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Simultaneous Synchronization and Topology Identification of Complex Dynamical Networks

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Abstract— **We propose a new method for simultaneous synchronization and topology identification of a complex dynamical network that relies on the edge-agreement framework and on adaptive-control approaches by design of an auxiliary network. Our method guarantees the identification of the unknown topology and it guarantees that once the topology is identified the complex network achieves synchronization. Under our identification algorithm we are able to provide stability results for the estimation errors in the form of uniform semiglobal practical asymptotic stability. Finally, we demonstrate the effectiveness of our approach with an illustrating example.**

I. Introduction

Complex dynamical networks exist everywhere and can represent multiple real networks, including social networks, biological networks, sensor systems and so on [1]. For instance, collaborative behaviors over dynamical networks can be used to model complex phenomena in biological systems or complex tasks in multi-robots scenarios. Synchronization is one of the fundamental problems of multi-agent coordination, aiming at rendering all agents to behave following the same dynamics. Plenty of results on how to design strategies for the synchronization of different systems have been achieved in recent years [1]–[4].

The topology of a complex network plays a key role in the control design for synchronization. Most of the existing methods, however, assume complete knowledge of the network topology and do not consider the relevant and realistic situations where such topology is, usually, partially or fully unknown. In view of this, lots of methods have been developed to identify the topology of complex networks, including knock-out methods [5], [6], optimization-based methods [7], and adaptive-control-based methods [8]–[11], to name a few. In this paper we focus on an adaptive-control based method to identify the network topology.

Adaptive-control-based designs have been used to identify the topology of various network models, where a *linear independence condition* is an important assumption for successful identification of the topology, see e.g. [8], [9]. This condition, however, is usually difficult to verify. In light of the latter, the authors in [10] propose an approach to address this concern by designing the control of the original network to track an auxiliary system, which guarantees the success of the topology identification of a network. The auxiliary system is designed such that the linear independence condition is inherently satisfied, thus avoiding the need to verify it explicitly. However, this method only deals with the topology identification problem, without considering the equally relevant problem of simultaneous achievement of synchronization and topology identification. Indeed, in real scenarios, the topology of complex systems being unknown, hinders the use of existing algorithms for synchronization. However, in the existing works mentioned above, in order to successfully identify the topology, the system cannot reach a synchronized state since the conditions guaranteeing the success of the identification algorithm are lost at synchronization. Therefore, in this paper we propose an approach to simultaneously identify the topology and reach synchronization.

The main contribution of this paper is two-fold. Firstly, we present a new approach to study the network topology identification problem based on the so-called *edge-agreement* framework [1]. The edge-agreement framework transforms the synchronization problem of complex networks into the stabilization problem of the edge dynamics, which allows the simultaneous synchronization and topology identification problem to be solved by well-known adaptive control methods. It relies on the definition of a virtual weighted dynamic complete graph, where the non-zero edge-weights correspond to the edges of the real topology. Hence, the topology is identified by estimating the weights of edges of the complete graph. Second, we propose a new topology identification algorithm based on adaptive control using the concept of δ-persistency of excitation [12]. For the topology identification of complex networks, we design an auxiliary system with δ-persistently exciting dynamics, thereby preventing the synchronization from happening before the topology has been identified. It guarantees the success of simultaneous identification of the topology and the synchronization of the network. Moreover, the δ-persistency of excitation condition enlarges the choice of possible auxiliary systems, compared to the approach in [10]. Different from the existing works studying topology identification using adaptive-control tools, in this work we establish uniform semi-global practical asymptotical stability for the topology identification errors. The latter are stronger stability results than the convergence properties usually established in the literature.

Notations: A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$

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if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is decreasing and $\sigma(s) \to 0$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot) \in \mathcal{L}$ for any $s \in \mathbb{R}_{\geq 0}$. We use $|\cdot|$ for the Euclidean norm of vectors and the induced L_2 norm of matrices. The set $\mathbb{B}(\delta) \subset \mathbb{R}^n$ is the closed ball of radius Δ centered at the origin, i.e. $\mathbb{B}(\Delta) := \{x \in \mathbb{R}^n \ : \ |x| \leq \Delta\}$. The notation $\mathbb{H}(\delta, \Delta) := \{x \in \mathbb{R}^n \ : \ \delta \leq |x| \leq \Delta\}$. We define $|x|_\delta := \inf_{y \in \mathbb{B}(\delta)} |x - y|$. We use $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ to denote a weighted graph defined by a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ with cardinality N, an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ with cardinality M such that an edge $e_k := (i,j) \in \mathcal{E}$, $k = \{1, \ldots, M\}$ indicating that agent j has access to information from node i, and a positive diagonal matrix $W \in \mathbb{R}^{M \times M}$, whose diagonal w_k entries represent the weights of the edges. A tree is a subgraph in which every node has exactly one parent except for one node, called the root, which has no parent and which has a path to every other node. A spanning tree is a tree subgraph containing all nodes in $\mathcal{V}$. A graph is said to be complete if there exists an edge between every pair of agents.

