ROBUSTNESS AND INVARIANCE OF CONNECTIVITY MAINTENANCE CONTROL FOR MULTI-AGENT SYSTEMS*

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Abstract. This paper is focused on a cooperative control design which guarantees robust connectivity and invariance of a multi-agent network inside a bounded domain, under the presence of additional bounded input terms in each agent's dynamics. In particular, under the assumptions that the domain is convex and has a smooth boundary, we can design a repulsion vector field near its boundary, which ensures invariance of the agents' trajectories and does not affect the robustness properties of the control part that is exploited for connectivity maintenance.

Key words. multi-agent systems, connectivity, robust control, tubular neighborhood

AMS subject classifications. 93A14, 93C10, 53A05

1. Introduction.

Multi-agent coordination has evolved in the last decades into a well established field of research with emerging applications ranging from robotics to social sciences [24]. From a control perspective, high interest is focused on the design of control protocols that are based on local network information for the accomplishment of a team goal. Typical objectives are the consensus problem, which aims at the agreement of the agents' states to a common value [16], [29], rendezvous to a common location [23], reference tracking [1] and formation control [17]. For application fields such as mobile robot coordination, it is of paramount importance to ensure network connectivity [41], due to the agents' limited sensing and communication capabilities which necessitate the satisfaction of certain relative distance constraints between communicating agents. The latter objective requires control designs which guarantee that the network topology will remain connected during the evolution of the system.

In [17], solutions to the rendezvous and formation control problems are provided while preserving connectivity by means of unbounded feedback laws. Other approaches to the problem of connectivity maintenance include [9], where controllers that additionally guarantee collision avoidance are designed, bounded potential field based control laws [1], decentralized navigation functions [8], [18], hybrid control policies [40], algorithmic solutions for discrete time second order agents [30] and optimization frameworks for the maximization of the second smallest Laplacian eigenvalue [11] (see also [2], [37], [38], [39]). A detailed literature review on the subject can be also found in the survey paper [41]. Furthermore, in the recent work [28], Lyapunov based barrier functions are constructed for the coordination of a multi-agent team with a leader under guaranteed collision avoidance, where connectivity to the leader is established by enforcing the team to operate inside a circular domain. Robustness of multi-agent coordination has been studied in particular with respect to the consensus problem, also due to the Input-to-State Stability property of consensus algorithms [20]. Results on consensus in the presence of disturbances can be found for instance in [33] for single integrator agents with general time-varying graph topologies, in [22] for systems with heterogeneous uncertainties, in [15] for agents with nonlinear dynamics and in [14], [26] for higher order systems. With respect to connectivity maintenance robustness issues have been addressed in [10], where flocking is studied in the pres-

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ence of disturbances for second order systems, and [31], which provides an algorithmic framework and considers robustness with respect to link failures.

In this paper we consider for each agent a control law comprising of a feedback component, which depends on the relative states of the agent and its neighbors and is responsible for keeping the network connected, and an extra bounded input term, which provides some additional control freedom to the agent. In particular, we design a bounded control law which results in network connectivity of the system for all future times provided that the initial relative distances of interconnected agents and the additional input terms satisfy appropriate bounds. Relevant feedback laws can be found in [6], where finite time consensus is guaranteed in the presence of a common unknown nonlinear drift term for all agents. However, the framework is based on the design of unbounded feedback laws and the dynamics of the drift vector field are the same for all agents, whereas in this paper, constraints on the additional input terms are only imposed on their magnitude. Also, in [10], where flocking is considered in the presence of disturbances, the latter evolve according to the dynamics of a known external system and are estimated through the applied feedback design.

