ABSTRACTIONS OF VARYING DECENTRALIZATION DEGREE FOR REACHABILITY OF COUPLED MULTI-AGENT SYSTEMS*

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Abstract. In this paper we present a decentralized abstraction framework for multi-agent systems with couplings in their dynamics, which arise in their popular coordination protocols. The discrete models are based on a varying decentralization degree, namely, the agents' individual abstractions are obtained by using discrete information up to a tunable distance in their network graph. Deriving these models at the agent level is essential in order to address scalability issues which appear in the discretization of systems with a high state dimension. The approach builds on the appropriate discretization of the agents' state space and the selection of a transition time step, which enable the construction of a nonblocking transition system for each agent with quantifiable transition possibilities. The transitions are based on the design of local feedback laws for the manipulation of the coupling terms, which guarantee the execution of the transitions by the continuous system. For a class of nonlinear agent interconnections, the derivation of such abstractions is always guaranteed, based on sufficient conditions which relate the agents' dynamics and the space/time quantization.

Key words. hybrid systems, multi-agent systems, abstractions, transition systems.

AMS subject classifications. 93A14, 93C10, 93C15

1. Introduction.

The analysis and control of multi-agent systems constitutes an active area of research with numerous applications, ranging from the analysis of power networks to the automatic deployment of robotic teams [10]. Of central interest in this field is the problem of high level planning by exploiting tools from formal verification [21]. In order to follow this approach for dynamical systems, a suitable discrete representation of the system is required, which enables the automatic synthesis of discrete plans that guarantee satisfaction of the high level specifications. Then, under appropriate relations between the continuous system and its discrete analogue, these plans can be converted to low level primitives such as sequences of feedback controllers, which enable the execution of the corresponding tasks by the continuous system.

The need for a formal approach to the aforementioned control synthesis problem has led to a considerable research effort for the extraction of discrete state symbolic models, also called abstractions. Results in this direction for the nonlinear single plant case have been obtained in the papers [27] and [36], which exploit approximate simulation and bisimulation relations. Exact variants of these relations were introduced in control by providing state space models of reduced dimension while preserving behavioral properties, originally for linear [26], and for nonlinear systems in [34], [16], and recently, also to address stability of dynamical and hybrid systems [29]. Symbolic models for piecewise affine systems on simplices and rectangles were introduced in [18] and have been further studied in [8]. Closer related to the control framework that we adopt here for the abstraction, are the papers [19], [20], which build on the notion of In-Block Controllability [9], a property of the form that every point in a block of

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states is reachable from any other point in the same block through a bounded control. Other abstraction techniques for nonlinear systems include [31], where discrete time systems are studied in a behavioral framework and [2], where box abstractions are studied for polynomial and other classes of systems. It is also noted that certain of the aforementioned approaches have been extended to switched [14], [15], and stochastic systems [35], allowing for the probabilistic verification of their reachability and safety properties [30], [1]. The abstraction of larger systems through the above approaches, which rely on the global representation of their dynamics, can become intractable due to the size of the discrete models, which grows exponentially with respect to the state dimension. Thus, recent research has been devoted to the construction of symbolic models for interconnected systems, based on the appropriate composition of abstractions for their components. Results towards this direction have been developed in [33] for stabilizable interconnected linear systems, in [28], where approximate bisimulations are obtained under a small gain hypothesis for the components, and in [32], which is focused on reducing the system dimension under quantitative bounds between the original and lower dimensional model. Small gain type assumptions are also leveraged in [11], where compositional results are provided for discrete state systems, whereas in [24] and [23], compositional abstractions for safety specifications are obtained for systems with monotone dynamics. It is worthwhile mentioning that in [23], a varying selection of subsystems for the abstraction is exploited, providing a tunable tradeoff between complexity reduction and model accuracy.

In this work, we focus on multi-agent systems, such as robotic networks, and consider continuous time models for the agents' dynamics. For such systems, a variety of decentralized protocols in the form of coupling feedback controllers between the agents can guarantee objectives such as collision avoidance, network connectivity, and formation in a distributed and autonomous manner [22]. Furthermore, it can be desirable that the agents fulfill additional higher level specifications, such as reachability within specific time bounds, periodic monitoring of workspace areas, and request response actions. This motivates our consideration of agent dynamics consisting of a class of feedback interconnection terms, which are encountered in typical multi-agent coordination schemes, and bounded additive input terms, which we call free inputs and provide certain control freedom under the coupled constraints. Our goal is to leverage this control freedom to construct an individual transition system for each agent based on discrete information from nearby agents, and use these discrete agent models for higher level control synthesis. It is noted that the compositional discrete abstraction approaches introduced in the above paragraph, are in principle not suitable for the systems we consider, where apart from Lipschitz continuity and boundedness, there are no other assumptions for the interconnection terms.

The abstraction is based on a partition of the workspace into cells and the selection of a time step, which guarantee that each agent's symbolic model has quantifiable transition capabilities from every discrete state. Selecting a common time step for the discretization allows for the synchronization of the agents' discrete transitions, whose appropriate composition can capture the behavior of the coupled system. This enables the synthesis of control sequences which satisfy high level tasks by working only with the discrete models. Furthermore, it provides a convenient setting for the synthesis of control strategies under timed specifications, that can be expressed through formal languages such as Metric Interval Temporal Logic, e.g., specifications of the form "always between 2 and 8 time units avoid region A and reach location B between 6 and 10 time units" [25]. Building the discrete models at the agent level also allows for the consideration of the specific dynamic properties of each team member, which in principle cannot be captured with sufficient detail in a global manner for a large scale system. In addition, through this framework it is possible to exploit a strict subset of the agents' abstractions for control synthesis. In this case it is no longer required to consider the composition of all subsystems, whose state grows exponentially with the number of agents and constitutes the computational bottleneck of the centralized case. For instance, such an approach turns out to be applicable for acyclic network structures such as trees, with the agents' tasks prioritized according to their inverse distance to the root node; this enables a "sequential" synthesis procedure, by first selecting satisfying plans for the root agent, then using them to specify the desired paths of the agents one layer below, and so forth.

In this paper, we generalize the results of our recent work [4], where each agent's abstraction is based on the knowledge of its neighbors' discrete positions, by allowing the agent to have this information for all members of the network up to a certain distance in the communication graph. This provides an improved estimate of its neighbors' evolution and allows for more accurate discrete models, due to the reduction of the control magnitude that is required to manipulate the coupling terms. In addition, the derived abstractions are coarser than the ones in [4], and hence, of reduced state complexity. Finally, we note that this paper includes the proofs of its companion conference version [6], which have been completely omitted therein due to space constraints, as well as certain additional results.

The rest of the paper is organized as follows. Notation and preliminaries are introduced in Section 2 and the problem is formulated in Section 3. In Section 4, we formally define well posed abstractions for multi-agent systems and prove consistency of the latter with the required bounds on the system's free inputs. Section 5 is devoted to the study of deviation bounds between reference trajectories of neighboring agents and their estimates. In Section 6 we derive space and time discretizations with quantifiable transition capabilities. The framework is illustrated through an example with simulation results in Section 7 and we conclude in Section 8.

2. Preliminaries and Notation. We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$ and $\mathrm{int}(S)$ for the interior of a set $S \subset \mathbb{R}^n$. Given $R > 0$ and $x \in \mathbb{R}^n$, we denote $B(x;R) := \{y \in \mathbb{R}^n : |x-y| \leq R\}$ and $B(R) := B(0;R)$.

Consider a multi-agent system with N agents. For each agent $i \in \mathcal{N} := \{1,\ldots,N\}$ we consider a fixed set of neighbors $\mathcal{N}_i \subset \mathcal{N} \setminus \{i\}$ and use the notation N_i for the cardinality of $\mathcal{N}_i$. We also consider an ordering of the agent's neighbors which is denoted by $j_1(i) \prec \cdots \prec j_{N_i}(i)$ and define the N_i-tuple $j(i) = (j_1(i),\ldots,j_{N_i}(i))$. Whenever it is clear from the context, the argument i will be omitted from the latter notation. The agents' network is represented by a directed graph $\mathcal{G} := (\mathcal{N},\mathcal{E})$, with vertex set $\mathcal{N}$ the agents' index set and edge set $\mathcal{E}$ the ordered pairs (ℓ,i) with $i,\ell \in \mathcal{N}$ and $\ell \in \mathcal{N}_i$. The sequence $i_0 i_1 \cdots i_m$ with $(i_{\kappa-1}, i_\kappa) \in \mathcal{E}$, $\kappa = 1,\ldots,m$, namely, consisting of m consecutive edges in $\mathcal{G}$, forms *a path of length* m in $\mathcal{G}$. For each $m \geq 1$, we denote by $\mathcal{N}_i^m$ the set of agents from which i is reachable through a path of length m and not by a shorter one, excluding also the possibility to reach itself through a cycle. Notice that $\mathcal{N}_i^1 = \mathcal{N}_i$. We also define $\mathcal{N}_i^0 := \{i\}$ and for each $m \geq 1$ the set $\bar{\mathcal{N}}_i^m := \bigcup_{\ell=0}^m \mathcal{N}_i^\ell$, consisting of all agents from which i is reachable by a path of length at most m, including i. With some abuse of language, we use the terminology *m-neighbor set* of agent i for the set $\bar{\mathcal{N}}_i^m$, since it always contains the agent itself and will also refer to the rest of the agents in $\bar{\mathcal{N}}_i^m$, i.e., to $\bar{\mathcal{N}}_i^m \setminus \{i\}$, as the *$m$-neighbors* of i. Finally, we denote by $\bar{N}_i^m$ the cardinality of agent i's m-neighbors, namely, of the set $\bar{\mathcal{N}}_i^m \setminus \{i\}$. Given an agent $i \in \mathcal{N}$ and its m-neighbor set $\bar{\mathcal{N}}_i^m$ for certain

$m \geq 1$, we consider an extension of its neighbors' ordering $j_1(i) \prec \cdots \prec j_{N_i}(i)$ to an ordering $i \prec j_1(i) \prec \cdots \prec j_{\bar{N}_i^m}(i)$ on $\bar{\mathcal{N}}_i^m$, which is fixed throughout the paper. Whenever it is clear from the context we remove the argument i from these ordered elements. Given an index set $\mathcal{I}$ and recalling that N is the total number of agents, we

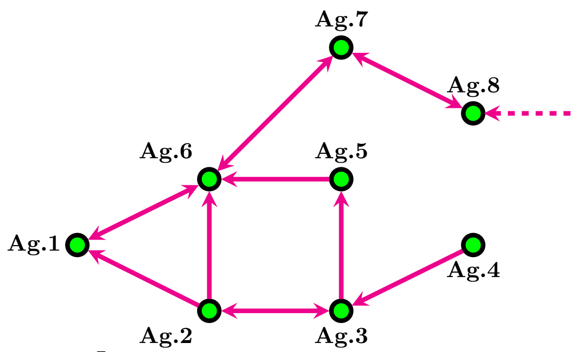


FIG. 1. *The sets* $\mathcal{N}_i^m$, $\bar{\mathcal{N}}_i^m$ *for agents 1, 5 up to paths of length 3 are: (Agent 1)* $\bar{\mathcal{N}}_1^1 = \{1,2,6\}$, $\mathcal{N}_1^1 = \{2,6\}$; $\bar{\mathcal{N}}_1^2 = \{1,2,3,5,6,7\}$, $\mathcal{N}_1^2 = \{3,5,7\}$; $\bar{\mathcal{N}}_1^3 = \{1,2,3,4,5,6,7,8\}$, $\mathcal{N}_1^3 = \{4,8\}$; *(Agent 5)* $\bar{\mathcal{N}}_5^1 = \{3,5\}$, $\mathcal{N}_5^1 = \{3\}$; $\bar{\mathcal{N}}_5^2 = \{2,3,4,5\}$, $\mathcal{N}_5^2 = \{2,4\}$; $\bar{\mathcal{N}}_5^3 = \{2,3,4,5\}$, $\mathcal{N}_5^3 = \emptyset$.

define the map $\mathrm{pr}_i : \mathcal{I}^N \to \mathcal{I}^{\bar{N}_i^m+1}$, which assigns to each N-tuple $(l_1,\ldots,l_N) \in \mathcal{I}^N$ the $\bar{N}_i^m + 1$-tuple $(l_i, l_{j_1}, \ldots, l_{j_{\bar{N}_i^m}}) \in \mathcal{I}^{\bar{N}_i^m+1}$, i.e., the indices of agent i and its m-neighbors in accordance to the ordering. We also define a *transition system* as a tuple $TS := (Q, Act, \longrightarrow)$, where: Q is a set of states; Act is a set of actions; $\longrightarrow$ is a transition relation with $\longrightarrow \subset Q \times Act \times Q$. The transition system is said to be finite, if Q and Act are finite sets. We denote by $q \xrightarrow{a} q'$ an element $(q,a,q') \in \longrightarrow$, and define $\mathrm{Post}(q;a) := \{q' \in Q : (q,a,q') \in \longrightarrow\}$ for every $q \in Q$ and $a \in Act$.

3. Problem Formulation.

In this section we provide the agent's dynamic model and formulate the basic requirements of their distributed discretizations.