II. Model and problem formulation

We consider a multi-agent system where the agents interact over an *unknown* topology described by an *undirected* graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, W)$. Each agent's dynamics is given by

$$\dot{x}_i = f_i(x_i) - \sum_{j=1}^{N} w_{ij} [x_i - x_j] + u_i \quad i \in \mathcal{V}, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent i, $f_i : \mathbb{R} \to \mathbb{R}$ is a smooth function, $w_{ij} \equiv w_k$ is the *unknown* weight of the interconnection between agents i and j such that $w_{ij} = 0$ if the edge $e_k \notin \mathcal{E}$ and $w_{ij} \neq 0$ if the edge $e_k \in \mathcal{E}$.

Let $E \in \mathbb{R}^{N \times M}$ denote the (unknown) incidence matrix of an arbitrary orientation of $\mathcal{G}$. This is a matrix with rows indexed by the nodes and columns indexed by the edges with its (i,k)th entry defined as follows: $[E]_{ik} := -1$ if i is the terminal node of edge e_k, $[E]_{ik} := 1$ if i is the initial node of edge e_k, and $[E]_{ik} := 0$ otherwise. Defining $x := [x_1 \ \ldots \ x_N]^\top$, $F(x) := [f(x_1) \ \ldots \ f(x_N)]^\top$, and $u := [u_1 \ \ldots \ u_N]^\top$, (1) may be written in compact form as

$$\dot{x} = F(x) - EWE^\top x + u. \tag{2}$$

Now, denote by $\bar{E}$ the incidence matrix of a *complete* graph $\mathcal{K}(\mathcal{V}, \mathcal{E}_c, \bar{W})$, where $\mathcal{E} \subseteq \mathcal{E}_c$, $|\mathcal{E}_c| = \frac{1}{2}N(N-1) =: \bar{M}$, and let $\bar{W} := \mathrm{diag}\{\bar{w}_k\}$ where $\bar{w}_k \equiv w_k$ if $\bar{e}_k \in \mathcal{E}$ and $\bar{w}_k = 0$ if $\bar{e}_k \in \mathcal{E}_c \backslash \mathcal{E}$. That is, the weight $\bar{w}_k$ is different from 0 if the edge $\bar{e}_k$ of the complete graph exists in the graph $\mathcal{G}$ to be identified. Therefore, akin to (2), we have that (1) can also be written as

$$\dot{x} = F(x) - \bar{E}\bar{W}\bar{E}^\top x + u. \tag{3}$$

In this paper, we rely on the edge-agreement framework [1] for the representation of multi-agent systems in which we consider the state of the edges of the graph rather than that of the nodes. Hence, we define the edge variable $z := \bar{E}^\top x$. Then, using (3) we obtain

$$\dot{z} = \bar{E}^\top F(x) - \bar{E}^\top \bar{E}\bar{W}z + \bar{E}^\top u. \tag{4}$$

Under the edge-based representation for networked systems it is possible to obtain an equivalent reduced system in terms of an arbitrary directed spanning tree. Indeed, using an appropriate labeling of the edges, the incidence matrix of the complete graph $\mathcal{K}$ may be expressed as

$$\bar{E} = [\ \bar{E}_\mathcal{T} \quad \bar{E}_\mathcal{C}\ ] \tag{5}$$

where $\bar{E}_\mathcal{T} \in \mathbb{R}^{N \times (N-1)}$ denotes the full-column-rank incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_\mathcal{T} \subset \mathcal{K}$ and $\bar{E}_\mathcal{C} \in \mathbb{R}^{N \times (\bar{M}-N+1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_\mathcal{T}$. Moreover, defining

$$R := [\ I_{N-1} \quad T\ ], \quad T := \left(\bar{E}_\mathcal{T}^\top \bar{E}_\mathcal{T}\right)^{-1} \bar{E}_\mathcal{T}^\top \bar{E}_\mathcal{C}, \tag{6}$$

with I_{N-1} denoting the $N-1$ identity matrix, one obtains the identity

$$\bar{E} = \bar{E}_\mathcal{T} R. \tag{7}$$

Similarly, the edge state may be divided as

$$z = \left[z_\mathcal{T}^\top \ \ z_\mathcal{C}^\top\right]^\top, \tag{8}$$

where $z_\mathcal{T} \in \mathbb{R}^{(N-1)}$ are the states of the edges corresponding to the spanning tree $\mathcal{G}_\mathcal{T}$ and $z_\mathcal{C} \in \mathbb{R}^{\bar{M}-N+1}$ are the states of the remaining edges. Moreover, using (7) we have that

$$z = R^\top z_\mathcal{T}. \tag{9}$$

Then, from (4), applying (7) and (9), we obtain a reduced-order model

$$\dot{z}_\mathcal{T} = \bar{E}_\mathcal{T}^\top F(x) - \bar{E}_\mathcal{T}^\top \bar{E}\bar{W}R^\top z_\mathcal{T} + \bar{E}_\mathcal{T}^\top u. \tag{10}$$

Note that, since $\bar{E}$ is the incidence matrix of a complete graph on N nodes, it is *known*. Moreover, any arbitrary spanning tree contained in the complete graph may be chosen for defining (5) and (8). Now, using the notation in (4), the topology identification problem reduces to estimating the diagonal entries of matrix $\bar{W}$, i.e. the weights $\bar{w}_k$. On the other hand, the synchronization problem is transformed into the stabilization of the origin for the reduced-order system (10). Indeed, from (9) $z_\mathcal{T} \to 0$ implies $z \to 0$, and the latter is equivalent to $x_i - x_j \to 0$ for all $i, j \in \mathcal{V}$.