Most existing works in the literature study connectivity in conjunction with additional multi-agent control goals, such as flocking [35], consensus [36], formation [3], [7], rendezvous [12], [34], containment [19] and leader follower control [32]. Our primary motivation for the control design in this paper comes from the exploitation of the extra input terms for high level planning, through the construction of finite symbolic agent models (abstractions) which can provide algorithmic solutions to reachability problems of the multi-agent system. A derivation of such discrete models has been studied in our recent work [4], which provides an appropriate discretization of the agents' workspace into cells and relies on the agents' dynamics bounds and the corresponding bounds on the additional input terms, which are exploited for the navigation of each agent to its successor cells. Thus, the results of this paper provide a suitable framework for the aforementioned approach to high-level planning, since the designed feedback terms are bounded, and additionally, inputs up to a certain bound do not affect the desired connectivity maintenance. Furthermore, we design an extra feedback term which ensures invariance of the system's solution inside a bounded domain and enables the derivation of finite abstractions, which in turn can ensure computational feasibility of discrete planning problems. Hence, the main contribution of this paper is the design of a control framework which can allow the synthesis of high level plans for multi-agent systems under guaranteed network connectivity and trajectory invariance. In particular, a rich variety of collaborative and individual goals can be addressed to the agents by exploiting the expressiveness of formal languages and satisfying plans can be found by leveraging the discrete agent models that can be derived in [4] together with appropriate algorithmic tools. This application has been considered in [27] which deals with multi-agent planning under timed temporal specifications, in the presence of coupling constraints between the agents.

In this work we extend our previous results in [5] where robust connectivity was studied in conjunction with invariance inside a spherical domain, to any convex domain with smooth boundary. We also provide proofs of technical details which were omitted in [5] due to space constraints. The invariance approach is based on the design of a repulsion vector field near the boundary of the domain, whose construction leverages the tubular neighborhood theorem [21]. It is noted that tubular neighborhoods have been also used for the construction of Lyapunov functions for asymptotic submanifold stabilization in the recent work [25]. Finally, in addition to the invariance result, we exploit the convexity assumption on the agents' workspace in order to prove that the robustness properties of the connectivity maintenance control law are unaffected by the superposition of the repulsion vector field. Thus, in terms of the theoretical analysis, the contribution of the paper is summarized in i) the derivation of sufficient conditions which guarantee quantifiable robustness of the connectivity control with respect to additional inputs, in terms of the agents' initial configurations, algebraic properties of the network graph and tunable nonlinearities of the applied feedback laws and ii) the proof of the fact that this robustness margin is unaffected by the superposition of the repulsion vector field through the exploitation of tools from differential geometry and convex analysis.

The rest of the paper is organized as follows. Section 2 introduces basic notation and preliminaries. In Section 3, results on robust connectivity maintenance are provided and explicit controllers which establish this property are designed. In Section 4, the corresponding controllers are appropriately modified, in order to additionally guarantee invariance of the solution for the case of a convex domain. An example with illustrative simulations is provided in Section 5. Finally, we summarize the results and discuss possible extensions in Section 6.

2. Preliminaries and Notation.

2.1. Notation. We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{m\times n}$ we use the notation $|A| := \max\{|Ax| : x \in \mathbb{R}^n\}$ for the induced Euclidean matrix norm and A^T for its transpose. For two vectors $x, y \in \mathbb{R}^n (= \mathbb{R}^{n\times 1})$ we denote their inner product by $\langle x, y\rangle := x^T y$. Given a subset S of $\mathbb{R}^n$, we denote by $\mathrm{cl}(S)$, $\mathrm{int}(S)$ and ∂S its closure, interior and boundary, respectively, where $\partial S := \mathrm{cl}(S) \setminus \mathrm{int}(S)$. For $R > 0$, we denote by $B(R)$ the closed ball with center $0 \in \mathbb{R}^n$ and radius R. Given a vector $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ we define the component operators $c_l(x) := x^l$, $l = 1, \dots, n$. Likewise, for a vector $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nn}$ we define the component operators $c_l(x) := (c_l(x_1), \dots, c_l(x_N)) \in \mathbb{R}^N$, $l = 1, \dots, n$.