3.1. System Dynamics.

We consider multi-agent systems of the form

$$\dot{x}_i = f_i(x_i, \mathbf{x}_j) + v_i, x_i \in \mathbb{R}^n, i \in \mathcal{N}, \tag{1}$$

that are governed by decentralized control laws. These consist of a feedback term $f_i(\cdot)$, that depends on the states of i and its neighbors, which we compactly denote by $\mathbf{x}_j (= \mathbf{x}_{j(i)}) := (x_{j_1}, \ldots, x_{j_{N_i}}) \in \mathbb{R}^{N_i n}$ (see Section 2 for the notation $j(i)$), and an additive input term v_i, which we call free input. The dynamics $f_i(x_i, \mathbf{x}_j)$ are encountered in a large set of multi-agent protocols [22], including consensus, connectivity maintenance, collision avoidance and formation control. In addition, they may represent internal dynamics of the system as for instance in the case of smart buildings (see e.g., [3]), where the temperature $x_i \in \mathbb{R}$, $i \in \mathcal{N}$ of each room evolves according to $\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i) + v_i$, with $\mathcal{N}_i$ denoting the rooms adjacent to i, a_{ij} the heat conductivity between rooms i and j, and v_i the heating/cooling capabilities of the room. We assume that the functions $f_i(\cdot)$ are bounded and globally Lipschitz, i.e., there exist constants $M > 0$, $L_1 > 0$, and $L_2 > 0$ such that

$$|f_i(x_i, \mathbf{x}_j)| \leq M, \tag{2}$$

$$|f_i(x_i, \mathbf{x}_j) - f_i(x_i, \mathbf{y}_j)| \leq L_1 |\mathbf{x}_j - \mathbf{y}_j|, \tag{3}$$

$$|f_i(x_i, \mathbf{x}_j) - f_i(y_i, \mathbf{x}_j)| \leq L_2 |x_i - y_i|, \tag{4}$$

$$\forall x_i, y_i \in \mathbb{R}^n, \mathbf{x}_j, \mathbf{y}_j \in \mathbb{R}^{N_i n},$$

for all agents $i \in \mathcal{N}$. Furthermore, we assume that each input v_i is piecewise continuous and satisfies the bound

$$|v_i(t)| \leq v_{\max} < M, \forall t \geq 0. \tag{5}$$

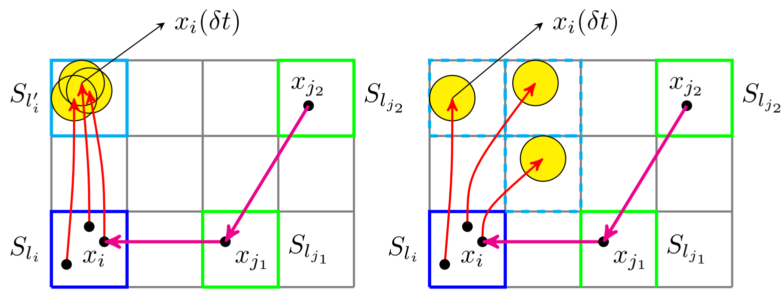
Assumption (5) is in part motivated from the design of cooperative multi-agent protocols which guarantee robustness of certain network properties with respect to the free inputs. This enables the exploitation of the free inputs for high level planning. A class of multi-agent systems of the form (1) which justifies this assumption has been studied in our companion works [5], [7]. These provide sufficient conditions to guarantee robust connectivity of a static agent network for an appropriate selection of $v_{\max}$, which necessitate the latter to satisfy (5).

3.2. Abstraction Requirements. Our aim is to derive a discrete transition system for each individual agent in the coupled system (1), through an appropriate state partition and a time discretization step $\delta t > 0$. For the space discretization we consider a *cell decomposition* $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of each agent's state space $\mathbb{R}^n$ (see also [17, p 129]), namely, a family of uniformly bounded and connected sets S_l, $l \in \mathcal{I}$, such that $\mathrm{int}(S_l) \cap \mathrm{int}(S_{\hat{l}}) = \emptyset$ for all $l \neq \hat{l}$ and $\cup_{l\in\mathcal{I}} S_l = \mathbb{R}^n$. Each agent's abstraction is based on the knowledge of its neighbors' discrete positions up to a distance $m \in \mathbb{N}$ in the network graph, which is fixed throughout the paper. This distance specifies the m-neighbor set of each agent and is called the *degree of decentralization*. Given a cell decomposition $\{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, we denote by $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{\bar{N}_i^m}}) \in \mathcal{I}^{\bar{N}_i^m+1}$ (or just $\mathbf{l}_i$) the indices of the cells where agent i and its m-neighbors belong at a certain time instant and call it the *m-cell configuration of i*. Analogously, we denote by $\mathbf{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$ the cell configuration of all agents, which projects to the m-cell configuration of each agent i through the operator $\mathrm{pr}_i(\cdot)$ from Section 2 applied to the cell-index set $\mathcal{I}$, i.e., $\mathbf{l}_i = \mathrm{pr}_i(\mathbf{l})$.

Informally, we consider for each agent i the transition system whose states are the cells of the decomposition, actions are the cells of the agents in its m-neighbor set, and transitions are defined as follows. A final cell is reachable from an initial one, *if for all states in the initial cell there is a free input such that the trajectory of i will reach the final cell at δt for all possible initial states of its m-neighbors in their cells and their corresponding free inputs*. To guarantee that the abstractions can generate infinite transition sequences, we require that the discretization is well posed, namely, that every agent can perform at least one transition from any cell.

We illustrate the concept of a well posed space-time discretization in Figure 2. For the depicted cases we consider the same 2-cell configuration for the 2-neighbor set $\bar{\mathcal{N}}_i^2 = \{i, j_1, j_2\}$ of i, but different dynamics. In case (i), it is possible to drive agent i to cell $S_{l_i'}$ at δt for all initial conditions in S_{l_i}, irrespectively from where j_1 and j_2 start in their cells and the inputs they choose. Assuming that this holds for all 2-cell configurations of i and for all the agents, we have a well posed discretization for System (i). On the other hand, for System (ii), there are distinct initial conditions of i in S_{l_i}, whose reachable sets at δt lie in different cells, and thus the discretization is not well posed for System (ii).

3.3. Discrete Transition Control Design. The derivation of the discrete models is based as in [4] on the design of appropriate hybrid feedback laws in place of the v_i's which enable the desired transitions. We next define specific control laws that will be used therefore in this paper. Consider first a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt and a selection of a reference point $x_{l,G}$ for each cell S_l, $l \in \mathcal{I}$. Also,



System (i):

$\dot{x}_i = f_{i,(\mathrm{i})}(x_i, x_{j_1}) + v_{i,(\mathrm{i})}$

$\dot{x}_{j_1} = f_{j_1,(\mathrm{i})}(x_{j_1}, x_{j_2}) + v_{j_1,(\mathrm{i})}$

System (ii):

$\dot{x}_i = f_{i,(\mathrm{ii})}(x_i, x_{j_1}) + v_{i,(\mathrm{ii})}$

$\dot{x}_{j_1} = f_{j_1,(\mathrm{ii})}(x_{j_1}, x_{j_2}) + v_{j_1,(\mathrm{ii})}$

FIG. 2. *The space-time discretization is well posed for System (i) but not for System (ii).*

pick an agent i, an m-cell configuration $\mathbf{l}_i$ of i, assume without any loss of generality that $\mathcal{N}_i^{m+1} \neq \emptyset$ (the general case will be provided in the next section), and consider the initial value problem (IVP)

$$\begin{aligned} \dot{\chi}_\ell(t) &= f_\ell(\chi_\ell(t), \chi_{j(\ell)_1}(t), \ldots, \chi_{j(\ell)_{N_\ell}}(t)), t \geq 0, \ell \in \bar{\mathcal{N}}_i^{m-1}, \\ \chi_\ell(0) &= x_{l_\ell,G}, \forall \ell \in \bar{\mathcal{N}}_i^{m-1}, \end{aligned} \tag{6}$$

with the terms $\chi_\ell(\cdot)$, $\ell \in \mathcal{N}_i^m$ defined as

$$\chi_\ell(t) := x_{l_\ell,G}, \forall t \geq 0, \ell \in \mathcal{N}_i^m. \tag{7}$$

The IVP (6)-(7) provides a solution of the unforced, i.e., without free inputs subsystem formed by the m-neighbor set of agent i. In addition, the agents are initiated from their reference points in their cells and the neighbors precisely m hops away are considered fixed at their corresponding reference points for all times. We will call the i-th component $\chi_i(\cdot)$ of the solution in (6) the reference trajectory of i. We also compactly denote as $\boldsymbol{\chi}_j(\cdot) := (\chi_{j_1}(\cdot), \ldots, \chi_{j_{N_i}}(\cdot))$ the corresponding components of i's neighbors. The latter provide an estimate of the neighbors' possible evolution over the time interval $[0, \delta t]$. Notice that agent i can move along its reference trajectory $\chi_i(\cdot)$ when initiated at $x_{l_i,G}$ by applying the time varying feedback law

$$k_{i,\mathbf{l}_i,1}(t, x_i, \mathbf{x}_j) := f_i(\chi_i(t), \boldsymbol{\chi}_j(t)) - f_i(x_i, \mathbf{x}_j), t \in [0, \infty), (x_i, \mathbf{x}_j) \in \mathbb{R}^{(N_i+1)n}, \tag{8}$$

in place of its free input v_i in (1). Next, consider a function

$$\zeta_i : \mathbb{R}_{\geq 0} \to [0, \bar{\lambda}], 0 < \bar{\lambda} < 1 \tag{9}$$

and select a vector w_i from

$$W := B(v_{\max}) \subset \mathbb{R}^n. \tag{10}$$

By adding to the control law $v_i = k_{i,\mathbf{l}_i,1}(\cdot)$ in (8) also the term

$$k_{i,\mathbf{l}_i,2}(t; w_i) := \zeta_i(t) w_i, t \in [0, \infty), w_i \in W, \tag{11}$$

it follows that agent i, when initiated from $x_{l_i,G}$, will move according to the trajectory $t \mapsto \chi_i(t) + w_i \int_0^t \zeta_i(s)ds$ and reach the point $x := \chi_i(\delta t) + w_i \int_0^{\delta t} \zeta_i(s)ds$ inside the ball depicted in Figure 3 at δt. In a similar way, it is possible to reach any point inside this ball by a different selection of w_i. Its radius

$$r_i := \int_0^{\delta t} \zeta_i(s)ds v_{\max}, \tag{12}$$

is determined through the maximum value of the control part $k_{i,\mathbf{l}_i,2}(\cdot)$ that is assigned to the free input to increase the agent's transition choices and is quantified through the design parameter $\bar{\lambda}$ in (9), which upper bounds the values of $\zeta_i(\cdot)$. Finally, we also augment the component

$$k_{i,\mathbf{l}_i,3}(x_{i0}) := \frac{1}{\delta t}(x_{l_i,G} - x_{i0}), x_{i0} \in S_{l_i}, \tag{13}$$

to the suggested control scheme, namely, we consider the feedback law

$$k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) := k_{i,\mathbf{l}_i,1}(t, x_i, \mathbf{x}_j) + k_{i,\mathbf{l}_i,2}(t; w_i) + k_{i,\mathbf{l}_i,3}(x_{i0}), \tag{14}$$

which is parameterized by the initial condition $x_{i0} \in S_{l_i}$ and the vector $w_i \in W$. Then, it follows from (1), (8), (11), (13), (14), and the IVP (6)-(7), that the agent's trajectory $x_i(\cdot)$ with $x_i(0) = x_{i0}$ and $v_i = k_{i,\mathbf{l}_i}$ will be given by

$$\begin{aligned}
x_i(t) &= x_{i0} + \int_0^t f_i(\chi_i(s), \boldsymbol{\chi}_j(s))ds + w_i \int_0^t \zeta_i(s)ds + \frac{t}{\delta t}(x_{l_i,G} - x_{i0}) \\
&= x_{l_i,G} + \int_0^t f_i(\chi_i(s), \boldsymbol{\chi}_j(s))ds + w_i \int_0^t \zeta_i(s)ds + \frac{\delta t - t}{\delta t}(x_{i0} - x_{l_i,G}) \\
&= \chi_i(t) + w_i \int_0^t \zeta_i(s)ds + \frac{\delta t - t}{\delta t}(x_{i0} - x_{l_i,G}).
\end{aligned}$$

The latter implies that $x_i(\delta t) = \chi_i(\delta t) + w_i \int_0^{\delta t} \zeta_i(s)ds$, and thus, that agent i can reach the point x in Figure 3 at δt *from any initial point* x_{i0} in the cell S_{l_i}. Analogously, i can reach any other point in $B(\chi_i(\delta t); r_i)$ from every initial condition in S_{l_i} by an alternative assignment of the parameter w_i in $k_{i,\mathbf{l}_i}(\cdot,\cdot,\cdot;\cdot,w_i)$, and thus, *perform a transition to any cell which has a nonempty intersection with* $B(\chi_i(\delta t); r_i)$. It remains though to verify that the magnitude of the feedback law evaluated along the trajectory of the agent does not violate the constraint (5) on the available control. Space and time discretizations which guarantee this requirement can always be obtained for system (1) and are provided in Section 6. It is also noted that due to the assumption $v_{\max} < M$ in (5), it is in principle not possible to cancel the interconnection terms. In addition, the feedback laws $k_{i,\mathbf{l}_i}(\cdot)$ depend on the cell of agent i, and specifically, on its m-cell configuration $\mathbf{l}_i$, through the reference point $x_{l_i,G}$ in (13) and the trajectories $\chi_i(\cdot)$ and $\boldsymbol{\chi}_j(\cdot)$ in (8), as provided by the initial value problem (6)-(7).

4. Abstractions of Varying Decentralization.

Based on the previous sections, we proceed with the formal definition of well posed discretizations and the individual discrete models that are associated to each agent. The formulation builds on the controller design introduced in Section 3.3. However, it is provided in a more abstract framework, in order to focus on the desirable properties of the control laws for the candidate discretizations, which do not require their precise formulas and allow the design of alternative feedback laws to the ones given in (14).