III. Main result

A. Simultaneous identification and synchronization

The control goal is to design the external input u in order to simultaneously identify the unknown graph topology $\mathcal{G}$ (the edge weights $\bar{w}_k$) and synchronize the dynamical systems (1). For that purpose, the main component of the design is to drive the network system in the edge representation (4) to converge to the known dynamics of an auxiliary system with state $\hat{x}$. More precisely, let us define $\bar{w}^\top := [\bar{w}_1 \ \cdots \ \bar{w}_{\bar{M}}] \in \mathbb{R}^{\bar{M}}$ as the vector of unknown weights, $\hat{w}^\top := [\hat{w}_1 \cdots \hat{w}_{\bar{M}}] \in \mathbb{R}^{\bar{M}}$ as the estimate of $\bar{w}$, and $\hat{W} := \operatorname{diag}\{\hat{w}\}$. Then, the external input is set to

$$u = -F(x) - c_1(x - \hat{x}(t)) + \dot{\hat{x}}(t) + \bar{E}\hat{W}(t)\hat{z}(t) \tag{11}$$

with the adaptive law

$$\dot{\hat{w}} = -\hat{Z}(t)\bar{E}^\top \bar{E}\tilde{z}, \tag{12}$$

where c_1 is a positive constant, $\tilde{z} := z - \hat{z}(t) = \bar{E}^\top(x - \hat{x}(t))$, $\hat{z}(t) := \bar{E}^\top \hat{x}(t)$, $\hat{Z}(t) := \operatorname{diag}\{\hat{z}(t)\}$.

Define $\tilde{w} := \bar{w} - \hat{w}$, $\xi^\top := \begin{bmatrix}\tilde{z}^\top & \tilde{w}^\top\end{bmatrix}$, and let $\phi(t,\xi) : \mathbb{R}_{\geq 0} \times \mathbb{R}^{2\bar{M}} \to \mathbb{R}^{\bar{M}}$, $(t,\xi) \mapsto \phi(t,\xi)$ be a function to be defined later. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous non-decreasing function. Assume that for all $\xi \in \mathbb{R}^{\bar{M}}$ and almost all $t \in \mathbb{R}_{\geq 0}$

$$\max\left\{|\phi(\cdot)|, \left|\frac{\partial\phi(\cdot)}{\partial t}\right|, \left|\frac{\partial\phi(\cdot)}{\partial \xi}\right|\right\} \leq \rho(|\xi|). \tag{13}$$

Then, the update law for the auxiliary variable $\hat{x}(t)$ (for $\hat{z}(t)$ in edge form) is given by

$$\dot{\hat{z}} = -c_2\hat{z} + \bar{E}^\top \phi(t,\xi) \tag{14}$$

where c_2 is a positive constant.

From (4), (11), and (12) we have that the closed-loop system is given by (14) and

$$\begin{bmatrix}\dot{\tilde{z}} \\ \dot{\tilde{w}}\end{bmatrix} = \begin{bmatrix}-c_1 I - \bar{E}^\top\bar{E}\bar{W} & -\bar{E}^\top\bar{E}\hat{Z}(t) \\ \hat{Z}(t)\bar{E}^\top\bar{E} & 0\end{bmatrix}\begin{bmatrix}\tilde{z} \\ \tilde{w}\end{bmatrix}, \tag{15}$$

or, equivalently, using the identities (7) and (9), by

$$\begin{bmatrix}\dot{\tilde{z}}_{\mathcal{T}} \\ \dot{\tilde{w}}\end{bmatrix} = \begin{bmatrix}-c_1 I - \bar{E}_{\mathcal{T}}^\top\bar{E}_{\mathcal{T}}R\bar{W}R^\top & -\bar{E}_{\mathcal{T}}^\top\bar{E}_{\mathcal{T}}R\hat{Z}(t) \\ \hat{Z}(t)R^\top\bar{E}_{\mathcal{T}}^\top\bar{E}_{\mathcal{T}}RR^\top & 0\end{bmatrix}\begin{bmatrix}\tilde{z}_{\mathcal{T}} \\ \tilde{w}\end{bmatrix}. \tag{16}$$

The main result, presented in Proposition 1 below, relies on designing the signal $\hat{z}(t)$ to be persistently exciting (PE) through the design of ϕ—see Appendix I for a definition. Indeed, systems in the form of (15) and (16) have been studied for decades in adaptive control and are known to be *globally exponentially stable* if $\hat{z}(t)$ is PE [13]. Although this would allow us to identify the unknown topology, as in [10] synchronization of the multi-agent system would be prevented even when the topology has been correctly estimated. In order to identify the topology and simultaneously achieve synchronization, in this paper we design ϕ to be uniformly δ-persistently exciting (uδ-PE)—see Appendix I.