Consider a multi-agent system with N agents. For each agent $i \in \{1, \dots, N\} =: \mathcal{N}$ we use the notation $\mathcal{N}_i$ for the set of its neighbors and N_i for its cardinality. We also consider an ordering of the agent's neighbors which we denote by $j_1, \dots, j_{N_i}$. The undirected network's edge set is denoted by $\mathcal{E}$ and $\{i, j\} \in \mathcal{E}$ iff $j \in \mathcal{N}_i$. The network graph $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ is connected if for each $i, j \in \mathcal{N}$ there exists a finite sequence $i_1, \dots, i_l \in \mathcal{N}$ with $i_1 = i$, $i_l = j$ and $\{i_k, i_{k+1}\} \in \mathcal{E}$, for all $k = 1, \dots, l-1$. Consider an arbitrary orientation of the network graph $\mathcal{G}$, which assigns to each edge $\{i, j\} \in \mathcal{E}$ precisely one of the ordered pairs (i, j) or (j, i). When selecting the pair (i, j) we say that i is the tail and j is the head of edge $\{i, j\}$. By considering a numbering $l = 1, \dots, M$ of the graph's edge set we define the $N \times M$ incidence matrix $D(\mathcal{G})$ corresponding to the particular orientation as follows:

$$D(\mathcal{G})_{kl} := \begin{cases} 1, & \text{if vertex } k \text{ is the head of edge } l, \\ -1, & \text{if vertex } k \text{ is the tail of edge } l, \\ 0, & \text{otherwise.} \end{cases}$$

The graph Laplacian $L(\mathcal{G})$ is the $N \times N$ positive semidefinite symmetric matrix $L(\mathcal{G}) := D(\mathcal{G})D(\mathcal{G})^T$. If we denote by $\mathbb{1}$ the vector $(1, \dots, 1) \in \mathbb{R}^N$, then $L(\mathcal{G})\mathbb{1} = D(\mathcal{G})^T\mathbb{1} = 0$. Let $0 = \lambda_1(\mathcal{G}) \le \lambda_2(\mathcal{G}) \le \cdots \le \lambda_N(\mathcal{G})$ be the ordered eigenvalues of $L(\mathcal{G})$, which correspond to a set of mutually orthogonal eigenvectors. In addition, $\lambda_2(\mathcal{G}) > 0$ iff $\mathcal{G}$ is connected.

2.2. Problem Statement. We focus on single integrator multi-agent systems with dynamics

$$\dot{x}_i = u_i, x_i \in \mathbb{R}^n, i \in \mathcal{N}. \tag{1}$$

We aim at designing decentralized control laws of the form

$$u_i := k_i(x_i, x_{j_1}, \ldots, x_{j_{N_i}}) + v_i, \tag{2}$$

which ensure that appropriate apriori bounds on the initial relative distances of interconnected agents guarantee network connectivity for all future times, for all inputs v_i bounded by a certain constant. In particular, *we assume that two agents form an edge as long as the maximum distance between them does not exceed a given positive constant* R. In addition, we make the following connectivity hypothesis for the initial states of the agents.

(ICH) We assume that the agents' communication graph is initially connected and that

$$\max\{|x_i(0) - x_j(0)| : \{i,j\} \in \mathcal{E}\} \leq \tilde{R} \text{ for certain constant } \tilde{R} \in (0, R). \tag{3}$$

2.3. Potential Functions. For the solution of the problem we will assign potential field-type controllers to the feedback terms (2), which depend on the relative positions of the interconnected agents. We proceed by defining certain mappings that will be exploited for the design of these control laws. Let $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous function satisfying the following property.

(P) $r(\cdot)$ is increasing and $r(0) > 0$.

Also, consider the integral

$$P(\rho) = \int_0^\rho r(s) s ds, \rho \in \mathbb{R}_{\geq 0}. \tag{4}$$

For each pair $\{i,j\} \in \mathcal{E}$ we define the potential function $V_{ij} : \mathbb{R}^{Nn} \to \mathbb{R}_{\geq 0}$ as

$$V_{ij}(x) := P(|x_i - x_j|), x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn}. \tag{5}$$

Notice that $V_{ij}(\cdot) = V_{ji}(\cdot)$. Furthermore, $V_{ij}(\cdot)$ is continuously differentiable and satisfies

$$D_{x_i} V_{ij}(x) = r(|x_i - x_j|)(x_i - x_j)^T, \forall x \in \mathbb{R}^{Nn}, \tag{6}$$

where D_{x_i} stands for the derivative with respect to the x_i-coordinates.