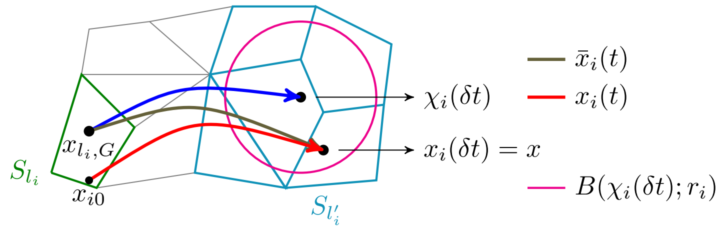


FIG. 3. *Consider any point x inside the ball $B(\chi_i(\delta t); r_i)$. Then, we can assign a parameter $w_i \in W$ to the feedback law $k_{i,l_i}(\cdot,\cdot,\cdot;\cdot,w_i)$ in (14), so that the trajectory $x_i(\cdot)$ of i with input $v_i = k_{i,l_i}$ satisfies $x_i(\delta t) = \chi_i(\delta t) + w_i \int_0^{\delta t} \zeta_i(s)ds = x \in S_{l_i'}$, for each initial condition $x_{i0} \in S_{l_i}$. Thus, through the selected w_i, the controller k_{i,l_i} enables a transition from cell S_{l_i} to $S_{l_i'}$, since the latter is reachable for every initial condition $x_{i0} \in S_{l_i}$.*

4.1. Agent Reference Trajectories. The following definition provides for each agent i its reference trajectory and the estimates of its neighbors' reference trajectories, based on i's m-cell configuration.

DEFINITION 1. *Given a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$ and a reference point $x_{l,G} \in S_l$ for each $l \in \mathcal{I}$, consider an agent $i \in \mathcal{N}$, its m-neighbor set $\bar{\mathcal{N}}_i^m$ and an m-cell configuration $\boldsymbol{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{|\bar{\mathcal{N}}_i^m|}})$ of i. We define the functions $\chi_i(t)$, $\boldsymbol{\chi}_j(t) := (\chi_{j_1}(t), \ldots, \chi_{j_{N_i}}(t))$, $t \geq 0$, through the solution of the following initial value problem, specified by Cases (i) and (ii) below:*
Case (i). $\mathcal{N}_i^{m+1} = \emptyset$. Then, we have the initial value problem

$$\begin{aligned} \dot{\chi}_\ell(t) =& f_\ell(\chi_\ell(t), \chi_{j(\ell)_1}(t), \ldots, \chi_{j(\ell)_{N_\ell}}(t)), t \geq 0, \ell \in \bar{\mathcal{N}}_i^m, \\ \chi_\ell(0) =& x_{l_\ell,G}, \forall \ell \in \bar{\mathcal{N}}_i^m, \end{aligned} \tag{15}$$

where $j(\ell)_1, \ldots, j(\ell)_{N_\ell}$ denote the corresponding neighbors of each agent $\ell \in \bar{\mathcal{N}}_i^m$.
Case (ii). $\mathcal{N}_i^{m+1} \neq \emptyset$. Then, we have the initial value problem (6)-(7). ◁

REMARK 2. *(i) In Case (i) for the IVP of Definition 1, the requirement $\mathcal{N}_i^{m+1} = \emptyset$ implies by Lemma 17(iii) in the Appendix, that for each agent $\ell \in \bar{\mathcal{N}}_i^m$ its neighbors $j(\ell)_1, \ldots, j(\ell)_{N_\ell}$ also belong to $\bar{\mathcal{N}}_i^m$. Hence, the subsystem formed by the agents in $\bar{\mathcal{N}}_i^m$ is decoupled from the other agents in the system and IVP (15) is well defined.*
(ii) In Case (ii), the subsystem formed by the agents in $\bar{\mathcal{N}}_i^m$ is not decoupled from the other agents in the system. However, by considering the agents in $\mathcal{N}_i^m$ fixed at their reference points by (7), the initial value problem (6)-(7) is again well defined.
(iii) Apart from the notation $\chi_i(\cdot)$ and $\boldsymbol{\chi}_j(\cdot)$ above, we use the notation $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$ for the trajectory of each agent $\ell \in \bar{\mathcal{N}}_i^m$, as specified by the IVP initial value problem of Definition 1 for the m-cell configuration $\boldsymbol{l}_i$ of i (with $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$ as defined by (7) when $\mathcal{N}_i^{m+1} \neq \emptyset$). We refer to $\chi_i(\cdot) \equiv \chi_i^{\boldsymbol{l}_i}(\cdot)$ as the reference trajectory of agent i and to each $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$ with $\ell \in \bar{\mathcal{N}}_i^m \setminus \{i\}$ as the estimate of ℓ's reference trajectory by i. Whenever a cell decomposition and a selection of reference points are given for system (1), the IVP of Definition 1 is uniquely determined by agent i and its m-cell configuration $\boldsymbol{l}_i$. In this case, we also refer to it as the $\boldsymbol{l}_i$-IVP.

EXAMPLE 3. *In this example we demonstrate the IVPs of Definition 1 for $m = 3$, as specified by Cases (i) and (ii) for agents 5 and 1 of Figure 1, respectively: Agent 5 (Case (i)) $\dot{\chi}_5(t) = f_5(\chi_5(t), \chi_3(t))$, $\dot{\chi}_3(t) = f_3(\chi_3(t), \chi_2(t), \chi_4(t))$, $\dot{\chi}_2(t) = f_2(\chi_2(t), \chi_3(t))$, $\dot{\chi}_4(t) = f_4(\chi_4(t))$, $\chi_\kappa(0) = x_{l_\kappa,G}, \kappa = 5,3,2,4$; Agent 1 (Case (ii)) $\dot{\chi}_1(t) = f_1(\chi_1(t), \chi_6(t), \chi_2(t))$, $\dot{\chi}_6(t) = f_6(\chi_6(t), \chi_1(t), \chi_2(t), \chi_5(t), \chi_7(t))$, $\dot{\chi}_2(t) = f_2(\chi_2(t), \chi_3(t))$, $\dot{\chi}_5(t) = f_5(\chi_5(t), \chi_3(t))$, $\dot{\chi}_3(t) = f_3(\chi_3(t), \chi_2(t), x_{l_4,G})$, $\dot{\chi}_7(t) = f_7(\chi_7(t), \chi_6(t), x_{l_8,G})$, $\chi_\kappa(0) = x_{l_\kappa,G}, \kappa = 1,6,2,5,3,7$.*

We next characterize bounds for the deviation between the reference trajectory of an agent's neighbor, and its estimate obtained from the agent's initial value problem. These refer to m-cell configurations of the agent and its neighbor where the corresponding common agents belong to the same cells. Specifically, given m-cell configurations $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{\bar{N}_i^m}})$ and $\mathbf{l}_\ell = (\bar{l}_\ell, \bar{l}_{j(\ell)_1}, \ldots, \bar{l}_{j(\ell)_{\bar{N}_\ell^m}})$ of agents $i \in \mathcal{N}$ and $\ell \in \mathcal{N}_i$, respectively, we say that $\mathbf{l}_i$ and $\mathbf{l}_\ell$ are *consistent*, if $l_\kappa = \bar{l}_\kappa$ for all $\kappa \in \bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m$.

Definition 4. *Given $i \in \mathcal{N}$, the continuous function $\alpha_i : [0, \delta t] \to \mathbb{R}_{\geq 0}$ is called a reference trajectory deviation bound for i, if for each $\ell \in \mathcal{N}_i$ and consistent m-cell configurations $\boldsymbol{l}_i$ and $\boldsymbol{l}_\ell$ of i and ℓ, respectively, it holds that*

$$|\chi_\ell^{\boldsymbol{l}_i}(t) - \chi_\ell^{\boldsymbol{l}_\ell}(t)| \leq \alpha_i(t), \forall t \in [0, \delta t]. \triangleleft \tag{16}$$

Specific reference trajectory deviation bounds for the agents will be provided in Section 5 by exploiting the bounds and Lipschitz constants of their coupling terms.

4.2. Individual Transition Requirements. We next provide the class of hybrid feedback laws that are assigned to the free inputs v_i to obtain the discrete transitions. As the control laws in (14), they depend on the cells each agent and its m-neighbors belong, and are parameterized by the agent's initial conditions and a set of auxiliary parameters, which are exploited to increase the transition choices. In particular, let $\mathcal{S} = \{S_l\}_{l \in \mathcal{I}}$ and W be a cell decomposition and a nonempty subset of $\mathbb{R}^n$, respectively. Given an agent $i \in \mathcal{N}$ and a cell configuration $\mathbf{l}_i$ of i, we consider feedback laws $k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) : [0, \infty) \times \mathbb{R}^{(N_i+1)n} \to \mathbb{R}^n$, parameterized by $x_{i0} \in S_{l_i}$, and $w_i \in W$, which are piecewise continuous on t and globally Lipschitz continuous on $(x_i, \mathbf{x}_j)$ (uniformly with respect to $t \in [0, \infty)$, $x_{i0} \in S_{l_i}$ and $w_i \in W$). We refer to each such control law as a *globally Lipschitz W-parameterized feedback law.* The motivation for this definition comes from the fact that different parameters w_i provide alternative transition possibilities to the agent, as also discussed in Section 3.3. Due to the uniform bound on the size of the cells in the decomposition, we choose its diameter $d_{\max}$, as the diameter of an open ball whose translation can cover each individual cell. Thus, we can select a reference point $x_{l,G}$ for each cell with

$$|x_{l,G} - x| < \frac{d_{\max}}{2}, \forall x \in S_l, l \in \mathcal{I}. \tag{17}$$

Motivated by the control design of the previous section, we provide conditions which enable an agent to perform a discrete transition based on the knowledge of its m-cell configuration. Recall that each agent aims to reach a point inside the ball with center the endpoint of its reference trajectory and radius r_i given by (12), which due to (9) satisfies $r_i \leq v_{\max}\delta t$. From the latter and (17), the agent's distance from its reference trajectory will be bounded by a continuous function $\beta : [0, \delta t] \to \mathbb{R}_{\geq 0}$ with

$$\frac{d_{\max}}{2} \leq \beta(0); \beta(\delta t) \leq v_{\max}\delta t. \tag{18}$$

In order to define the agents' individual transitions, we consider for each $i \in \mathcal{N}$ the following system with disturbances:

$$\dot{x}_i = f_i(x_i, \mathbf{d}_j) + v_i, \tag{19}$$

where $d_{j_1}, \ldots, d_{j_{N_i}} : [0, \infty) \to \mathbb{R}^n$ (also denoted d_ℓ, $\ell \in \mathcal{N}_i$) are continuous functions. This approach is inspired by [13], where a nonlinear system is modeled by means of a piecewise affine system with disturbances. The following definition provides the desired transition requirement, based on the auxiliary dynamics (19).

Definition 5. *Consider a cell decomposition $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt, a nonempty subset W of $\mathbb{R}^n$, an agent $i\in\mathcal{N}$, and continuous functions $\beta(\cdot)$, $\alpha_i(\cdot)$, satisfying (18) and (16) of Definition 4, respectively. Given an m-cell configuration $\boldsymbol{l}_i$ of i, a globally Lipschitz W-parameterized feedback law $v_i=k_{i,\mathbf{l}_i}(t,x_i,\boldsymbol{x}_j;x_{i0},w_i)$, a vector $w_i\in W$, and a cell index $l_i'\in\mathcal{I}$, we say that $k_{i,\mathbf{l}_i}$, w_i, l_i' satisfy the Transition Requirement (TR), if the following hold. For each initial condition $x_{i0}\in S_{l_i}$ and selection of continuous functions $d_\ell:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, $\ell\in\mathcal{N}_i$ satisfying*

$$|d_\ell(t)-\chi_\ell^{\boldsymbol{l}_i}(t)|\leq\alpha_i(t)+\beta(t),\forall t\in[0,\delta t], \tag{20}$$

with $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$, $\ell\in\mathcal{N}_i$ as provided by the $\boldsymbol{l}_i$-IVP, the solution $x_i(\cdot)$ of (19) with $v_i=k_{i,\mathbf{l}_i}(t,x_i,\boldsymbol{d}_j;x_{i0},w_i)$, satisfies

$$|x_i(t)-\chi_i^{\boldsymbol{l}_i}(t)|<\beta(t),\forall t\in[0,\delta t), \tag{21}$$

$$x_i(\delta t)\in S_{l_i'}, \tag{22}$$

$$|k_{i,\mathbf{l}_i}(t,x_i(t),\boldsymbol{d}_j(t);x_{i0},w_i)|\leq v_{\max},\forall t\in[0,\delta t]. \quad \triangleleft \tag{23}$$

Note that when the Transition Requirement is satisfied, agent i can be driven to $S_{l_i'}$ precisely at δt under the feedback law $k_{i,\mathbf{l}_i}(\cdot)$ corresponding to the given parameter w_i. The latter is possible for all disturbances satisfying (20), which capture the evolution of i's neighbors over the time interval $[0,\delta t]$, given the knowledge of i's m-cell configuration. Indeed, notice that $\beta(\cdot)$ bounds the distance of each agent from its reference trajectory over $[0,\delta t]$. Also, recall that the deviation between the reference trajectory of each $\ell\in\mathcal{N}_i$ and its estimate by i is bounded by $\alpha_i(\cdot)$. Thus, the distance between ℓ's trajectory $x_\ell(\cdot)$ and the estimate $\chi_\ell^{\mathbf{l}_i}(\cdot)$ of its reference trajectory by i is bounded by $\alpha_i(\cdot)+\beta(\cdot)$. Some additional intuition behind the Transition Requirement is given in the example of Figure 4, with degree of decentralization $m=3$.

4.3. Well Posed Discretizations. We next define a well posed space-time discretization, for which the individual transition system of each agent i is non blocking, i.e., there is an outgoing transition from each state. This is formulated through the condition that for each m-cell configuration of i, there exists a control law and a successor state, such that the Transition Requirement of Definition 5 is fulfilled.

Definition 6. *Consider a cell decomposition $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, a time step δt, a nonempty subset W of $\mathbb{R}^n$, and continuous functions $\alpha_i:[0,\delta t]\to\mathbb{R}_{\geq 0}$, $i\in\mathcal{N}$ and $\beta:[0,\delta t]\to\mathbb{R}_{\geq 0}$ satisfying (16) and (18), respectively. We say that the space-time discretization $\mathcal{S}-\delta t$ is well posed (for system (1)), if for each agent $i\in\mathcal{N}$ and m-cell configuration $\boldsymbol{l}_i$ of i, there exist a globally Lipschitz W-parameterized feedback law $v_i=k_{i,\mathbf{l}_i}(t,x_i,\boldsymbol{x}_j;x_{i0},w_i)$, a vector $w_i\in W$, and a cell index $l_i'\in\mathcal{I}$, which satisfy the Transition Requirement.*

Remark 7. *Due to the Transition Requirement, the definition of a well posed discretization $S-\delta t$ is associated with a selection of the set W, and the mappings $\alpha_i(\cdot)$ and $\beta(\cdot)$. This selection will be often assumed implicitly in subsequent statements and invoked when necessary.*

Given a well posed space-time discretization $\mathcal{S}-\delta t$ and based on Definition 6, we next define the discrete transition system of each agent.