Proposition 1: Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{2\bar{M}} \to \mathbb{R}^{\bar{M}}$, $(t,\xi) \mapsto \phi(t,\xi)$ satisfying (13). Then, if ϕ is uδ-PE in the sense of Definition 2 in Appendix I, the origin of the closed-loop system (16) with $\hat{z}(t)$ given by the update law (14) is *uniformly globally asymptotically stable* and the topology of network (4) is identified by the estimate $\hat{w}$, that is, $\bar{w} = \lim_{t\to\infty} \hat{w}(t)$ with the update law (12). Moreover, the system achieves synchronization, i.e. $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{V}$ or, equivalently, $z_k(t) \to 0$ as $t \to \infty$ for all $k \leq \bar{M}$. □

Proof: The total derivative of the Lyapunov function $V_1(\tilde{z}_{\mathcal{T}}, \tilde{w}) = \frac{1}{2}\left[|\tilde{z}_{\mathcal{T}}|^2 + |\tilde{w}|^2\right]$ along (16) yields

$$\dot{V}_1(\tilde{z}_{\mathcal{T}}, \tilde{w}) \leq -c_1|\tilde{z}_{\mathcal{T}}|^2 \leq 0. \tag{17}$$

From (17) we conclude that the system (16) is uniformly globally stable. Therefore, the solutions $\xi(t, t_0, \xi_0)$ are uniformly globally bounded.

Now, for the auxiliary system (14) consider the Lyapunov function $\hat{V}(\hat{z}) := 0.5|\hat{z}|^2$ whose total derivative satisfies

$$\begin{aligned}\dot{\hat{V}}(\hat{z}) &\leq -c_2|\hat{z}|^2 + |\hat{z}||\bar{E}||\phi(t,\xi)| \\ &\leq -c_2'|\hat{z}|^2 + |\rho(|\xi|)|^2 \\ &\leq -c_2'|\hat{z}|^2 + \sigma,\end{aligned} \tag{18}$$

where the last inequality follows from the fact that the solutions $\xi(t, t_0, \xi_0)$ are uniformly globally bounded, hence, there exists a positive constant σ such that $|\rho(|\xi(t, t_0, \xi_0)|)|^2 \leq \sigma$ for all t. Similarly, from (18) we conclude that the solutions $\hat{z}(t, t_0, \hat{z}_0)$ are uniformly globally bounded. Therefore, since all the assumptions in Lemma 3 in Appendix I are satisfied, $\hat{z}(t)$ is uδ-PE with respect to ξ.

Then, invoking Lemma 4 in Appendix I we conclude that the origin of (16), and therefore that of (15), is uniformly globally asymptotically stable. Therefore, $\lim_{t\to\infty} \hat{w} = \bar{w}$. Moreover, since $\lim_{t\to\infty} \xi(t) = 0$ we have that, from (18), $\lim_{t\to\infty} \hat{z}(t) = 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$, i.e., the system asymptotically reaches synchronization. ■

Remark 1: The proof of Proposition 1 relies on the fact that ϕ in (14) is uδ-PE with respect to ξ. As a matter of fact, Proposition 1 still holds even if ϕ is uδ-PE "only" with respect to $\tilde{w}$—cf. [14]. However, for this purpose, one would have to design ϕ dependent on $\tilde{w}$, hence, on $\bar{w}$ which are the *unknown* weights to estimate. Therefore, in the following, we study the stability properties of (16) and (14) when ϕ is uδ-PE "only" with respect to $\tilde{z}$. •

Proposition 2: Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{\bar{M}} \times \mathbb{R}^{n_\theta} \to \mathbb{R}^{\bar{M}}$, $(t, \tilde{z}, \theta) \mapsto \phi(t, \tilde{z}, \theta)$, be parameterized by the free constants $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$. Moreover, let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous non-decreasing function and denote $\phi_\theta(t, \tilde{z}) := \phi(t, \tilde{z}, \theta)$ satisfies

$$\max\left\{|\phi_\theta(\cdot)|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial t}\right|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial \tilde{z}}\right|\right\} \leq \rho(|\tilde{z}|). \tag{19}$$

Then, if ϕ_θ is uδ-PE with respect to $\tilde{z}$, the origin of the closed-loop system (16) with $\hat{z}(t)$ given by the update law

$$\dot{\hat{z}} = -c_3\hat{z} + \bar{E}^\top\phi_\theta(t, \tilde{z}) \tag{20}$$

is *uniformly semiglobally practically asymptotically stable* on Θ. □

Proof: We begin by introducing the following result.