REMARK 1. *Notice, that we are only interested in the values of the mappings* $r(\cdot)$ *and* $P(\cdot)$ *in the interval* $[0, R]$*, which stands for the maximum distance that two interconnected agents may achieve before losing connectivity. Yet, defining them on the whole positive line provides us certain technical flexibilities for the analysis employed in the subsequent proofs.*

3. Robust Connectivity Analysis. In this section, we will design the feedback terms in (2) and provide bounds on the maximum initial relative distances of the agents and the input terms v_i, which will guarantee connectivity of the multi-agent network. In particular, based on the potential functions $V_{ij}(\cdot)$ in (5) (corresponding to certain continuous $r(\cdot)$ that satisfies property (P)), we will assign to each agent the control law

$$u_i = -\sum_{j\in\mathcal{N}_i} \nabla_{x_i} V_{ij}(x) + v_i = -\sum_{j\in\mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i, \tag{7}$$

where $\nabla_{x_i} V_{ij}(x)$ is the gradient of $V_{ij}(x)$ at x with respect to the x_i-coordinates, namely, $\nabla_{x_i} V_{ij}(x) = (D_{x_i} V_{ij}(x))^T$. Our approach is inspired by the analysis employed in [17] (see also [24, Section 7.2]) and relies on the selection of a tension energy type function, whose derivative along the solutions of the system becomes negative for all possible appropriately bounded inputs v_i, when the relative distances between interconnected agents exceed a certain threshold. We consider the energy function

$$V(x) := \frac{1}{2}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i} V_{ij}(x), x \in \mathbb{R}^{Nn}, \tag{8}$$

where the mappings $V_{ij}(\cdot)$, $\{i,j\} \in \mathcal{E}$ are given in (5). Then, it follows from (6) that

$$D_{x_i} V(x) = \sum_{j\in\mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j)^T. \tag{9}$$

Also, in accordance with [24, Section 7.2] we have for $l = 1, \ldots, n$ that

$$c_l\left(\sum_{j\in\mathcal{N}_1} r(|x_1 - x_j|)(x_1 - x_j), \ldots, \sum_{j\in\mathcal{N}_N} r(|x_N - x_j|)(x_N - x_j)\right) = L_w(x)c_l(x). \tag{10}$$

The weighted Laplacian matrix $L_w(x)$ in (10) is given as

$$L_w(x) = D(\mathcal{G})W(x)D(\mathcal{G})^T, \tag{11}$$

where $D(\mathcal{G})$ is the incidence matrix of the communication graph (see Notation) and

$$W(x) := \mathrm{diag}\{w_1(x), \ldots, w_M(x)\} := \mathrm{diag}\{r(|x_i - x_j|), \{i,j\} \in \mathcal{E}\} \tag{12}$$

(recall that $M = \mathrm{card}(\mathcal{E})$, where $\mathrm{card}(\cdot)$ is used to denote the cardinality of a set). Before proceeding to the main result of this section, we provide a bound on the derivative of the energy function $V(\cdot)$ along the vector field $u := (u_1, \ldots, u_N)$ (parameterized by the v_i's) with the feedback laws u_i, $i \in \mathcal{N}$ as given by (7). Therefore, we also introduce some additional notation. Let Y be the subspace

$$Y := \{x \in \mathbb{R}^{Nn} : x_1 = x_2 = \cdots = x_N\}.$$

For a vector $x \in \mathbb{R}^{Nn}$ we denote by $\bar{x}$ its projection to the subspace Y, and $x^\perp$ its orthogonal complement with respect to that subspace, namely $x^\perp := x - \bar{x}$. By taking into account that for all $y \in Y$ we have $D(\mathcal{G})^T c_l(y) = 0$ and hence, due to (11), that $c_l(y) \in \ker(L_w(x))$, it follows that for every vector $x \in \mathbb{R}^{Nn}$ with $x = \bar{x} + x^\perp$ it holds

$$L_w(x)c_l(\bar{x}) = 0 \Longrightarrow L_w(x)c_l(x) = L_w(x)c_l(x^\perp). \tag{13}$$