Definition 8. *For each agent i, its individual transition system $TS_i:=(Q_i,Act_i,\longrightarrow_i)$ is defined as follows:*

- *$Q_i:=\mathcal{I}$ (the indices of the cell decomposition)*

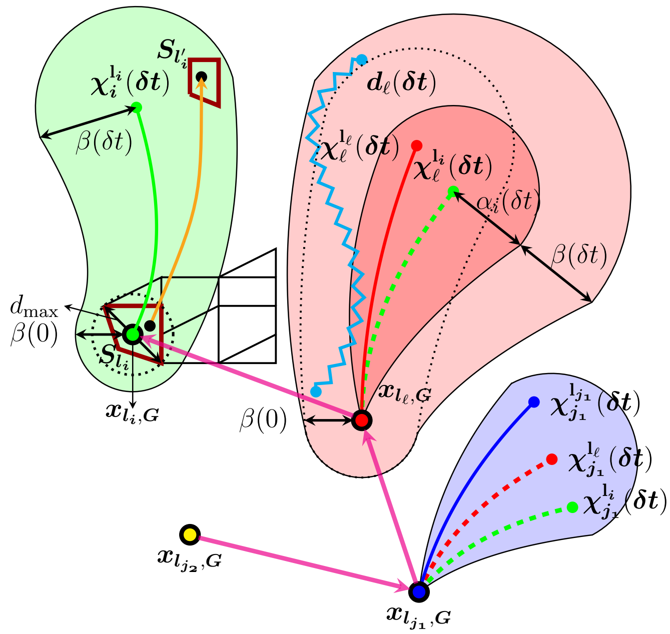


FIG. 4. *This figure illustrates i's reference trajectory $\chi_i^{l_i}(\cdot)$ on the left. The area enclosing the agent corresponds to all positions of i that satisfy (21). The restriction (20) imposed on the acceptable disturbances for i is depicted through the larger area enclosing the reference point $x_{l_\ell,G}$ of ℓ. The darker part of this area comprises of the points with distance from the (dashed) estimate of ℓ's reference trajectory $\chi_\ell^{l_i}(t)$ by i no more than $\alpha_i(t)$. Thus, given that the distance of ℓ from its own reference trajectory $\chi_\ell^{l_\ell}(t)$ is bounded by $\beta(t)$ (i.e., lies inside the closed dashed curve), ℓ will remain within the larger red area.*

- *$Act_i := \mathcal{I}^{\bar{N}_i^m+1}$ (the set of all m-cell configurations of i)*
- *For any $l_i, l'_i \in Q_i$ and $\boldsymbol{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{\bar{N}_i^m}}) \in Act_i$, $l_i \xrightarrow{\boldsymbol{l}_i}_i l'_i$ iff there exist $k_{i,\boldsymbol{l}_i}$, w_i with $k_{i,\boldsymbol{l}_i}$, w_i, l'_i satisfying the Transition Requirement.* $\triangleleft$

In Definition 8, considering the m-cell configurations of each agent in its action set, indicates that the agent's transitions are affected by the discrete positions of its m-neighbors. This is in accordance with the intuition provided in Section 3.3, because the agent's m-cell configuration affects the endpoint of its reference trajectory and hence, the successor cells which intersect the ball in Figure 3.

REMARK 9. *Given a well posed discretization $\mathcal{S} - \delta t$ and an initial cell configuration $\boldsymbol{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$, it follows from Definitions 6 and 8 that $\mathrm{Post}_i(l_i; \mathrm{pr}_i(\boldsymbol{l})) \neq \emptyset$ for each agent $i \in \mathcal{N}$ ($\mathrm{Post}_i(\cdot)$ refers to the transition system TS_i of agent i).*

According to Definition 6, a well posed space-time discretization requires the existence of a transition for each agent i and m-cell configuration of i. The latter reduces to the selection of an appropriate feedback controller for i, which guarantees that the auxiliary system with disturbances (19) satisfies the Transition Requirement. We next establish correctness of the abstraction, by showing that individual agent transitions which are initiated from compatible cell configurations are correctly executed by the continuous system (1). In particular, given a discrete configuration of all agents and a corresponding transition for each of them, one can assign a local feedback law to every agent and guarantee that the resulting closed-loop system will simultaneously execute all these transitions. At the same time, the magnitude of the agents' control laws evaluated along the closed-loop solution will not exceed the bound $v_{\max}$ in (5) during the transition interval.

PROPOSITION 10. *Assume that the space-time discretization $S - \delta t$ is well posed for system (1), consider an initial cell configuration $\boldsymbol{l} = (l_1, \ldots, l_N) \in \mathcal{I}^N$, and select a successor state $l'_i \in \mathrm{Post}_i(l_i; \mathrm{pr}_i(\boldsymbol{l}))$ for each agent. Then, there exist globally Lipschitz W-parameterized feedback laws*

$$v_i = k_{i,\mathrm{pr}_i(\boldsymbol{l})}(t, x_i, \boldsymbol{x}_j; x_{i0}, w_i), i \in \mathcal{N}, \tag{24}$$

and $w_1, \ldots, w_N \in W$, such that for each initial condition $x(0) \in \mathbb{R}^{Nn}$ with $x_i(0) = x_{i0} \in S_{l_i}$, $i \in \mathcal{N}$, the solution of the closed-loop system (1), (24) is well defined on $[0, \delta t]$, and satisfies

$$x_i(\delta t, x(0)) \in S_{l'_i}, \forall i \in \mathcal{N}, \tag{25}$$

$$|k_{i,\mathrm{pr}_i(\boldsymbol{l})}(t, x_i(t), \boldsymbol{x}_j(t); x_{i0}, w_i)| \leq v_{\max}, \forall t \in [0, \delta t], i \in \mathcal{N}. \tag{26}$$

Proof. Indeed, consider the successor states (cell indices) l'_i, $i \in \mathcal{N}$ selected in the statement of the proposition. Since the discretization is well posed, there exist continuous functions $\alpha_i : [0, \delta t] \to \mathbb{R}_{\geq 0}$, $i \in \mathcal{N}$ and $\beta : [0, \delta t] \to \mathbb{R}_{\geq 0}$ satisfying (16) and (18), respectively, such that the requirements of Definition 6 are fulfilled. Thus, by the definition of the operators $\mathrm{Post}_i(\cdot)$, $i \in \mathcal{N}$ we can select for each agent $i \in \mathcal{N}$ a globally Lipschitz W-parameterized feedback $k_{i,\mathrm{pr}_i(\mathbf{l})}(\cdot)$ and a vector $w_i \in W$, such that $k_{i,\mathbf{l}_i}$, w_i, l'_i satisfy the Transition Requirement.

Next, we pick for each agent i an initial condition $x_{i0} \in S_{l_i}$ and notice that due to the Lipschitz assumption for the control laws, the solution of the closed loop system is defined for all $t \geq 0$. Furthermore, it follows from (17) that $|x_{i0} - x_{l_i,G}| < \frac{d_{\max}}{2}$ for all $i \in \mathcal{N}$. Hence, by continuity of the solution of the closed-loop system (1), (24) we deduce from (18) that there exists $\delta \in (0, \delta t]$, such that

$$|x_i(t) - \chi_i^{\mathbf{l}_i}(t)| < \beta(t), \forall t \in [0, \delta], \tag{27}$$

for all $i \in \mathcal{N}$, where $x_i(\cdot)$ is the i-th component of the solution and $\chi_i^{\mathbf{l}_i}(\cdot)$ is the reference trajectory of i corresponding to the m-cell configuration $\mathbf{l}_i = \mathrm{pr}_i(\mathbf{l})$ of i, with initial condition $\chi_i^{\mathbf{l}_i}(0) = x_{l_i,G}$. We claim that for each $i \in \mathcal{N}$, (27) holds for all $t \in [0, \delta t)$. Indeed, suppose on the contrary that there exist an agent $\iota \in \mathcal{N}$ and a time $T \in (0, \delta t)$ such that $|x_\iota(T) - \chi_\iota^{\mathbf{l}_\iota}(T)| \geq \beta(T)$, where $\mathbf{l}_\iota = \mathrm{pr}_\iota(\mathbf{l})$, and define $\tau := \sup\{\bar{t} \in (0, \delta t] : |x_i(t) - \chi_i^{\mathbf{l}_i}(t)| < \beta(t), \forall t \in [0, \bar{t}], i \in \mathcal{N}\}$. From the latter and (27), it follows that τ is well defined, $0 < \tau < \delta t$, and that there exists $\ell \in \mathcal{N}$ such that

$$|x_\ell(\tau) - \chi_\ell^{\mathbf{l}_\ell}(\tau)| = \beta(\tau). \tag{28}$$

Next, notice that by the definition of τ, it holds that

$$|x_\kappa(t) - \chi_\kappa^{\mathbf{l}_\kappa}(t)| \leq \beta(t), \forall t \in [0, \tau], \kappa \in \mathcal{N}_\ell. \tag{29}$$

Also, since for each $\kappa \in \mathcal{N}_\ell$ the m-cell configurations $\mathbf{l}_\kappa = \mathrm{pr}_\kappa(\mathbf{l})$ of κ and $\mathbf{l}_\ell = \mathrm{pr}_\ell(\mathbf{l})$ of ℓ are consistent, it follows from (16) and the fact that $\tau < \delta t$, that $|\chi_\kappa^{\mathbf{l}_\ell}(t) - \chi_\kappa^{\mathbf{l}_\kappa}(t)| \leq \alpha_\ell(t)$, for all $t \in [0, \tau]$, $\kappa \in \mathcal{N}_\ell$. Hence, we obtain from the latter and (29) that

$$|x_\kappa(t) - \chi_\kappa^{\mathbf{l}_\ell}(t)| \leq \beta(t) + \alpha_\ell(t), \forall t \in [0, \tau], \kappa \in \mathcal{N}_\ell. \tag{30}$$

By setting $d_\kappa(t) := x_\kappa(t)$, $t \geq 0$, $\kappa \in \mathcal{N}_\ell$, it follows from standard uniqueness results from ODE theory, that $x_\ell(\cdot)$ is also the solution of the system with disturbances (19), with $i = \ell$, $v_i = k_{\ell,\mathbf{l}_\ell}(t, x_\ell, \mathbf{d}_{j(\ell)}; x_{\ell 0}, w_\ell)$, and initial condition $x_{\ell 0} \in S_{l_\ell}$. Thus, by exploiting causality of (19) with respect to the disturbances and observing that due to (30) the disturbances satisfy (20) for all $t \in [0, \tau)$, it follows from (21) and the fact that $\tau < \delta t$, that $|x_\ell(\tau) - \chi_\ell^{\mathbf{l}_\ell}(\tau)| < \beta(\tau)$, which contradicts (28). Hence, we conclude that (27) holds for all $t \in [0, \delta t)$.

Next, by using the same arguments as above, we can deduce that for each agent i, the i-th component of the solution of the closed loop system (1), (24), is the same as the solution of system (19) for i, with disturbances $d_\kappa(\cdot)$, $\kappa \in \mathcal{N}_i$ being the components $x_\kappa(\cdot)$, $\kappa \in \mathcal{N}_i$ of the solution corresponding to i's neighbors. Furthermore, it follows that the disturbances satisfy (20). Hence, from the Transition Requirement, and the fact that the components of the solution of the closed loop system and the corresponding solutions of the systems with disturbances are identical, we obtain that (25) and (26) are satisfied. The proof is now complete. ∎

4.4. Exploitation of the Individual Subsystems. We next elaborate on the exploitation of the agents' individual models by discussing a case where control synthesis can be performed under guaranteed computational complexity reduction. A rigorous framework to address this problem will be given in a subsequent work.

Consider a tree (or more generally acyclic) network structure and assume that the agents' tasks are prioritized according to their inverse distance from the root node. Hence, the tasks of the root agent are prioritized compared to its children nodes and so forth. Then, by assuming agent i to be the root of the tree, it follows that it remains unaffected by the coupling dynamics and its transition requirement reduces to a simplified variant of the Transition Requirement without disturbances. Thus, we can first select the set of discrete paths of i which satisfy its specification and as a next step, use all these paths as actions for the transition systems of i's children in order to determine the paths which satisfy their plans. Note that for any selection of the decentralization degree, the m-neighbor set of i's children consists exclusively of i. Proceeding analogously, and considering a descendant ℓ of i, we use all the selected paths of the ancestors of ℓ up to m hops up in the tree, in order to determine all the satisfying paths of its specification. This approach can reduce significantly the memory storage required for the transitions compared to the centralized case. In addition, the specifications restrict the agent's acceptable transitions and hence, the possible actions in the m-cell configurations of the descendant agents.

5. Reference Trajectory Deviation Bounds. This section is devoted to the derivation of explicit reference trajectory deviation bounds, that are introduced in Definition 4. In Lemma 11 below, we provide conditions on the network structure in a neighborhood of an agent i, which guarantee that for consistent cell configurations, the reference trajectories of i's neighbors coincide with their estimates by i.