Consider the parameterized nonlinear time-varying system

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A & B\phi_\theta(t, x_1)^\top \\ -\phi_\theta(t, x_1)B^\top & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \tag{21}$$

where $x^\top := \begin{bmatrix}x_1^\top & x_2^\top\end{bmatrix}$, $\phi_\theta(t, x_1) := \phi(t, x_1, \theta)$, $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}^n$ is piece-wise continuous in t and continuous in x_1, with $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ a constant free parameter.

Lemma 1: For system (21) suppose there exists a continuous non-decreasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\max\left\{|\phi_\theta(\cdot)|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial t}\right|, \left|\frac{\partial\phi_\theta(\cdot)}{\partial x_1}\right|\right\} \leq \rho(|x_1|). \tag{22}$$

Furthermore, assume that for any $Q = Q^\top > 0$ there exist $P = P^\top > 0$ such that $A^\top P + PA = -Q$. Then, if $\phi_\theta(t, x_1)$ is uδ-PE with respect to x_1, the origin of (21) is *uniformly semiglobally practically asymptotically stable* on Θ. □

Proof: First consider the Lyapunov function $V_1 := x^\top P x$, which satisfies

$$\alpha_1 |x|^2 \leq V_1(t, x) \leq \alpha_2 |x|^2 \tag{23}$$

$$\dot{V}_1 \leq -\alpha_3 |x_1|^2 \tag{24}$$

with $\alpha_1 := \lambda_{min}(P)$, $\alpha_2 := \lambda_{max}(P)$, and $\alpha_3 := \lambda_{min}(Q)$.

Next, let us define the functions

$$V_2(t, x) := -x_1^\top B \phi_\theta(t, x_1)^\top x_2 \tag{25}$$

$$V_3(t, x) := -\int_t^\infty e^{(t-\tau)} \left| B\phi_\theta(\tau, x_1)^\top x_2 \right|^2 d\tau \tag{26}$$

$$V_4(t, x) := V_2(t, x) + V_3(t, x). \tag{27}$$

Note that for any $\Delta > 0$, in view of ϕ_θ being uδ-PE, (22), and Lemma 2, (25) satisfies, for all $(t, x) \in \mathbb{R} \times \mathbb{B}(\Delta)$

$$V_4(t, x) \leq b|x_1|\rho(|x_1|)|x_2| - \gamma_\theta(|x_1|)|x_2|^2 \tag{28}$$

where $b := |B|$ and $\gamma_\theta(|x_1|) := e^{\vartheta_\Delta(|x_1|)}\gamma_\Delta(|x_1|, \theta)$.

We proceed now to evaluate the total derivative of $V_4(t, x)$ along the trajectories of the system. First, we have

$$\begin{aligned}\dot{V}_2(t, x) &= x_1^\top B \phi_\theta(t, x_1)^\top \phi_\theta(t, x_1) B^\top x_1 \\ &\quad - x_2^\top \phi_\theta(t, x_1)^\top B^\top B \phi_\theta(t, x_1) x_2 \\ &\quad - x_2^\top \phi_\theta(t, x_1)^\top B^\top A x_1 - x_2^\top \overbrace{\phi_\theta(t, x_1)}^{\cdot} B^\top x_1 \\ &= V_2(t, x) - |B\phi_\theta(t, x_1) x_2|^2 + \left|\phi_\theta(t, x_1) B^\top x_1\right|^2 \\ &\quad - x_2^\top \phi_\theta(t, x_1)^\top B^\top (A - I) x_1 \\ &\quad - x_2^\top \overbrace{\phi_\theta(t, x_1)}^{\cdot} B^\top x_1.\end{aligned} \tag{29}$$

Next, we write

$$\frac{\partial V_3}{\partial x_1} = -\int_t^\infty 2e^{(t-\tau)} x_2^\top \phi_\theta(\tau, x_1) B^\top B \left[\frac{\partial \phi_\theta(\tau, x_1)}{x_1}^\top x_2\right] d\tau$$

$$\frac{\partial V_3}{\partial x_2} = -\int_t^\infty 2e^{(t-\tau)} \phi_\theta(\tau, x_1) B^\top B \phi_\theta(\tau, x_1)^\top x_2 d\tau$$

$$\frac{\partial V_3}{\partial t} = \left|B\phi_\theta(\tau, x_1)^\top x_2\right|^2 - \int_t^\infty \frac{\partial}{\partial t}\left[e^{(t-\tau)} \left|B\phi_\theta(\tau, x_1)^\top x_2\right|^2\right] d\tau.$$

Finally, from (22) and (28) we can obtain a bound for the derivative of (25). Define $b_\rho := b\rho(\Delta)$ and $\bar{\rho}(r, s) := b_\rho \left[5rs + b_\rho r^2 + b_\rho s^2\right]$, then, for $(t, x) \in \mathbb{R} \times \mathbb{B}(\Delta)$,

$$\dot{V}_4(t, x) \leq \bar{\rho}(|x_1|, |x_2|) - \gamma_\theta(|x_1|)|x_2|^2. \tag{30}$$