We also denote by $\Delta x \in \mathbb{R}^{Mn}$ the stack column vector of the vectors $x_i - x_j$, $\{i,j\} \in \mathcal{E}$ with the edges ordered as in the case of the incidence matrix. Thus, it follows that for all $x \in \mathbb{R}^{Nn}$ it holds

$$D(\mathcal{G})^T c_l(x) = c_l(\Delta x). \tag{14}$$

We are now in position to state the lemma which provides the desired bounds on the derivative of the energy function.

LEMMA 2. *Consider the energy function $V(\cdot)$ as defined in (8) and the feedback laws u_i, $i \in \mathcal{N}$ in (7). Then, the derivative of $V(\cdot)$ along the vector field $u = (u_1, \dots, u_N)$ satisfies the bound*

$$DV(x)u \leq -[\lambda_2(\mathcal{G})r(0)]^2|x^{\perp}|^2 + \sqrt{N}\sqrt{\lambda_N(\mathcal{G})}|\Delta x|r(|\Delta x|_\infty)|v|_\infty, \tag{15}$$

where $\lambda_2(\mathcal{G})$ and $\lambda_N(\mathcal{G})$, are the second and largest eigenvalues of the network's graph Laplacian, respectively, $v = (v_1, \dots, v_N)$ with each v_i, $i \in \mathcal{N}$ as given in (7), $x^{\perp}, \Delta x$ are defined above, and $|\Delta x|_\infty$, $|v|_\infty$ are given as

$$|v|_\infty := \max\{|v_i|, i \in \mathcal{N}\}, \tag{16}$$

$$|\Delta x|_\infty := \max\{|\Delta x_i|, i = 1, \dots, M\}. \tag{17}$$

Proof. By evaluating the derivative of $V(\cdot)$ along the vector field given by u and taking into account (8), (9) and (10) we get

$$\begin{aligned}
DV(x)u &= \sum_{l=1}^{n} c_l\left(DV(x)\right)c_l(u) \\
&= -\sum_{l=1}^{n} c_l(x)^T L_w(x)(L_w(x)c_l(x) - c_l(v)) \\
&\leq -\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right|. && (18)
\end{aligned}$$

First, we provide certain useful inequalities between the eigenvalues of the weighted Laplacian $L_w(x)$ and the Laplacian matrix of the graph $L(\mathcal{G})$. Notice, that due to (12), for each $i = 1, \dots, M$ we have $w_i(x) = r(|x_k - x_\ell|)$ for certain $\{k, \ell\} \in \mathcal{E}$ and hence, by virtue of Property (P), it holds

$$0 < r(0) \leq w_i(x) \leq \max_{\{k,\ell\}\in\mathcal{E}} r(|x_k - x_\ell|). \tag{19}$$

In addition, since $L_w(x)$ is also a symmetric positive semidefinite matrix satisfying $L_w(x)\mathbb{1} = 0$, it follows from (19) that

$$\lambda_2(x) \geq \lambda_2(\mathcal{G})r(0), \tag{20}$$

where $0 = \lambda_1(x) < \lambda_2(x) \leq \dots \leq \lambda_N(x)$ and $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \dots \leq \lambda_N(\mathcal{G})$ are the eigenvalues of $L_w(x)$ and the Laplacian matrix of the graph $L(\mathcal{G})$, respectively. Indeed, in order to show (20), notice that

$$\begin{aligned}
L_w(x) &= D(\mathcal{G})\mathrm{diag}\{w_1(x), \dots, w_M(x)\}D(\mathcal{G})^T \\
&= D(\mathcal{G})\mathrm{diag}\{r(0), \dots, r(0)\}D(\mathcal{G})^T \\
&\quad + D(\mathcal{G})\mathrm{diag}\{w_1(x) - r(0), \dots, w_M(x) - r(0)\}D(\mathcal{G})^T = r(0)L(\mathcal{G}) + B,
\end{aligned}$$