LEMMA 11. *Assume that for agent $i \in \mathcal{N}$ it holds $\mathcal{N}_i^{m+1} = \emptyset$, and let $\boldsymbol{l}_i$ be an m-cell configuration of i. Then, for every $\ell \in \mathcal{N}_i$ with $\mathcal{N}_\ell^{m+1} = \emptyset$, and m-cell configuration $\boldsymbol{l}_\ell$ of ℓ consistent with $\boldsymbol{l}_i$, it holds that $\chi_\ell^{\boldsymbol{l}_\ell}(t) = \chi_\ell^{\boldsymbol{l}_i}(t)$, for all $t \geq 0$, with $\chi_\ell^{\boldsymbol{l}_\ell}(\cdot)$ and $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$ as determined by the IVP of Definition 1 for $\boldsymbol{l}_\ell$ and $\boldsymbol{l}_i$, respectively.*

Proof. The proof is given in the Appendix. ∎

Despite the result of Lemma 11, in principle, the reference trajectory of each agent's neighbor and its estimate through the initial value problem for the reference trajectory of the specific agent do not coincide. Explicit bounds for their deviation are given in Proposition 13 below, whose proof requires the following auxiliary lemma.

LEMMA 12. *Let $i \in \mathcal{N}$, $\ell \in \mathcal{N}_i$, and $\boldsymbol{l}_i$, $\boldsymbol{l}_\ell$ be any consistent m-cell configurations of agents i and ℓ, respectively. Also, let t^* be the unique positive solution of*

$$e^{L_2 t^*} - \left(L_2 + \frac{L_2^2}{L_1\sqrt{N_{\max}}}\right)t^* - 1 = 0, \tag{31}$$

with

$$N_{\max} := \max\{N_i : i \in \mathcal{N}\}. \tag{32}$$

Then, for each $\kappa \in \bar{\mathcal{N}}_\ell^{m-1}$ ($\bar{\mathcal{N}}_\ell^{m-1} \subset \bar{\mathcal{N}}_i^m$ by Lemma 17(i)) it holds that:

$$|\chi_\kappa^{\boldsymbol{l}_i}(t) - \chi_\kappa^{\boldsymbol{l}_\ell}(t)| \leq Mt, \forall t \in [0, t^*], \tag{33}$$

where $\chi_\kappa^{\boldsymbol{l}_i}(\cdot)$ and $\chi_\kappa^{\boldsymbol{l}_\ell}(\cdot)$ are determined by the initial value problem of Definition 1 for the m-cell configurations $\boldsymbol{l}_i$ and $\boldsymbol{l}_\ell$, respectively.

Proof. The proof is given in the Appendix. ◻

PROPOSITION 13. *Consider the agents $i \in \mathcal{N}$, $\ell \in \mathcal{N}_i$, and let $\boldsymbol{l}_i$ and $\boldsymbol{l}_\ell$ be any consistent m-cell configurations of i and ℓ, respectively. Then, it holds that*

$$|\chi_\ell^{\boldsymbol{l}_i}(t) - \chi_\ell^{\boldsymbol{l}_\ell}(t)| \leq H_m(t), \forall t \in [0, t^*], \tag{34}$$

where t^* is given in (31), the functions $H_\kappa(\cdot)$, $\kappa \geq 1$, are defined recursively as

$$H_1(t) := Mt, t \geq 0; \quad H_\kappa(t) := \int_0^t e^{L_2(t-s)} L_1\sqrt{N_{\max}} H_{\kappa-1}(s) ds, t \geq 0 \tag{35}$$

and $\chi_\ell^{\boldsymbol{l}_i}(\cdot)$, $\chi_\ell^{\boldsymbol{l}_\ell}(\cdot)$ are determined by the initial value problem of Definition 1.

Proof. The proof is given in the Appendix. ◻

We next provide some linear upper bounds for the functions $H_m(\cdot)$ above, which are used for the derivation of acceptable discretizations. Let $\bar{c} \in (0, 1)$ and define

$$\bar{t} := \sup\left\{t > 0 : e^{L_2 t} - \left(L_2 + \bar{c}\frac{L_2^2}{L_1\sqrt{N_{\max}}}\right)t - 1 < 0\right\}. \tag{36}$$

Then, it follows that

$$0 < \bar{t} < t^*, \tag{37}$$

where t^* is defined in (31), and the function $H_m(\cdot)$ given in Proposition 13 satisfies

$$H_m(t) \leq \bar{c}^{m-1} Mt, \forall t \in [0, \bar{t}]. \tag{38}$$

Hence, if we select each function $\alpha_i(\cdot)$ in Definition 4 as $\alpha_i(\cdot) \equiv \alpha(\cdot)$, with

$$\alpha(t) := cMt, \forall t \in [0, \delta t]; \quad c := \bar{c}^{m-1}, \tag{39}$$

it follows from (37), (38), and Proposition 13, that the neighbor reference trajectory deviation bound (16) is satisfied for all $\delta t \in (0, \bar{t}]$.

REMARK 14. *Due to (39), for any fixed $\bar{c} \in (0, 1)$, the reference trajectory deviation bound decreases exponentially with respect to the degree of decentralization.*

6. Well Posed Space-Time Discretizations. In this section, we exploit the controllers from Section 3.3 to provide sufficient conditions for well posed space-time discretizations. In particular, it is shown that for any system with bounded and globally Lipschitz dynamics of the form (1), and the hard input constraints (5), we can always select a well posed discretization $\mathcal{S}-\delta t$. This requires that each agent's transition system is nonblocking, namely, that for each m-cell configuration of the agent, the Transition Requirement is fulfilled for at least one successor cell. To verify this for the suggested discretizations, we select the function $\beta(\cdot)$ in Definition 5 as

$$\beta(t) := \frac{d_{\max}(\delta t - t)}{2\delta t} + \bar{\lambda} v_{\max} t, t \in [0, \delta t], \tag{40}$$

where the parameter $\bar{\lambda}$ is introduced in (9) and provides the upper part of the free input that is used for reachability purposes.

6.1. Sufficient Conditions. As in the previous sections, given a cell decomposition $\{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, we consider a reference point $x_{l,G}$ satisfying (17) for each cell, and the associated reference solutions $\chi_i(\cdot)$ of the initial value problems for the m-cell configurations of each agent i. Leveraging the corresponding feedback laws (14) we derive sufficient conditions for well posed discretizations in the following theorem.

THEOREM 15. *Consider a cell decomposition $\mathcal{S}$ of $\mathbb{R}^n$ with diameter $d_{\max}$, a time step δt, the constant r_i defined in (12), the parameter $\bar{\lambda}$ in (9), and let $\lambda \in [0, \bar{\lambda}]$. We assume that $d_{\max}$ and δt satisfy the following restrictions:*

$$\delta t \in \left(0, \min\left\{\bar{t}, \frac{(1-\lambda)v_{\max}}{L_1\sqrt{N_{\max}}(c+\bar{\lambda}v_{\max}) + \lambda L_2 v_{\max}}\right\}\right) \tag{41}$$

$$d_{\max} \in \left(0, \min\left\{\frac{2(1-\lambda)v_{\max}\delta t}{1+(L_1\sqrt{N_{\max}}+L_2)\delta t}, 2(1-\lambda)v_{\max}\delta t\right.\right.$$
$$\left.\left. -2(L_1\sqrt{N_{\max}}(cM+\bar{\lambda}v_{\max}) + \lambda L_2 v_{\max})\delta t^2\right\}\right], \tag{42}$$

with L_1, L_2, M, $v_{\max}$, c, and $\bar{t}$, as given in (3), (4), (2), (5), (39), and (36), respectively. Then, the space-time discretization is well posed for (1). In particular, for each agent $i \in \mathcal{N}$ and cell configuration $\mathbf{l}_i$ of i we have

$$\mathrm{Post}_i(l_i; \mathbf{l}_i) \supset \{l \in \mathcal{I} : S_l \cap B(\chi_i(\delta t); r_i) \neq \emptyset\}, \tag{43}$$

where r_i is defined in (12) with

$$\zeta_i(t) := \lambda. \tag{44}$$

Proof. According to Definition 6, to prove that the discretization is well posed, we specify continuous functions $\beta(\cdot)$ and $\alpha_i(\cdot)$, $i \in \mathcal{N}$, satisfying (18) and (16), respectively, so that (43) holds for all $i \in \mathcal{N}$ and $\mathbf{l}_i \in \mathcal{I}^{\bar{N}_i^m+1}$. We pick $\beta(\cdot)$ as in (40) and $\alpha_i(\cdot) \equiv \alpha(\cdot)$, for all $i \in \mathcal{N}$, with $\alpha(\cdot)$ as given in (39). Notice first that $\beta(\cdot)$ satisfies (18). Due to the requirement that $\delta t \leq \bar{t}$ in (41) and the discussion below (39), the functions $\alpha_i(\cdot)$ satisfy (16). Thus, the requirements of a well posed discretization for $\beta(\cdot)$ and $\alpha_i(\cdot)$ are fulfilled. Next, let $i \in \mathcal{N}$ and $\mathbf{l}_i \in \mathcal{I}^{\bar{N}_i^m+1}$. To verify (43), we need to show that for each $l_i' \in \mathcal{I}$ with $S_{l_i'} \cap B(\chi_i(\delta t); r_i) \neq \emptyset$, there exists a transition $l_i \xrightarrow{\mathbf{l}_i}_i l_i'$ in TS_i. Therefore, consider the globally Lipschitz W-parameterized feedback law $k_{i,\mathbf{l}_i}(\cdot)$ given by (14) and let $l_i' \in \mathcal{I}$ with $S_{l_i'} \cap B(\chi_i(\delta t); r_i) \neq \emptyset$. According to Definition 8, to show that $l_i \xrightarrow{\mathbf{l}_i}_i l_i'$, we need to pick $w_i \in W$ so that $k_{i,\mathbf{l}_i}$, w_i, l_i' satisfy the Transition Requirement. Let $x \in S_{l_i'} \cap B(\chi_i(\delta t); r_i)$ and select

$$w_i := \frac{x - \chi_i(\delta t)}{\lambda \delta t}, \tag{45}$$

with λ as in (9). From (12) and (44), $|w_i| \leq \frac{r_i}{\lambda \delta t} \leq v_{\max}$. Hence, by (10), $w_i \in W$. To verify the Transition Requirement let $x_{i0} \in S_{l_i}$. We show that the solution $x_i(\cdot)$ of (19) with $v_i = k_{i,\mathbf{l}_i}(t, x_i, \mathbf{d}_j; x_{i0}, w_i)$ satisfies (21), (22), and (23), for any continuous $d_{j_1}, \ldots, d_{j_{N_i}}$ satisfying (20). We break the proof in the following steps.

STEP 1: Proof of (21) *and* (22). By taking into account (19), (14), (8), (13), (11) and (44) we obtain for any continuous $d_{j_1}, \ldots, d_{j_{N_i}}$ the solution $x_i(\cdot)$ of (19) with $v_i = k_{i,\mathbf{l}_i}$ as $x_i(t) = x_{i0} + \int_0^t (f_i(x_i(s), \mathbf{d}_j(s)) + k_{i,\mathbf{l}_i}(s, x_i(s), \mathbf{d}_j(s); x_{i0}, w_i))ds = x_{i0} + \int_0^t \left(f_i(\chi_i(s), \boldsymbol{\chi}_j(s))ds + \frac{1}{\delta t}(x_{l_i,G} - x_{i0}) + \lambda w_i\right) ds = x_{i0} + \chi_i(t) - x_{l_i,G} + \frac{t}{\delta t}(x_{l_i,G} - x_{i0}) + t\lambda w_i = \chi_i(t) + \frac{\delta t - t}{\delta t}(x_{i0} - x_{l_i,G}) + t\lambda w_i, t \geq 0$. Hence, we deduce from (17) that

$$|x_i(t) - \chi_i(t)| < \frac{(\delta t - t) d_{\max}}{2\delta t} + t\lambda v_{\max}, \forall t \in [0, \delta t), \tag{46}$$

which by (40) and (9) establishes (21). Furthermore, we get from (45) that $x_i(\delta t) = \chi_i(\delta t) + \delta t \lambda w_i = x \in S_{l_i'}$, and thus, (22) also holds.

STEP 2: Estimation of bounds on $k_{i,\mathbf{l}_i,1}(\cdot)$, $k_{i,\mathbf{l}_i,2}(\cdot)$, and $k_{i,\mathbf{l}_i,3}(\cdot)$ along the solution $x_i(\cdot)$ of (19) *with $v_i = k_{i,\mathbf{l}_i}$ and $d_{j_1}, \ldots, d_{j_{N_i}}$ satisfying* (20). We first show that

$$\begin{aligned} |k_{i,\mathbf{l}_i,1}(t, x_i(t), \mathbf{d}_j(t))| \leq & L_1 \sqrt{N_{\max}} \left(\frac{d_{\max}(\delta t - t)}{2\delta t} + (cM + \bar{\lambda} v_{\max})t \right) \\ & + L_2 \left(\frac{(\delta t - t) d_{\max}}{2\delta t} + \lambda v_{\max} t \right), \forall t \in [0, \delta t]. \end{aligned} \tag{47}$$

Indeed, due to (8), we have that

$$\begin{aligned} k_{i,\mathbf{l}_i,1}(t, x_i(t), \mathbf{d}_j(t)) = & [f_i(\chi_i(t), \boldsymbol{\chi}_j(t)) - f_i(x_i(t), \boldsymbol{\chi}_j(t))] \\ & + [f_i(x_i(t), \boldsymbol{\chi}_j(t)) - f_i(x_i(t), \mathbf{d}_j(t))]. \end{aligned} \tag{48}$$