Now, consider the candidate Lyapunov function $V(t, x) := V_1(t, x) + \varepsilon V_4(t, x)$, with ε a small positive constant to be defined. Notice that in view of (28), $V_4(t, x)$ satisfies on $\mathbb{R} \times \mathbb{H}(\delta, \Delta)$

$$\begin{aligned}-\varepsilon\gamma_\theta(\Delta)|x_2|^2 - \varepsilon b_\rho |x_1||x_2| \leq \varepsilon V_4(t, x) \leq &\varepsilon b_\rho |x_1||x_2| \\ &- \varepsilon\gamma_\theta(\delta)|x_2|^2.\end{aligned} \tag{31}$$

So, from (23) and (31), for any $\Delta > \delta > 0$ and for a sufficiently small ε, there exist $\underline{\alpha}_{\delta,\Delta} > 0$ and $\overline{\alpha}_{\delta,\Delta} > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{H}(\delta, \Delta)$

$$\underline{\alpha}_{\delta,\Delta}|x|^2 \leq V(t, x) \leq \overline{\alpha}_{\delta,\Delta}|x|^2. \tag{32}$$

Using (24) and (30) the total derivative of $V(t, x)$ satisfies, for all $(t, x) \in \mathbb{R} \times \mathbb{H}(\delta, \Delta)$,

$$\begin{aligned}\dot{V}(t, x) \leq &- \alpha_3|x_1|^2 - 5\varepsilon b_\rho |x_1||x_2| + \varepsilon b_\rho^2 |x_1|^2 + \varepsilon b_\rho^2 |x_2|^2 \\ &- \varepsilon\gamma_\theta(\delta)|x_2|^2.\end{aligned}$$

Choosing $\theta^*(\delta, \Delta)$ such that $\gamma_\theta(\delta) \geq 2b_\rho^2$ we have

$$\dot{V}(t, x) \leq -(\alpha_3 - \varepsilon b_\rho^2)|x_1|^2 - 5\varepsilon b_\rho |x_1||x_2| - \varepsilon b_\rho^2 |x_2|^2, \tag{33}$$

and, for a sufficiently small ε, there exists $c > 0$ such that

$$\dot{V}(t, x) \leq -c|x|^2. \tag{34}$$

The result follows from (32), (34), and [15, Theorem 10]. ■

Following the same analysis as in the proof of Proposition 1 we can conclude that $\hat{z}(t)$, given by the update law (20), is uδ-PE with respect to $\tilde{z}$. Therefore, the proof follows from Lemma 1 replacing x_1, x_2, A, B, and ϕ_θ by $\tilde{z}$, $\tilde{w}$, $-c_1 I - \bar{E}^\top \bar{E}\bar{W}$, $-\bar{E}^\top \bar{E}$, and $\hat{Z}$, respectively. ■

Remark 2: An example of the parameterized uδ-PE function ϕ_θ in (20) can be given as

$$\phi_\theta(t, \tilde{z}) = \tanh\left(\kappa\bar{E}\tilde{z}\right)\sin(t), \tag{35}$$

where $\kappa \equiv \theta^*(\delta, \Delta)$ is the design parameter that can be chosen large enough so that the ultimate bound of the estimation error is as small as desired. Note that (35) is uδ-PE with respect to $\tilde{z}$ as per Definition 2 in Appendix I. •

IV. Numerical Example

We consider a network of $N = 10$ agents modeled by (1) with an interaction determined by the *unknown* topology $\mathcal{G}(\mathcal{V}, \mathcal{E}, W)$, represented in Fig. 1. The objective is to estimate the *unknown* weights $\bar{w}_{ij}$. For this purpose, the external input u is given by (11) with (12) and (20), where

$$\begin{aligned}\phi_\theta(t, \tilde{z}) &= \tanh\left(\kappa\bar{E}\tilde{z}\right) p_e(t) \\ p_e(t) &= 2\sin(\pi t) + 0.3\cos(6\pi t) - 0.5\sin(8\pi t)\end{aligned}$$

and the control gains are taken as $c_1 = 1$, $c_3 = 0.5$, $\kappa = 200$.

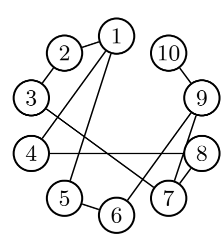

Fig. 1: Connected undirected topology graph of the network.

The simulation results are presented in Figs. 2-4. As can be clearly seen from Fig. 2, the estimation errors of the weights $\bar{w}_{ij}$ converge to the origin, successfully identifying the unknown topology, which is given by the values of the estimated weights in Fig. 3. Note that, as predicted by the theoretical results, the estimation error of the weights does not converge exactly to the origin, but rather to a small neighborhood of the origin, hence exact identification is not achieved. However, this small neighborhood can be made arbitrarily small by tuning the parameter κ in the uδ-PE function ϕ_θ. Indeed, in Fig. 5 we present the estimation errors with $\kappa = 20$ and it is clear that the steady state errors are larger than for the case in Fig. 5 with $\kappa = 200$. Finally, note that the topology is successfully identified (up to a small tunable error), as predicted by the theory, despite the system reaching synchronization, and therefore not satisfying the Linear Independence Condition, as can be seen from Fig. 4.

