complex the polytope representation of the predicate level sets $\mathcal{B}_{ij}$ is, the more information has to be exchanged.

We highlight that while the edge-computing graph is a convenient abstraction for our decomposition, either agent $r \in \mathcal{V}$ or $s \in \mathcal{V}$ will be effectively solving (55) over the edge $(r, s) \in \mathcal{E}_\pi = \mathcal{V}_\Theta$. Since the communication graph $\mathcal{G}_c$ is acyclic and $\mathcal{E}_\pi \subseteq \mathcal{E}_c$, then it is possible to select the computing nodes by selecting the leaf nodes of $\mathcal{G}_c$ as computing and iteratively select computing nodes along the paths from the leaf agent to the root of the graph. In this way the case in which a single agent r solves an instance of (55) for both the edges (r, s') and (r, s'') for some $((r, s'), (r, s'')) \in \mathcal{E}_\Theta$, is avoided. Moreover, consider again two neighbouring edges $((s', r), (r, s'')) \in \mathcal{E}_\Theta$ over the computing graph $\mathcal{G}_\Theta$ such that the node r is shared. Then exchanging the consensus vectors and Lagrangian parameters ${}^t\mu_{rs'}$, ${}^t\mu_{rs''}$, ${}^t\lambda_{rs',rs''}$ and ${}^t\lambda_{rs'',rs'}$ among the neighbouring edges (s', r) and (r, s'') over the communication graph $\mathcal{G}_c$, takes either 1 communication hop (if s' and r are the computing agents for (s', r) and (r, s'') , respectively), or 2 communication hops (if s' and s'' are the computing agents for (s', r) and (r, s'') , respectively).

VII. SIMULATIONS

We apply the developed algorithm for a space exploration task. A team of 15 agents with single integrator dynamics $\dot{\mathbf{x}}_i = \mathbf{u}_i \in \mathbb{R}^2, \forall i = 1, \dots, 15$ is considered for the mission, where $\mathbf{x}_i$ represents the position of each agent and such that $\|\mathbf{u}_i\| \leq 1.8 \text{ km/h}$. The exploration consists of taking three panoramic images of the Thira crater on Mars' southern hemisphere from three different sides of the crater and visiting smaller craters around Thira. The interest points are represented by red solid circles in Figure 6, with Thira being the largest and marked by a yellow dot. A cyan square represents the location of the main base station from which the agents initially depart from to accomplish the mission. Each agent is assumed to have a maximum communication range of $r_c = 8.5 \text{ km}$. The mission consists of a first exploration phase from the base station located at $[0, 0]$ to the interest points, which lasts 20 hours. During the exploration phase, agent 1 can be considered the global leader of the team and the one holding the information about the interest points location, such that the other agents 2-15 can reach their target location by staying in formation with agent 1. This phase is expressed by the tasks $G_{[10,20]} \langle -{}^4A_1(\mathbf{x}_1 - [5 \ 0]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$, $F_{[10,15]} \langle -{}^6A_{1,8}(\mathbf{e}_{1,8} - [10 \ 4]^T) + \mathbf{1}_6 \cdot 0.4 \rangle$, $F_{[10,15]} \langle -{}^6A_{1,4}(\mathbf{e}_{1,4} - [10 \ -4]^T) + \mathbf{1}_6 \cdot 0.4 \rangle$, $G_{[13,15]} \langle -{}^4A_{1,14}(\mathbf{e}_{1,14} - [-12 \ 5.5]^T) + \mathbf{1}_4 \cdot 0.7 \rangle$, $F_{[10,15]} \langle -{}^5A_{1,13}(\mathbf{e}_{1,13} - [-6 \ -6]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$, $F_{[13,15]} \langle -{}^5A_{1,5}(\mathbf{e}_{1,5} - [-3.5 \ -4]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$, $G_{[10,15]} \langle -{}^5A_{1,6}(\mathbf{e}_{1,6} - [0 \ 2]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$, $G_{[10,15]} \langle -{}^5A_{1,2}(\mathbf{e}_{1,2} - [0 \ -2]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$,