For the second difference on the right hand side of (48), we obtain from (3), (20), (32), (39), and (40), that $|f_i(x_i(t), \boldsymbol{\chi}_j(t)) - f_i(x_i(t), \mathbf{d}_j(t))| \leq L_1 (\sum_{\kappa=1}^{N_i} (\alpha(t) + \beta(t))^2)^{\frac{1}{2}} \leq L_1 \sqrt{N_{\max}} (\frac{d_{\max}(\delta t - t)}{2\delta t} + (cM + \bar{\lambda} v_{\max})t)$. For the other difference in (48), it follows from (4) that $|f_i(x_i(t), \boldsymbol{\chi}_j(t)) - f_i(\chi_i(t), \boldsymbol{\chi}_j(t))| \leq L_2 |(\chi_i(t) + t\lambda w_i + \left(1 - \frac{t}{\delta t}\right)(x_{i0} - x_{l_i,G})) - \chi_i(t)| \leq L_2 (\frac{(\delta t - t) d_{\max}}{2\delta t} + \lambda v_{\max} t)$, where $x_i(\cdot)$ is evaluated in Step 1. Consequently, from the derived bounds on the differences of the right hand side of (48), we get (47). Next, for $k_{i,\mathbf{l}_i,2}(\cdot)$ we have from (10), (11), and (44), that

$$|k_{i,\mathbf{l}_i,2}(t; w_i)| = |\lambda w_i| \leq \lambda v_{\max}, \forall t \in [0, \delta t], w_i \in W. \tag{49}$$

Finally, by recalling that $x_{l_i,G}$ satisfies (17), it follows from (13) that

$$|k_{i,\mathbf{l}_i,3}(x_{i0})| = \frac{1}{\delta t} |x_{i0} - x_{l_i,G}| \leq \frac{d_{\max}}{2\delta t}, \forall x_{i0} \in S_{l_i}. \tag{50}$$

STEP 3: Verification of (23). In this step we exploit the bounds from Step 2 to show (23) for any $d_{j_1}, \ldots, d_{j_{N_i}}$ satisfying (20). Due to (14), (47), (49), and (50), it suffices to show that $L_1\sqrt{N_{\max}}(\frac{d_{\max}(\delta t-t)}{2\delta t}+(cM+\bar{\lambda}v_{\max})t)+\frac{d_{\max}}{2\delta t}+L_2(\frac{(\delta t-t)d_{\max}}{2\delta t}+\lambda v_{\max}t)+\lambda v_{\max}\le v_{\max}$, for each $t\in[0,\delta t]$. Since the left hand side of this inequality is linear with respect to t, we only need to verify it for $t=0$ and $t=\delta t$, which follows in both cases from (42). Hence, (23) is also fulfilled and the proof is complete. ∎

From the involvement of the parameter c given by (39) in the acceptable δt and $d_{\max}$, it follows that an increasing degree of decentralization allows the selection of coarser discretizations. For instance, assume that the system parameters and selected $\bar{\lambda}$, λ, and $\bar{t}$ are such that $\bar{t}\ge\frac{(1-\lambda)v_{\max}}{L_1\sqrt{N_{\max}}\bar{\lambda}v_{\max}+\lambda L_2 v_{\max}}$ and the maximum value of $d_{\max}$ over all possible δt is obtained through the second element of the min in (42), i.e., $d_{\max}=2(1-\lambda)v_{\max}\delta t-2(L_1\sqrt{N_{\max}}(cM+\bar{\lambda}v_{\max})+\lambda L_2 v_{\max})\delta t^2$. Then, it follows from (41) and (42) that this value will be $d_{\max}=\frac{(1-\lambda)v_{\max}^2}{L_1\sqrt{N_{\max}}(\bar{c}^{m-1}M+\bar{\lambda}v_{\max})+\lambda L_2 v_{\max}}$ and increase close to $\frac{(1-\lambda)v_{\max}^2}{L_1\sqrt{N_{\max}}\bar{\lambda}v_{\max}+\lambda L_2 v_{\max}}$ for large degrees of decentralization m. This observation suggests that it is not desirable to select m very large, in the sense that beyond some value, increasing it to $m+1$ results to an additional state dimension that is not sufficiently compensated by a small improvement of the discretization diameter.

We also present an improved version of Theorem 15, when the conditions of Lemma 11 are satisfied for all agents, namely, when for any m-cell configuration of each agent, the estimated reference trajectories of its neighbors coincide with their reference trajectories for consistent configurations.

THEOREM 16. *Assume that $\mathcal{N}_i^{m+1}=\emptyset$ holds for all $i\in\mathcal{N}$. Then, the result of Theorem 15 remains valid for any δt and $d_{\max}$ satisfying*

$$\delta t\in\left(0,\frac{(1-\lambda)v_{\max}}{L_1\sqrt{N_{\max}}\bar{\lambda}v_{\max}+\lambda L_2 v_{\max}}\right) \tag{51}$$

$$\begin{aligned} d_{\max}\in\Bigg(0,\min\Bigg\{&\frac{2(1-\lambda)v_{\max}\delta t}{1+(L_1\sqrt{N_{\max}}+L_2)\delta t}, \\ &2(1-\lambda)v_{\max}\delta-2(L_1\sqrt{N_{\max}}\bar{\lambda}v_{\max}+\lambda L_2 v_{\max})\delta t^2\Bigg\}\Bigg]. \end{aligned} \tag{52}$$

Proof. Since by hypothesis $\mathcal{N}_i^{m+1}=\mathcal{N}_\ell^{m+1}=\emptyset$ for each pair of agents $i\in\mathcal{N}$, $\ell\in\mathcal{N}_i$, it follows from Lemma 11 that for corresponding consistent cell configurations $\mathbf{l}_i$ and $\mathbf{l}_\ell$, it holds that $\chi_\ell^{\mathbf{l}_i}(t)=\chi_\ell^{\mathbf{l}_\ell}(t)$, $\forall t\ge 0$. Thus, we can select for each agent $i\in\mathcal{N}$ the reference trajectory deviation bound $\alpha_i(\cdot)\equiv 0$. The remaining proof follows the same arguments employed for the proof of Theorem 15 and is therefore omitted. ∎

6.2. Exploiting the Abstractions for Control Synthesis. Here we describe how Theorems 15, 16, and Proposition 10 can be used to synthesize discrete plans and project them to sequences of local feedback controllers to enable their correct low level execution by the agents.

Step 1. Consider the agents' Lipschitz constants L_1, L_2, dynamics bounds M, $v_{\max}$, and select a degree of decentralization m and parameters $\bar{\lambda}$, λ. Depending on the selection of m and the network structure, use either Theorem 15 or Theorem 16 to obtain the acceptable discretization values. If Theorem 15 is invoked, then select also a constant $\bar{c}\in(0,1)$ and evaluate $\bar{t}$ and c by (36) and (39), respectively. Finally, select $d_{\max}-\delta t$ and a corresponding cell decomposition $\mathcal{S}$.

Step 2. Fix a reference point for every cell of the decomposition and derive the transition system TS_i of each agent i as follows. For each m-cell configuration $\mathbf{l}_i$ solve the IVP of Definition 1 to obtain the endpoint $\chi_i(\delta t)$ of agent i's reference trajectory and specify the cells which intersect $B(\chi_i(\delta t);r_i)$, i.e., the transitions to the cells given by the right hand side of (43). It is noted that the reference trajectories need not be stored and the control laws for the transitions are not calculated at this step.

Step 3. Given a high level specification for each agent, find a discrete plan $\mathbf{l}^0\mathbf{l}^1\mathbf{l}^2\cdots$, i.e., a sequence of cell configurations $\mathbf{l}^\kappa=(l_1^\kappa,\ldots,l_N^\kappa)$, $\kappa=0,1,\ldots$ such that $l_i^{\kappa+1}\in \mathrm{Post}_i(l_i^\kappa;\mathbf{l}_i^\kappa)$, with $\mathbf{l}_i^\kappa=\mathrm{pr}_i(\mathbf{l}^\kappa)$ for all $i\in\mathcal{N}$, corresponding to a sequence of transitions $l_i^0\xrightarrow{\mathbf{l}_i^0}_i l_i^1\xrightarrow{\mathbf{l}_i^1}_i l_i^2\cdots$ for each agent which satisfies its specification.

Step 4. Determine the continuous control laws for the implementation of each agent's discrete plan as follows. For each transition $l_i^\kappa\xrightarrow{\mathbf{l}_i^\kappa}_i l_i^{\kappa+1}$, find again the solution of the IVP corresponding to the m-cell configuration $\mathbf{l}_i^\kappa$, and obtain the reference trajectories $\chi_i(\cdot)$ and $\boldsymbol{\chi}_i(\cdot)$ of i and its neighbors. Then, pick any $x\in B(\chi_i(\delta t);r_i)\cap S^i_{l_i^{\kappa+1}}$, compute the parameter w_i from (45) and use $\chi_i(\cdot)$, $\boldsymbol{\chi}_i(\cdot)$ to determine the corresponding feedback law $k_{i,\mathbf{l}_i^\kappa}(\cdot,\cdot,\cdot;\cdot,w_i)$ from (14), (8), (11), and (13).

Due to Proposition 10, the satisfying plan from Step 3 is correctly executed by the controllers evaluated in Step 4. Furthermore, Step 4 can considerably alleviate memory storage requirements, since the reference trajectories and control laws are stored only for the satisfying plans and not for the whole transition systems.

7. Example and Simulation Results. As an illustrative example we consider the coordination of five interconnected agents in $\mathbb{R}^2$ which need to fulfill certain reachability goals. Agent 3 is an autonomous surface vehicle (ASV) inside a straight river of width $2L^{\mathrm{riv}}$, with a sinusoidal velocity profile of maximum speed $v_{\max}^{\mathrm{riv}}$ and direction $q^{\mathrm{riv}}\in\mathbb{R}^2$, with $|q^{\mathrm{riv}}|=1$ (motivated by the single agent example in [12]). Its motion 3 is governed by the dynamics $\dot{x}_3=v_{\max}^{\mathrm{riv}}\cos\left(\frac{\pi|x_3-\langle x_3,q^{\mathrm{riv}}\rangle q^{\mathrm{riv}}|}{2L^{\mathrm{riv}}}\right)q^{\mathrm{riv}}+v_3$ and it is always possible to assign inputs $v_3(\cdot)$ to constrain the agent inside the river. The other agents are unmanned aerial vehicles (UAVs) operating at the same height close to the ground. Agents 2 and 4 are coupled with agent 3 through the dynamics $\dot{x}_i=\mathrm{sat}_{3\rho}(x_3-x_i)+v_i$, $i=2,4$, and agents 1 and 5 are coupled with 2 and 4 by $\dot{x}_i=\mathrm{sat}_{5\rho}(x_j-x_i)+v_i$, $(i,j)=(1,2),(5,4)$, where $\rho>0$ and $\mathrm{sat}_\rho(x):=x$, if $|x|<\rho$, $\mathrm{sat}_\rho(x):=\frac{\rho}{|x|}x$, if $|x|\ge\rho$. Assuming that the additive inputs v_i and the river velocity are bounded by $v_{\max}=v_{\max}^{\mathrm{riv}}=\rho$, it is not hard to show that if agents 2, 3 and 3, 4 are initially located within a distance of at most 3ρ, they will maintain this property during the system's evolution. This allows them to exchange information and measure their relative states which are used for their feedback loop. Analogously, agents 1, 2 and 4, 5 will maintain a distance of at most 5ρ. Thus, the network will remain connected during the system's evolution, allowing the agents to exchange information on the missions. We assign the team specification that every agent should reach a target box precisely at the end of the common mission horizon, which is the time interval $[0,2.5]$ (see Figure 5). The agents' dynamics have the form (1) and are globally Lipschitz and bounded. Thus, by selecting the degree of decentralization $m=2$, the conditions of Theorem 16 are satisfied. Assuming also that $\frac{\pi v_{\max}^{\mathrm{riv}}}{2L^{\mathrm{riv}}}\le 1$, we obtain the dynamics bounds and Lipschitz constants $M:=5\rho$, $L_1:=1$, and $L_2:=1$ for the agents. We next choose the constant $\bar{\lambda}=1$ in (9), $\lambda=0.4$ and obtain from (51) and (52) that $0<\delta t<\frac{1-\lambda}{1+\lambda}$ and $0<d_{\max}\le\min\{\frac{2(1-\lambda)\delta t}{1+2\delta t}\rho,2((1-\lambda)\delta t-(1+\lambda)\delta t^2)\rho\}$. For the simulation results we pick $\rho=5$ and focus on the system's behavior for $t\in[0,2.5]$. Since $d_{\max}$ is maximized for $\delta t=\frac{1-\lambda}{2(1+\lambda)}$, we select δt as the closest value to $\frac{1-\lambda}{2(1+\lambda)}$ to obtain an integer number of time steps $NT:=\frac{2.5}{\delta t}$, and the maximum corresponding $d_{\max}$ to partition $\mathbb{R}^2$ using square cells.