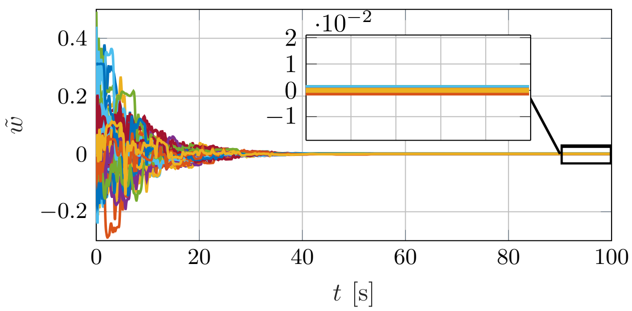

Fig. 2: Estimation errors of the graph weights ($\kappa = 200$).

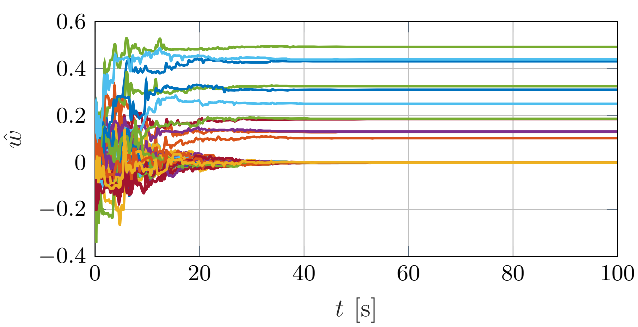

Fig. 3: Estimated values of the graph weights.

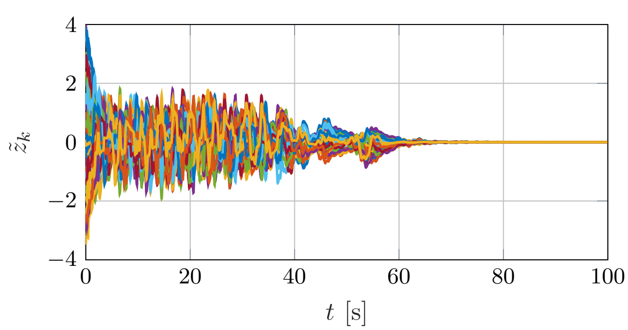

Fig. 4: Evolution of the synchronization errors.

V. Conclusions

We presented a new method for simultaneous synchronization and identification of the topology of a complex dynamical network. The design is based on the edge-agreement framework and on an known auxiliary network which is designed to be δ-persistently exciting. Under the proposed algorithm, the topology is successfully identified without requiring the verification of the Linear Independence Condition, and synchronization is achieved once the actual topology has been estimated. It is important to emphasize that, beyond mere convergence, we establish uniform semiglobal practical asymptotical stability of the estimation error. We believe that these properties may be useful to extend the current approach to more complex dynamical models and to establish finite- or fixed-time estimation and synchronization. The latter is the topic of current and further research.

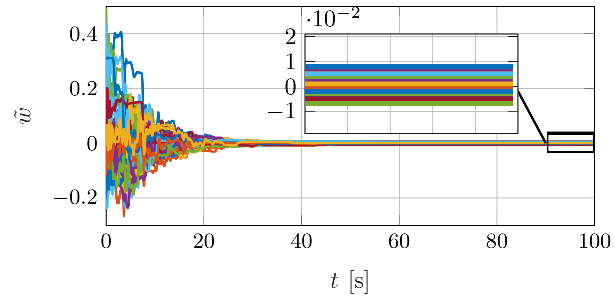

Fig. 5: Estimation errors of the graph weights ($\kappa = 20$).

Appendix I
On persistency of excitation and δ-persistency of excitation

Definition 1 (Persistency of excitation): A function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is said to be PE if there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$

$$\int_t^{t+T} |\phi(\tau)| d\tau \geq \mu. \tag{36}$$

The following is adapted from [16] to better fit the content of this paper. See also [12]. Let $x \in \mathbb{R}^n$ be partitioned as $x^\top := \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Define the column vector function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and the set $\mathcal{D}_1 := (\mathbb{R}^{n_1} \backslash \{0\}) \times \mathbb{R}^{n_2}$.

Definition 2 (δ-persistency of excitation): A function $\phi(\cdot,\cdot)$ where $t \mapsto \phi(t,x)$ is locally integrable and $x \mapsto \phi(t,x)$ is uniformly continuous in t, is said to be uδ-PE with respect to x_1 if and only if for each $x \in \mathcal{D}_1$ there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$

$$\int_t^{t+T} |\phi(\tau, x)| d\tau \geq \mu. \tag{37}$$

If $\phi(\cdot,\cdot)$ is uδ-PE with respect to the whole state x the it is simply said that $\phi(\cdot,\cdot)$ is uδ-PE.