where (19) implies that $B := D(\mathcal{G})\mathrm{diag}\{w_1(x) - r(0), \dots, w_M(x) - r(0)\}D(\mathcal{G})^T$ is positive semidefinite. Hence, it holds $L_w(x) \succeq r(0)L(\mathcal{G})$, with $\succeq$ being the partial order on the set of symmetric $N \times N$ matrices and thus, we deduce from Corollary

7.7.4(c) in [13, page 495] that (20) is fulfilled. Furthermore, due to (12) and (19), we get that

$$|W(x)| \leq r(|\Delta x|_\infty). \tag{21}$$

For the sequel, we will also use the following facts, whose proofs can be found in the Appendix. In particular, for the vectors $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{R}^{Nn}$ the following properties hold.

Fact I.

$$|L_w(x)c_l(x^\perp)| \geq \lambda_2(x)|c_l(x^\perp)|, \forall l = 1, \dots, n. \tag{22}$$

Fact II.

$$\sum_{l=1}^{n} |c_l(x)||c_l(y)| \leq |x||y|. \tag{23}$$

We are now in position to bound the two terms involved in the derivative of $V(\cdot)$.

Bound for the first term in (18). By taking into account (13), it follows that

$$\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) = \sum_{l=1}^{n} \left|L_w(x)c_l(x^\perp)\right|^2 \tag{24}$$

and by exploiting Fact I and (20), we get

$$\begin{aligned}
\sum_{l=1}^{n} \left|L_w(x)c_l(x^\perp)\right|^2 &\geq \sum_{l=1}^{n} \lambda_2(x)^2|c_l(x^\perp)|^2 \\
&\geq \sum_{l=1}^{n}[\lambda_2(\mathcal{G})r(0)]^2|c_l(x^\perp)|^2 = [\lambda_2(\mathcal{G})r(0)]^2|x^\perp|^2.
\end{aligned} \tag{25}$$

Thus, it follows from (24) and (25) that

$$\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) \geq [\lambda_2(\mathcal{G})r(0)]^2|x^\perp|^2. \tag{26}$$

Bound for the second term in (18). For this term, we have from (11) and (14) that

$$\begin{aligned}
\left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right| &\leq \sum_{l=1}^{n} |c_l(x)^T D(\mathcal{G}) W(x) D(\mathcal{G})^T c_l(v)| \\
&= \sum_{l=1}^{n} |c_l(\Delta x)^T W(x) D(\mathcal{G})^T c_l(v)| \\
&\leq \sum_{l=1}^{n} |c_l(\Delta x)||W(x)||D(\mathcal{G})^T||c_l(v)|.
\end{aligned} \tag{27}$$

By taking into account (21), and the fact that $|D(\mathcal{G})^T| = \sqrt{\lambda_{\max}(D(\mathcal{G})D(\mathcal{G})^T)} = \sqrt{\lambda_N(\mathcal{G})}$ we obtain

$$\sum_{l=1}^{n} |c_l(\Delta x)||W(x)||D(\mathcal{G})^T||c_l(v)| \leq \sum_{l=1}^{n} |c_l(\Delta x)| r(|\Delta x|_\infty)\sqrt{\lambda_N(\mathcal{G})}|c_l(v)|. \tag{28}$$

Also, by exploiting Fact II, we get that

$$\sum_{l=1}^{n}|c_l(\Delta x)|r(|\Delta x|_\infty)\sqrt{\lambda_N(\mathcal{G})}|c_l(v)| \leq r(|\Delta x|_\infty)\sqrt{\lambda_N(\mathcal{G})}|\Delta x||v|$$

$$\leq r(|\Delta x|_\infty)\sqrt{\lambda_N(\mathcal{G})}|\Delta x|\sqrt{N}|v|_\infty, \tag{29}$$

with $|v|_\infty$ as given in the statement of the lemma. Hence, it follows from (27)-(29) that

$$\left|\sum_{l=1}^{n}c_l(x)^T L_w(x)c_l(v)\right| \leq \sqrt{N}\sqrt{\lambda_N(\mathcal{G})}|\Delta x|r(|\Delta x|_\infty)|v|_\infty. \tag{30}$$