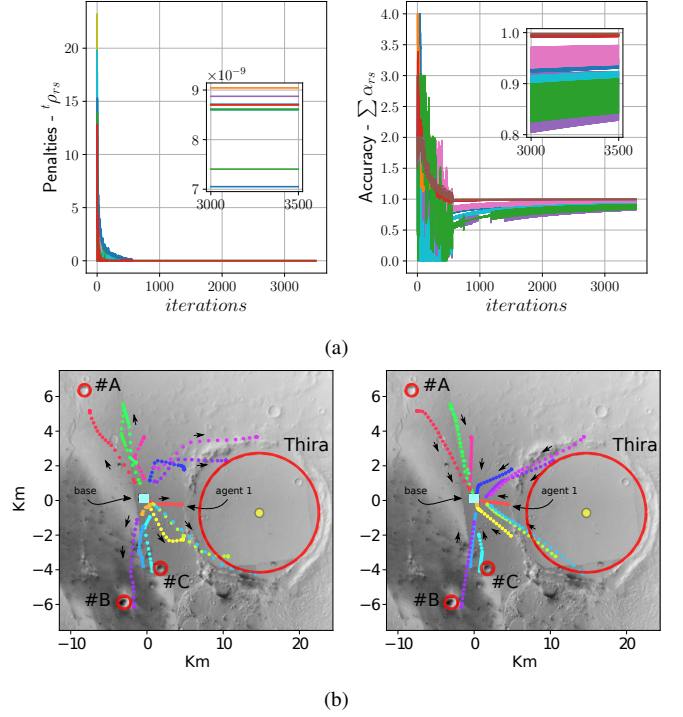


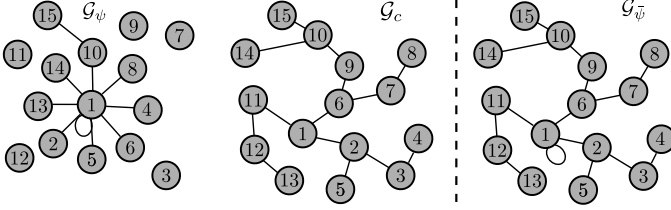
Fig. 6: (a) Optimal penalty values ${}^t\rho_{rs} \forall (r, s) \in \mathcal{E}_\pi$ and decomposition accuracy $\sum_{l \in \mathcal{K}_{rs}} \alpha_{rs}^l$ as per Alg. 1. (b) Agent's trajectories in the exploration phase (left) and return phase (right). Black arrows represent the direction of movement. A cyan square represents the main base. Red circles represent interest points to be visited.

$G_{[10,15]} \langle -{}^5A_{1,11}(\mathbf{e}_{1,11} - [-4 \ 0]^T) + \mathbf{1}_5 \cdot 0.4 \rangle$, $G_{[18,20]} \langle -{}^5A_{10,15}(\mathbf{e}_{10,15} - [0 \ 3]^T) + \mathbf{1}_5 \cdot 0.3 \rangle$. The notation ${}^nA_{ij}$ indicates the matrix A_{ij} associated with a regular polytope with n sides i.e for $n = 5$ then ${}^nA_{ij}$ is the matrix with 5 rows, where each row is the normal vector to the side of a regular pentagon. It is not necessary for our approach that the polytopes $\mathcal{P}$ have any regularity. Informally, the conjunction of the previous tasks implies that agents 1, 8 and 4 are involved in taking three panoramic images of Thira, with agent 1 (red trajectory in Fig. 6) approaching Thira from the west side leaving from the base station, while agents 4 and 8 reach the southern and northern sides of the crater respectively by maintaining an angle of 45 degrees with agents 1, thus reaching a triangular formation around Thira. Agent 14 visits the upper-left crater (marked as #A), while agents 13 and 5 visit the lower craters (#B and #C respectively). At the same time, agents 11, 6, and 2 should achieve a triangular formation around 1. Lastly, agents 15 and 10 should remain within 200 m of each other. After the exploration phase, the agents should group again around agent 1 in a radius of 2 km, while agent 1 returns to the base station, effectively "dragging" the team back to base. This phase is described via the tasks $G_{[30,35]} \langle -{}^8A_i(\mathbf{x}_1 - [0 \ 0]^T) + \mathbf{1}_8 \cdot 0.2 \rangle$, $G_{[30,40]} \langle -{}^8A_{1,i}(\mathbf{e}_{1,i} - [0 \ 0]^T) + \mathbf{1}_8 \cdot 2 \rangle, \forall i = 2, \dots, 15$.