To derive a satisfying discrete plan we use the agents' individual transition systems through the following sequential process. First, we exploit the transition system of agent 3 which is not affected by any coupling constraints to obtain all the discrete paths, i.e., cell index sequences $l_3^0 l_3^1 \cdots l_3^{NT}$ of consecutive transitions which satisfy its reachability specification. To obtain the relevant transitions, we evaluate the indices $Q_3^{\kappa} = \mathrm{Post}_3(Q_3^{\kappa-1})$ of the agent's reachable cells at each time $\kappa\delta t$, $\kappa = 0, 1, \ldots, NT$, using the notation $Q_3^0 := \{l_3^0\}$ and the convention $\mathrm{Post}_3(Q_3) := \cup_{l_3 \in Q_3} \mathrm{Post}_3(l_3; l_3)$ (l_3 is a 2-cell configuration of agent 1, since the agent has no neighbors). Thus, $\cup_{\kappa=0}^{NT} Q_3^{\kappa}$ is captured through the blue cells in Figure 5(i). Then, we pick all cells from Q_3^{NT} which lie in T3 to determine the agent's satisfying paths, depicted with the yellow cells in Figure 5(i), with a backward reachability algorithm. We next use the satisfying

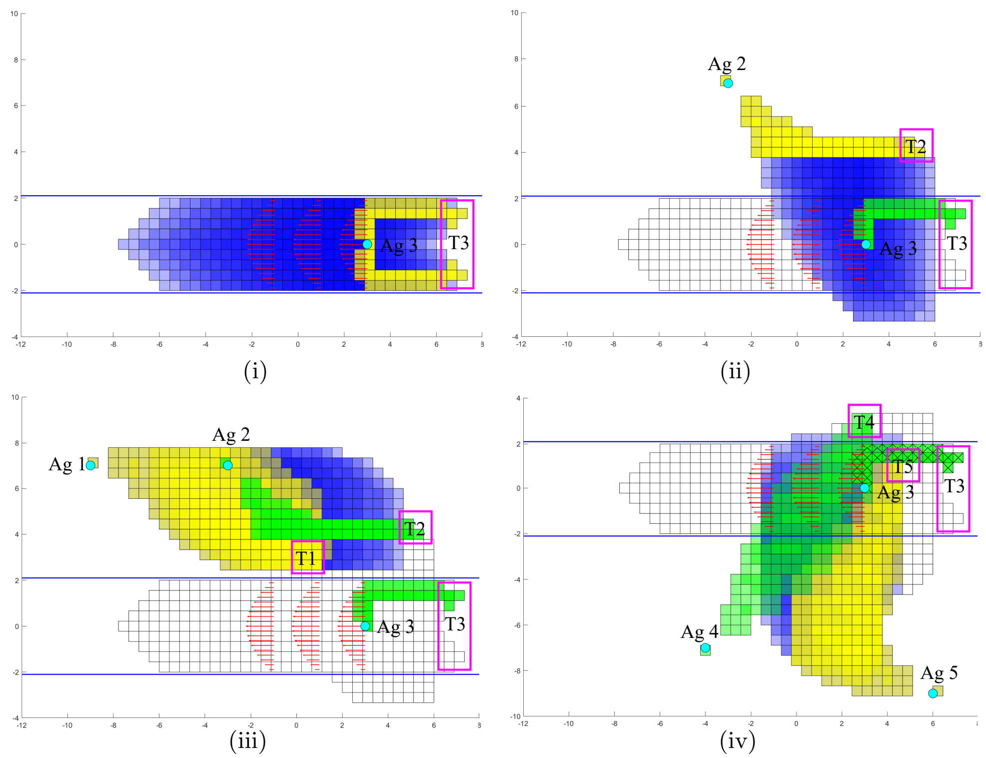


FIG. 5. *(i) The reachable cells of agent 3 are depicted in blue and the ones which reach its target box in yellow. (ii) All reachable cells of agent 2 based on the satisfying ones of agent 3 in (i) are in blue. The corresponding cells which satisfy the agents' specifications are in yellow and green, respectively. (iii) All reachable cells of agent 1 based on the satisfying ones of agents 2 and 3 in (ii) are shown in blue. The cells of agents 1, and 2, 3, depicted in yellow, and green, respectively, lead to the simultaneous satisfaction of their reachability goals. (iv) Corresponding simulation results for agents 5, 4, and 3, with the satisfying cells of agent 3 marked, to be distinguished from those of 4.*

paths of 3 to obtain the reachable cells Q_2^{κ} of agent 2 at each time $\kappa\delta t$. The cells with indices $\cup_{\kappa=0}^{NT} Q_2^{\kappa}$ are shown with blue in Figure 5(ii). Then, we determine the paths of agent 2 which lead to its target box, and the corresponding paths of 3, depicted in Figure 5(ii) with the yellow and green cells, respectively. Analogously, we use the satisfying paths of agents 2 and 3 as actions in the transition system of 1 to determine its reachable cells. Then, we obtain through backward reachability the satisfying paths of 1, depicted with the yellow cells in Figure 5(iii), and the corresponding ones of agents 2 and 3 shown in green. The exact same procedure is performed for the specifications of agents 4 and 5, as shown in Figure5(iv). It is observed from

Figures 5(iii) and 5(iv) that we can find a common path among the ones of agent 3 which provide satisfying plans for agents 1 and 2 (green cells of 3 in Figures 5(iii)) and the ones of agent 3 which provide satisfying plans for agents 4 and 5 (green marked cells of 3 in Figure 5(iv)). This implies that the specifications are satisfied by all agents simultaneously, namely, the team mission is feasible. The agents' successor cells were evaluated using Step 2 of the procedure outlined in Section 6.2, without storing their reference trajectories or evaluating the control laws for the transitions. As described in Step 4 of the same procedure, the latter are computed once a satisfying discrete plan is selected. The simulation results were implemented in MATLAB with a running time of approximately five minutes, on a PC with an Intel(R) Core(TM) i7-4600U CPU @ 2.10GHz processor.

8. Conclusions. We have provided abstractions for multi-agent systems under a varying degree of decentralization and modeled their transitions by exploiting a system with disturbances that capture the evolution of each agent's neighbors. Sufficient conditions for the space and time discretization quantify the reachability capabilities of the symbolic models. Their transitions are realized by hybrid feedback laws which modify a part of the agents' couplings and navigate them to their successor cells. Ongoing work includes the formulation of online abstractions for heterogeneous agents with updated choices of the discretization and the consideration of higher order dynamics. We also aim at studying robustness of the approach with respect to measurement and actuator errors, both from a continuous and a discrete perspective.

9. Appendix. The Appendix includes omitted proofs from Section 5 and the following Lemma, which establishes useful properties of the agents' m-neighbor sets.

LEMMA 17. *(i) For each agent $i \in \mathcal{N}$, neighbor $\ell \in \mathcal{N}_i$ of i, and $m \geq 1$, it holds that $\bar{\mathcal{N}}_\ell^{m-1} \subset \bar{\mathcal{N}}_i^m$.*
(ii) For each $i \in \mathcal{N}$, $m \geq 1$, and $\ell \in \bar{\mathcal{N}}_i^m$, it holds that $\mathcal{N}_\ell \subset \bar{\mathcal{N}}_i^{m+1}$.
(iii) Assume that for certain $i \in \mathcal{N}$ and $m \geq 1$ it holds $\mathcal{N}_i^{m+1} = \emptyset$. Then, for each $\ell \in \bar{\mathcal{N}}_i^m$ it holds that $\mathcal{N}_\ell \subset \bar{\mathcal{N}}_i^m$.

Proof. For the proof of Part (i), let any $i' \in \bar{\mathcal{N}}_\ell^{m-1}$. Then, since $\bar{\mathcal{N}}_\ell^{m-1} = \cup_{\kappa=0}^{m-1}\mathcal{N}_\ell^\kappa$, either $i' = \ell$, which implies that $i' \in \mathcal{N}_i$, and hence, also that $i' \in \bar{\mathcal{N}}_i^m$, or $i' \in \mathcal{N}_\ell^{m'}$ for certain $m' \in \{1, \ldots, m-1\}$. If $i' = i$, then $i' \in \bar{\mathcal{N}}_i^m$ and hence, it remains to consider the case where $i' \neq i$ and $i' \in \mathcal{N}_\ell^{m'}$ for some $m' \in \{1, \ldots, m-1\}$. The latter implies that there exists a shortest path $i_0 \ldots i_{m'}$ with $i_0 = i'$ and $i_{m'} = \ell$ from i' to ℓ. Then, since $i_{m'} = \ell \in \mathcal{N}_i$, it follows that $i_0 \ldots i_{m'} i$ is a path of length $m'+1 \leq m$ from i' to i. Thus, either it is a shortest path, implying that $i' \in \mathcal{N}_i^{m'+1} \subset \bar{\mathcal{N}}_i^m$, or, since $i' \neq i$, there exist $0 < m'' < m'+1$ and a shortest path of length m'' joining i' and i. In the latter case, it follows that $i' \in \mathcal{N}_i^{m''}$, and hence, again that $i' \in \bar{\mathcal{N}}_i^m$. The proof of Part (i) is now complete.

For the proof of Part (ii), let $\ell \in \bar{\mathcal{N}}_i^m$ and $i' \in \mathcal{N}_\ell$. If $i' = i$ then $i' \in \bar{\mathcal{N}}_i^{m+1}$. Otherwise, since $\ell \in \bar{\mathcal{N}}_i^m$, there exists $1 \leq m' \leq m$ and a path $i_0 \ldots i_{m'}$ of length m' with $i_0 = \ell$ and $i_{m'} = i$, implying that $i' i_0 \ldots i_{m'}$ is a path of length $m'+1 \leq m+1$ from i' to i. Thus, it follows that $i' \in \bar{\mathcal{N}}_i^{m+1}$.

For the proof of Part (iii), let $\ell \in \bar{\mathcal{N}}_i^m$ and $i' \in \mathcal{N}_\ell$. If $\ell = i$ or $i' = i$, then the result follows directly from the facts that $\mathcal{N}_i \subset \bar{\mathcal{N}}_i^m$ and $i \in \bar{\mathcal{N}}_i^m$, respectively. Otherwise, there exists $1 \leq m' \leq m$ such that $\ell \in \mathcal{N}_i^{m'}$, implying that there exists a path $i_0 \ldots i_{m'}$ of length m' with $i_0 = \ell$ and $i_{m'} = i$. Thus, since $i' \in \mathcal{N}_\ell$, we get that $i' i_0 \ldots i_{m'}$ is a path of length $m'+1$ from i' to i. If it is a shortest path, then it follows that $m' < m$, because otherwise, since $i' \neq i$, we would have a shortest path of length

$m+1$ joining i' and i, implying that $\mathcal{N}_i^{m+1}\neq\emptyset$ and contradicting the hypothesis of Part (iii). Hence, it holds in this case that $i'\in\mathcal{N}_i^{m'+1}$ for certain $1\leq m'<m$ and thus, that $i'\in\bar{\mathcal{N}}_i^m$. Finally, if $i'i_0\ldots i_{m'}$ is not a shortest path, then there exists a shortest path of length $1\leq m''<m$ joining i' and i, which implies that $i'\in\mathcal{N}_i^{m''}$, and hence, again that $i'\in\bar{\mathcal{N}}_i^m$. ∎

We next give the proofs of Lemma 11, Lemma 12, and Proposition 13 in Section 5.

Proof of Lemma 11. Since $\mathcal{N}_i^{m+1}=\emptyset$, it follows that the initial value problem which corresponds to the m-cell configuration of agent i and specifies the trajectory $\chi_\ell^{\mathbf{l}_i}(\cdot)$ of ℓ is provided by Case (i) of Definition 1, namely, by (15). We rewrite (15) in the compact form

$$\dot{X}=F(X);\ F=(f_{\kappa_1},\ldots,f_{\kappa_{\bar{N}_i^m+1}}),\ X=(\chi_{\kappa_1},\ldots,\chi_{\kappa_{\bar{N}_i^m+1}}), \tag{53}$$

with initial condition $\chi_{\kappa_\nu}(0)=x_{l_{\kappa_\nu},G}$, $\nu=1,\ldots,\bar{N}_i^m+1$, and κ_ν being the ν-th index of $\bar{\mathcal{N}}_i^m$ according to the total ordering $\prec$ of $\bar{\mathcal{N}}_i^m$ (see Section 2). Next, since $\mathcal{N}_\ell^{m+1}=\emptyset$, we similarly obtain the initial value problem associated to the m-cell configuration of agent ℓ to specify $\chi_\ell^{\mathbf{l}_\ell}(\cdot)$, as

$$\dot{X}_1=F_1(X_1);\ F_1=(f_{\kappa_1'},\ldots,f_{\kappa'_{\bar{N}_\ell^m+1}}),\ X_1=(\chi_{\kappa_1'},\ldots,\chi_{\kappa'_{\bar{N}_\ell^m+1}}), \tag{54}$$

with initial condition $\chi_{\kappa_\nu'}(0)=x_{l'_{\kappa_\nu},G}$, $\nu=1,\ldots,\bar{N}_\ell^m+1$, and κ_ν' being the corresponding ν-th index of $\bar{\mathcal{N}}_\ell^m$. Taking into account that $\mathcal{N}_i^{m+1}=\emptyset$, we get from Lemma 17(i) that $\bar{\mathcal{N}}_\ell^m\subset\bar{\mathcal{N}}_i^{m+1}=\bar{\mathcal{N}}_i^m\cup\mathcal{N}_i^{m+1}=\bar{\mathcal{N}}_i^m$. Thus, assuming without loss of generality that the inclusion is strict and reordering the components χ_{κ_ν}, $\nu=1,\ldots,\bar{N}_i^m+1$, (53) can be cast in the form

$$\dot{X}_1=F_1(X_1),\dot{X}_2=F_2(X_1,X_2), \tag{55}$$

with X_1, $F_1(\cdot)$ as in (54), $X_2=(\chi_{\kappa_{\bar{N}_\ell^m+2}},\ldots,\chi_{\kappa_{\bar{N}_i^m+1}})$, $F_2=(f_{\kappa_{\bar{N}_\ell^m+2}},\ldots,f_{\kappa_{\bar{N}_i^m+1}})$, and the same initial condition as for (54) for the X_1 part, due to the consistency of $\mathbf{l}_\ell$ with $\mathbf{l}_i$. Hence, since the X_1 part of the solution in (55) is independent of X_2, the reference trajectory $\chi_\ell^{\mathbf{l}_\ell}(\cdot)$ of agent ℓ given by (54) and its estimate $\chi_\ell^{\mathbf{l}_i}(\cdot)$ obtained from the first subsystem in (55) coincide. ∎

Proof of Lemma 12. For the proof, it suffices to show that (33) holds for all agents $\kappa\in\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$, since by Lemma 17(i) we have $\bar{\mathcal{N}}_\ell^{m-1}\subset\bar{\mathcal{N}}_i^m$. We distinguish the following cases.

Case (i). $\mathcal{N}_i^{m+1}\neq\emptyset$ and $\mathcal{N}_\ell^{m+1}\neq\emptyset$. For Case (i) we consider the following subcases for each agent $\kappa\in\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$.