Thus, we get from (18), (26) and (30) that (15) is fulfilled and the proof is complete.□

Having established this auxiliary result, we provide in the following proposition a control law (2) and an upper bound on the magnitude of the input terms $v_i(\cdot)$ which guarantee connectivity of the multi-agent network.

PROPOSITION 3. *For the multi-agent system* (1)*, assume that (ICH) is fulfilled and pick the control law* (7) *for certain continuous* $r(\cdot)$ *satisfying Property (P). Define*

$$K := \frac{2\sqrt{N(N-1)}\sqrt{\lambda_N(\mathcal{G})}}{\lambda_2(\mathcal{G})^2}, \tag{31}$$

and consider a constant $\delta > 0$*. Assume that* δ*,* $\tilde{R}$ *and* $r(\cdot)$ *satisfy the restrictions*

$$\delta \leq \frac{1}{K}r(0)^2\frac{s}{r(s)}, s \geq \tilde{R}, \tag{32}$$

with K *as given in* (31) *and*

$$MP(\tilde{R}) \leq P(R), \tag{33}$$

where $P(\cdot)$ *is given in* (4)*, and* $M = \mathrm{card}(\mathcal{E})$*. Then, the system remains connected for all positive times, provided that the input terms* $v_i(\cdot)$*,* $i \in \mathcal{N}$ *satisfy*

$$|v_i(t)| \leq \delta, \forall t \geq 0. \tag{34}$$

Proof. For the proof we exploit the result of Lemma 2, which provides bounds for the derivative of the energy function $V(\cdot)$ in (8) along the vector field $u = (u_1, \ldots, u_N)$ as specified by the feedback laws u_i, $i \in \mathcal{N}$ in (7). In particular, we want to provide bounds for the right hand side of (15) which guarantee that the sign of $DV(x)u$ is negative whenever the maximum distance between two agents exceeds the bound $\tilde{R}$ on the maximum initial distance as given in (3), and for appropriate bounds on the v_i terms. Therefore, we will also use the following facts, which are proved in the Appendix. In particular, for each $x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn}$ the following hold.

Fact III.

$$|x^\perp| \geq \frac{1}{\sqrt{2(N-1)}}|\Delta x|. \tag{35}$$

Fact IV.

$$|x^\perp| \geq \frac{1}{\sqrt{2}}|\Delta x|_\infty. \tag{36}$$

By exploiting Facts III and IV, we get from (15) that

$$
\begin{aligned}
DV(x)u &\leq -[\lambda_2(\mathcal{G})r(0)]^2 \frac{1}{\sqrt{2(N-1)}}|\Delta x|\frac{1}{\sqrt{2}}|\Delta x|_\infty \\
&\quad + \sqrt{N}\sqrt{\lambda_N(\mathcal{G})}|\Delta x| r(|\Delta x|_\infty)|v|_\infty \\
&= |\Delta x|\left(-\frac{1}{2\sqrt{N-1}}[\lambda_2(\mathcal{G})r(0)]^2|\Delta x|_\infty + \sqrt{N}\sqrt{\lambda_N(\mathcal{G})}r(|\Delta x|_\infty)|v|_\infty\right).
\end{aligned}
$$