Figure 6 shows the exploration phase (left panel) from time 0h to 20h, and the return phase (right panel) from time 20h to 40h. The task graph ($\mathcal{G}_\psi$), communication graph ($\mathcal{G}_c$) and the new task graph ($\mathcal{G}_{\psi'}^*$) obtained implementing the distributed task decomposition algorithm presented in Section VI-B are

TABLE I: Combined Edge-computing graph data

$(r, s) \in \mathcal{E}_\pi$	(10,9)	(15,10)	(2,1)	(4,3)	(14,10)	(1,11)	(13,12)	(12,11)	(6,1)	(5,2)	(9,6)	(8,7)	(3,2)	(7,6)
$ \Pi_{rs} $	4	1	5	2	2	3	2	3	8	2	5	2	3	3
$\dim \mathcal{X}_{rs} $	14	3	21	6	6	13	6	11	30	6	17	6	11	11
$\sum_{(i,j) \in \Pi_{rs}} m_{ij}$	208	64	253	100	80	153	89	153	436	89	272	100	164	164
$ \mathcal{Q}_{rs} $	1	0	3	0	0	2	0	1	3	0	1	0	1	1

Fig. 7: From the left: Original task graph $\mathcal{G}_\psi$, communication graph $\mathcal{G}_c$ and new task graph $\mathcal{G}_{\bar{\psi}}$.

shown in Figure 7. To achieve the task satisfaction, the agents are controlled via the Control Barrier Functions-based controller proposed in [40], exploiting only the state information of the neighbours in $\mathcal{N}_c(i)$ while enforcing collision avoidance. The controller in [40] requires input-affine nonlinear dynamics, but more general dynamics can be considered, as long as a valid controller leveraging only information for the agents in $\mathcal{N}_c(i)$ is designed to satisfy $\bar{\psi}$. As of now, feedback controllers that can handle general nonlinear dynamics, STL constraints, collision avoidance, and communication constraints simultaneously are not widely studied and are an active area of research.

The edge computing graph $\mathcal{G}_\Theta(\mathcal{V}_\Theta, \mathcal{E}_\Theta)$ is not reported due to space limitations, but it can be computed recalling that $\mathcal{V}_\Theta = \mathcal{E}_\pi$ and $\mathcal{E}_\Theta$ is as per (54). The task graph $\mathcal{G}_{\bar{\psi}}$ is computed before the start of the mission at time 0h, where Alg. 1 is run for 3500 iterations by each agent in $\mathcal{G}_\Theta$ with an average computational time of 52.133 s/agent (assuming no delay in exchanging the consensus and Lagrangian multiplier variables). In Table I we list the number of decomposition paths passing through a specific edge $(r, s) \in \mathcal{E}_\pi$ ($|\Pi_{rs}|$), the dimension of the variables $\mathcal{X}_{rs}$, the total number of shared constraints ($\sum_{(i,j) \in \Pi_{rs}} m_{ij}$) and the number of conflicting conjunctions sets of tasks ($|\mathcal{Q}_{rs}|$). Figure 6a shows the evolution of the penalty values $t_{\rho_{r,s}} \forall (r, s) \in \mathcal{E}_\pi$ and the accuracy of decomposition for each task over a range of 3500 iterations. In particular, the time evolution of the optimal penalties $t_{\rho_{r,s}}$ indicates that a valid task decomposition is found after 600 iterations of Alg. 1, while the decomposition accuracy as per (35) keeps improving before stabilizing at 3500 iterations. Since the optimization program (55) is only quasi-convex, the solution of the task decomposition is not unique. We used the open-source library CasADi to solve (55), leveraging an interior-point method solver on an Intel-Core i7-1265U.

VIII. CONCLUSIONS

We proposed a decentralized task decomposition approach to decompose collaborative STL tasks defined over communication inconsistent edges on the task graph, as conjunctions of tasks defined over communication consistent edges on the communication graph. Future work will involve expanding the

proposed results to consider more generic types of collaborative tasks, different metrics of connectivity (e.g. field of view connectivity) and include the impact of the agents' dynamics within the proposed decomposition framework.

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