Case (ia). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\cap(\mathcal{N}_i^m\cup\mathcal{N}_\ell^m)$. In this case, it follows from (7) that either $\chi_\kappa^{\mathbf{l}_i}(\cdot)\equiv x_{l_\kappa,G}$ or $\chi_\kappa^{\mathbf{l}_\ell}(\cdot)\equiv x_{l_\kappa,G}$. Without loss of generality we assume that $\kappa\in\bar{\mathcal{N}}_\ell^{m-1}$, and thus, $\chi_\kappa^{\mathbf{l}_\ell}(\cdot)$ is specified by the IVP (6), and $\chi_\kappa^{\mathbf{l}_i}(\cdot)\equiv x_{l_\kappa,G}$. Then, we get from (2) and consistency of $\mathbf{l}_\ell$ with $\mathbf{l}_i$, which implies that $\chi_\kappa^{\mathbf{l}_i}(0)=\chi_\kappa^{\mathbf{l}_\ell}(0)=x_{l_\kappa,G}$, that

$$|\chi_\kappa^{\mathbf{l}_i}(t)-\chi_\kappa^{\mathbf{l}_\ell}(t)|=|x_{l_\kappa,G}-\chi_\kappa^{\mathbf{l}_\ell}(t)|\leq\int_0^t|f_\kappa(\chi_\kappa^{\mathbf{l}_\ell}(s),\boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_\ell}(s))|ds\leq Mt,\forall t\geq 0. \tag{56}$$

Case (ib). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus(\mathcal{N}_i^m\cup\mathcal{N}_\ell^m)$. Notice first, that $(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus(\mathcal{N}_i^m\cup\mathcal{N}_\ell^m)\subset\bar{\mathcal{N}}_i^{m-1}\cap\bar{\mathcal{N}}_\ell^{m-1}$ and thus, $\kappa\in\bar{\mathcal{N}}_i^{m-1}$ and $\kappa\in\bar{\mathcal{N}}_\ell^{m-1}$. Hence, we obtain from Lemma 17(ii) that $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_i^m$ and $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_\ell^m$, respectively, implying that $\mathcal{N}_\kappa\subset$

$\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m$. Consequently, it follows from Definition 1 that both $\boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_i}(\cdot)$ and $\boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_\ell}(\cdot)$ are well defined. To show (33) for all κ of Case (ib) we will prove the following claim.

Claim I. There exists $\delta \in (0, t^*)$, so that (33) holds for all $t \in [0, \delta]$ and κ of Case (ib).

To show Claim I, let $\delta \in (0, t^*)$ with

$$\delta \leq \min\left\{\frac{1}{4L_1\sqrt{N_{\max}}}, \frac{\ln 2}{L_2}\right\}, \tag{57}$$

and $\kappa \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$. Since $\mathbf{l}_\ell$ is consistent with $\mathbf{l}_i$ and $\mathcal{N}_\kappa \subset \bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m$, we have that $\chi_\nu^{\mathbf{l}_\ell}(0) = \chi_\nu^{\mathbf{l}_i}(0) = x_{l_\nu, G}$, for all $\nu \in \mathcal{N}_\kappa$. Thus, by exploiting (2), (32), and (57), we deduce that

$$|\boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_i}(t) - \boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_\ell}(t)| \leq 2M\sqrt{N_{\max}}t \leq \frac{M}{2L_1}, \forall t \in [0, \delta]. \tag{58}$$

Next, we obtain from (3) and (4) that $|\chi_\kappa^{\mathbf{l}_i}(t) - \chi_\kappa^{\mathbf{l}_\ell}(t)| \leq \int_0^t (L_2 |\chi_\kappa^{\mathbf{l}_i}(s) - \chi_\kappa^{\mathbf{l}_\ell}(s)| + L_1 |\boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_i}(s) - \boldsymbol{\chi}_{j(\kappa)}^{\mathbf{l}_\ell}(s)|) ds$, which due to (58), implies that

$$|\chi_\kappa^{\mathbf{l}_i}(t) - \chi_\kappa^{\mathbf{l}_\ell}(t)| \leq \tfrac{Mt}{2} + \int_0^t L_2 |\chi_\kappa^{\mathbf{l}_i}(s) - \chi_\kappa^{\mathbf{l}_\ell}(s)| ds, \forall t \in [0, \delta]. \tag{59}$$

To bound $|\chi_\kappa^{\mathbf{l}_i}(\cdot) - \chi_\kappa^{\mathbf{l}_\ell}(\cdot)|$, we use the following version of the Gronwall Lemma.

Fact I. Let $\lambda : [a, b] \to \mathbb{R}$ be a continuously differentiable function with $\lambda(a) = 0$ and μ a nonnegative constant. If a continuous function $y(\cdot)$ satisfies $y(t) \leq \lambda(t) + \int_a^t \mu y(s) ds$ on $[a, b]$, then, on the same interval it holds that $y(t) \leq \int_a^t e^{\mu(t-s)} \dot{\lambda}(s) ds$. $\triangleleft$

By exploiting Fact I, we obtain from (59) and (57) that

$$|\chi_\kappa^{\mathbf{l}_i}(t) - \chi_\kappa^{\mathbf{l}_\ell}(t)| \leq \int_0^t e^{L_2(t-s)} \frac{M}{2} ds \leq Mt, \forall t \in [0, \delta], \tag{60}$$

which concludes the proof of Claim I. We next show that (33) also holds for all $t \in [0, t^*]$ and κ of Case (ib). Assume on the contrary that

$$|\chi_{\kappa'}^{\mathbf{l}_i}(T) - \chi_{\kappa'}^{\mathbf{l}_\ell}(T)| > MT, \tag{61}$$

for some $\kappa' \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$ and $T \in (0, t^*]$, and define

$$\tau := \max\{\bar{t} \in [0, T] : |\chi_\kappa^{\mathbf{l}_i}(t) - \chi_\kappa^{\mathbf{l}_\ell}(t)| \leq Mt, \forall t \in [0, \bar{t}], \kappa \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)\}. \tag{62}$$

Then, it follows from (60) and (61) that

$$|\chi_{\kappa''}^{\mathbf{l}_i}(\tau) - \chi_{\kappa''}^{\mathbf{l}_\ell}(\tau)| = M\tau, \tag{63}$$

for some $\kappa'' \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$. Also, from Claim I and (62) we get that

$$0 < \tau < t^*. \tag{64}$$

Since $\kappa'' \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$, it holds that $\mathcal{N}_{\kappa''} \subset \bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m$. Thus, for each neighbor $\nu \in \mathcal{N}_{\kappa''}$ of κ'', either $\nu \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \cap (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$, or $\nu \in (\bar{\mathcal{N}}_i^m \cap \bar{\mathcal{N}}_\ell^m) \setminus (\mathcal{N}_i^m \cup \mathcal{N}_\ell^m)$, and we deduce from (56) and (62), respectively, that

$$|\chi_\nu^{\mathbf{l}_i}(t) - \chi_\nu^{\mathbf{l}_\ell}(t)| \leq Mt, \forall t \in [0, \tau], \nu \in \mathcal{N}_{\kappa''}. \tag{65}$$

It then follows from (3), (4) and (65) that $|\chi^{\mathbf{l}_i}_{\kappa''}(\tau)-\chi^{\mathbf{l}_\ell}_{\kappa''}(\tau)|\leq\int_0^\tau L_2|\chi^{\mathbf{l}_i}_{\kappa''}(s)-\chi^{\mathbf{l}_\ell}_{\kappa''}(s)|ds+\int_0^\tau L_1M\sqrt{N_{\kappa''}}sds$. Hence, we get from Fact I and (32) that

(66)
$$|\chi^{\mathbf{l}_i}_{\kappa''}(\tau)-\chi^{\mathbf{l}_\ell}_{\kappa''}(\tau)|\leq\int_0^\tau e^{L_2(\tau-s)}L_1M\sqrt{N_{\kappa''}}sds\leq\frac{L_1}{L_2}M\sqrt{N_{\max}}\left(\frac{e^{L_2\tau}}{L_2}-\tau-\frac{1}{L_2}\right).$$

In addition, it can be checked by elementary calculations that

$$e^{L_2t}-\left(L_2+\frac{L_2^2}{L_1\sqrt{N_{\max}}}\right)t-1<0,\forall t\in(0,t^*), \tag{67}$$

with t^* specified by (31). However, from (63) and (66), $e^{L_2\tau}-(L_2+\frac{L_2^2}{L_1\sqrt{N_{\max}}})\tau-1\geq0$, which contradicts (67), since by (64), $0<\tau<t^*$. Thus, (33) holds for Case (ib).

Case (ii). $\mathcal{N}_i^{m+1}\neq\emptyset$ and $\mathcal{N}_\ell^{m+1}=\emptyset$. For Case (ii) we consider the following subcases for each agent $\kappa\in\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$.

Case (iia). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\cap\mathcal{N}_i^m$. In this case, it follows from (7) that $\chi^{\mathbf{l}_i}_\kappa(\cdot)\equiv x_{l_\kappa,G}$ and thus, by using similar arguments with Case (ia) that (56) is fulfilled.

Case (iib). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_i^m$. Notice that $(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_i^m\subset\bar{\mathcal{N}}_i^{m-1}\cap\bar{\mathcal{N}}_\ell^m$ and thus, for each agent $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_i^m$ we have that $\kappa\in\bar{\mathcal{N}}_i^{m-1}$ and $\kappa\in\bar{\mathcal{N}}_\ell^m$. Hence, we obtain from Lemma 17(ii) and the fact that $\mathcal{N}_\ell^{m+1}=\emptyset$, that $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_i^m$ and $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_\ell^{m+1}=\bar{\mathcal{N}}_\ell^m\cup\mathcal{N}_\ell^{m+1}=\bar{\mathcal{N}}_\ell^m$, respectively, implying that $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$. The remaining proof for this case follows similar arguments with the proof of Case (ib) and is omitted.

Case (iii). $\mathcal{N}_i^{m+1}=\emptyset$ and $\mathcal{N}_\ell^{m+1}\neq\emptyset$. We consider again the following subcases for each agent $\kappa\in\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$.

Case (iiia). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\cap\mathcal{N}_\ell^m$. In this case, it follows from (7) that $\chi^{\mathbf{l}_\ell}_\kappa(\cdot)\equiv x_{l_\kappa,G}$ and thus, by using again similar arguments with Case (ia) that (56) is fulfilled.

Case (iiib). $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_\ell^m$. Notice that $(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_\ell^m\subset\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^{m-1}$ and thus, for each agent $\kappa\in(\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m)\setminus\mathcal{N}_\ell^m$ we have that $\kappa\in\bar{\mathcal{N}}_i^m$ and $\kappa\in\bar{\mathcal{N}}_\ell^{m-1}$. Hence, we obtain from Lemma 17(ii) and the fact that $\mathcal{N}_i^{m+1}=\emptyset$, that $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_i^{m+1}=\bar{\mathcal{N}}_i^m\cup\mathcal{N}_i^{m+1}=\bar{\mathcal{N}}_i^m$ and $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_\ell^m$, respectively, implying that $\mathcal{N}_\kappa\subset\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$. The remaining proof for Case (iiib) follows again similar arguments with the proof of Case (ib) and is omitted.

Case (iv). $\mathcal{N}_i^{m+1}=\emptyset$ and $\mathcal{N}_\ell^{m+1}=\emptyset$. In this case the result follows from the proof of Lemma 11, which implies that the trajectories $\chi^{\mathbf{l}_i}_\kappa(\cdot)$ and $\chi^{\mathbf{l}_\ell}_\kappa(\cdot)$ coincide for all $\kappa\in\bar{\mathcal{N}}_i^m\cap\bar{\mathcal{N}}_\ell^m$. The proof is now complete. ∎

Proof of Proposition 13. The proof is carried out by induction and is based on the result of Lemma 12. We will show the following induction hypothesis:

IH. For each $m'\in\{1,\dots,m\}$ and $\iota\in\bar{\mathcal{N}}_\ell^{m-m'}$, it holds that

$$|\chi^{\mathbf{l}_i}_\iota(t)-\chi^{\mathbf{l}_\ell}_\iota(t)|\leq H_{m'}(t),\forall t\in[0,t^*]. \tag{68}$$

Note that for $m'=m$ the Induction Hypothesis implies (34). Also, by Lemma 12, IH is valid for $m'=1$. To prove the general step, assume that IH is true for $m'\in\{1,\dots,m-1\}$ and let $\iota\in\bar{\mathcal{N}}_\ell^{m-(m'+1)}$. Since $m-(m'+1)\leq m-2$, both $\boldsymbol{\chi}^{\mathbf{l}_i}_{j(\iota)}(\cdot)$ and $\boldsymbol{\chi}^{\mathbf{l}_\ell}_{j(\iota)}(\cdot)$ are well defined and the differences $|\chi^{\mathbf{l}_i}_\nu(\cdot)-\chi^{\mathbf{l}_\ell}_\nu(\cdot)|$, $\nu\in\mathcal{N}_{j(\iota)}$, of their respective components satisfy (68) with m'. It then follows that $|\boldsymbol{\chi}^{\mathbf{l}_i}_{j(\iota)}(t)-\boldsymbol{\chi}^{\mathbf{l}_\ell}_{j(\iota)}(t)|\leq\sqrt{N_\iota}H_{m'}(t)$ for all $t\in(0,t^*)$. Thus, by evaluating $|\chi^{\mathbf{l}_i}_\iota(\cdot)-\chi^{\mathbf{l}_\ell}_\iota(\cdot)|$ as in the proof of Lemma 12, we obtain that $|\chi_\iota^{\mathbf{l}_i}(t) - \chi_\iota^{\mathbf{l}_\ell}(t)| \le \int_0^t L_1\sqrt{N_\iota}H_{m'}(s)ds + \int_0^t L_2|\chi_\iota^{\mathbf{l}_i}(s) - \chi_\iota^{\mathbf{l}_\ell}(s)|ds$. By exploiting Fact I used in the proof of Lemma 12, we obtain from (68) and the recursive definition of $H_{m'+1}(\cdot)$ that $|\chi_\iota^{\mathbf{l}_i}(t) - \chi_\iota^{\mathbf{l}_\ell}(t)| \le \int_0^t e^{L_2(t-s)}L_1\sqrt{N_{\max}}H_{m'}(s)ds = H_{m'+1}(t), \forall t \in [0,t^*]$, which establishes the general induction step. The proof is complete. ∎

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