The following characterization of uδ-PE, presented in [16], is a technical tool used in the proof of convergence.

Lemma 2: For each $\Delta > 0$ there exist $\gamma_\Delta \in \mathcal{K}$ and $\vartheta_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ continuous strictly decreasing such that

$$\{|x_1|, |x_2| \in [0, \Delta] \backslash \{x_1 = 0\}\}$$
$$\implies \int_t^{t+\vartheta_\Delta(|x_1|)} |\phi(\tau, x)| d\tau \geq \gamma_\Delta(|x_1|), \tag{38}$$

for all $t \in \mathbb{R}$. □

The next Lemma establishes that the output of a strictly proper stable filter driven by a uδ-PE input conserves such property. A reminiscent version of the lemma was originally presented in [14]. For completeness we rewrite it here with a different proof.

Lemma 3 (Filtration property): Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ and consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} f(t, x, \omega) \\ f_1(t, \omega) + f_2(t, x)\omega + \phi(t, x) \end{bmatrix} \tag{39}$$

with $f_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz in ω uniformly in t and measurable in t and satisfies $|f_1(\cdot)| \leq |\omega|$ for all t; $f_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$ is locally Lipschitz in x uniformly in t and measurable in t. Assume that $\phi(t, x)$ is uδ-PE with respect to x. Assume also that ϕ is locally Lipschitz and there exists a non-decreasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that, for almost all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$:

$$\max\left\{|\phi(\cdot)|, |f(\cdot)|, |f_2(\cdot)|, \left|\frac{\partial\phi(\cdot)}{\partial t}\right|, \left|\frac{\partial\phi(\cdot)}{\partial x}\right|\right\} \leq \alpha(|x|). \tag{40}$$

Assume further that all solutions $t \mapsto x_\phi$ of (39), with $x_\phi^\top = \begin{bmatrix} x^\top & \omega^\top \end{bmatrix}$, are defined in $[t_0, \infty)$ and satisfy

$$|x_\phi(t, t_0, x_{\phi 0})| \leq r \quad \forall t \geq t_0, \tag{41}$$

then ω is uδ-PE with respect to x. □

Proof: Defining $\rho = -\omega^\top \phi$ we have that

$$\begin{aligned} \dot{\rho} &= -|\phi|^2 - \phi^\top f_1 - \omega^\top \left[f_2\phi + \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} f\right] \\ &\leq -|\phi|^2 + 2|\omega| \left[\alpha^2(r) + \alpha(r)\right] =: -|\phi|^2 + c(r)|\omega|. \end{aligned}$$

Inverting the sign and integrating both sides from t to $t+T_f$, with $T_f := (k+1)T$, we obtain that

$$\begin{aligned} &\omega(t)^\top \phi(t, x) - \omega(t + T_f)^\top \phi(t + T_f, x) \\ \geq &\int_t^{t+T_f} |\phi(\tau, x)|^2 d\tau + \int_t^{t+(k+1)T} c(r)|\omega(\tau)| d\tau \end{aligned} \tag{42}$$

Using the bounds in (40) and (41) on the left-hand side of inequality (42), the latter is equivalent to

$$2\alpha(r)r \geq \int_t^{t+T_f} |\phi(\tau, x)|^2 d\tau + \int_t^{t+T_f} c(r)|\omega(\tau)| d\tau.$$

Since $\phi(t, x)$ is uδ-PE, there exists μ such that

$$\int_t^{t+T_f} |\phi(\tau, x)|^2 d\tau \geq (k+1)\mu^2.$$

Then we obtain

$$\int_t^{t+T_f} |\omega(\tau)| d\tau \geq \frac{k+1}{c(r)}\mu^2 - \frac{2\alpha(r)r}{c(r)}.$$

Finally, choosing k large enough so that $k \geq 2\alpha(r)r$ we obtain that $\omega(t)$ is uδ-PE with respect to x. ■

The following lemma presented originally in [12] is included here for completeness. Consider the nonlinear time-varying system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & B\phi(t, x(t))^\top \\ -\phi(t, x(t))C^\top & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{43}$$

where $x^\top := \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}$, the function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^m$ is piece-wise continuous in t and continuous in x.

Lemma 4: For the system (43) suppose that there exist continuous non-decreasing functions $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, such that

$$\begin{aligned} |\phi(t, x)| &\leq \alpha_1(|x|), \\ \max\left\{\left|\frac{\partial\phi(t, x)}{\partial t}\right|, \left|\frac{\partial\phi(t, x)}{\partial \xi}\right|\right\} &\leq \alpha_2(|x|) \end{aligned}$$

and assume that $\phi(t, x(t))$ is uδ-PE with respect to x. If the triple $(A,\ B,\ C)$ satisfies the Kalman-Yakubovich-Popov (KYP) lemma, i.e. there exist $P = P^\top > 0$ and $Q = Q^\top > 0$ such that $A^\top P + PA = -Q$ and $PB = C$, then the origin of (43) is uniformly globally asymptotically stable. □

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