By using the notation $|\Delta x|_\infty := s$, in order to guarantee that the above right hand side is non-positive for $s \geq \tilde{R}$, it is required that

$$
\begin{aligned}
&-\frac{\lambda_2(\mathcal{G})^2}{2\sqrt{(N-1)}}r(0)^2 s + \sqrt{N}\sqrt{\lambda_N(\mathcal{G})}r(s)|v|_\infty \leq 0, \forall s \geq \tilde{R} \iff \\
&\frac{2\sqrt{N(N-1)}\sqrt{\lambda_N(\mathcal{G})}}{\lambda_2(\mathcal{G})^2}|v|_\infty \leq r(0)^2\frac{s}{r(s)}, \forall s \geq \tilde{R},
\end{aligned}
$$

or equivalently

$$
|v|_\infty \leq \frac{1}{K}r(0)^2\frac{s}{r(s)}, \forall s \geq \tilde{R}, \tag{37}
$$

with K as given in (31). Hence, we have shown that for v satisfying (37) the following implication holds

$$
|\Delta x|_\infty \geq \tilde{R} \Longrightarrow DV(x)u \leq 0. \tag{38}
$$

By assuming that conditions (34), (32) and (33) in the statement of the proposition are fulfilled and recalling that according to (ICH) (3) holds, we can show that the system will remain connected for all future times. Indeed, let $x(\cdot)$ be the solution of the closed loop system (1)-(7) with initial condition satisfying (3), defined on the maximal right interval $[0, T_{\max})$. We claim that the system remains connected on $[0, T_{\max})$, namely, that $\max\{|x_i(t)-x_j(t)| : \{i,j\} \in \mathcal{E}\} \leq R$ for all $t \in [0, T_{\max})$, which by boundedness of the dynamics on the set $\mathcal{F} := \{x \in \mathbb{R}^{Nn} : |x_i - x_j| \leq R, \forall \{i,j\} \in \mathcal{E}\}$ implies that $T_{\max} = \infty$. In order to prove the last assertion, assume on the contrary that $T_{\max} < \infty$. Then, by taking into account that $x(t)$ remains in $\mathcal{F}$ for all $t \in [0, T_{\max})$ and that the dynamics are bounded in $\mathcal{F}$, it follows that $x(t)$ remains in a compact subset of $\mathbb{R}^{Nn}$ for all $t \in [0, T_{\max})$ and hence, that it can be extended, contradicting maximality of $[0, T_{\max})$. We proceed with the proof of connectivity. First, notice that due to (3) and (33), it holds

$$
\begin{aligned}
V(x(0)) &= \frac{1}{2}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}P(|x_i(0)-x_j(0)|) \\
&\leq \frac{1}{2}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}P(\tilde{R}) = \frac{M}{2}P(\tilde{R}) \leq \frac{1}{2}P(R).
\end{aligned}\tag{39}
$$

In order to prove our claim, it suffices to show that

$$
V(x(t)) \leq \frac{1}{2}P(R), \forall t \in [0, T_{\max}), \tag{40}
$$

because if $|x_i(t) - x_j(t)| > R$ for certain $t \in [0, T_{\max})$ and $\{i, j\} \in \mathcal{E}$, then $V(x(t)) \geq \frac{1}{2}P(|x_i(t) - x_j(t)|) > \frac{1}{2}P(R)$. We prove (40) by contradiction. Indeed, suppose on the contrary that there exists $T \in (0, T_{\max})$ such that

$$V(x(T)) > \frac{1}{2}P(R) \tag{41}$$

and define

$$\tau := \min\{t \in [0, T] : V(x(\bar{t})) > \tfrac{1}{2}P(R), \forall \bar{t} \in (t, T]\}, \tag{42}$$

which due to (41) and continuity of $V(x(\cdot))$ is well defined. Then it follows from (39) and (42) that

$$V(x(\tau)) = \frac{1}{2}P(R), V(x(t)) > \frac{1}{2}P(R), \forall t \in (\tau, T], \tag{43}$$

hence, there exists $\bar{\tau} \in (\tau, T)$ such that

$$\dot{V}(x(\bar{\tau})) = \frac{V(x(T)) - V(x(\tau))}{T - \tau} > 0. \tag{44}$$

On the other hand, due to (43), it holds

$$V(x(\bar{\tau})) > \frac{1}{2}P(R), \tag{45}$$

which implies that there exists $\{i, j\} \in \mathcal{E}$ with

$$|x_i(\bar{\tau}) - x_j(\bar{\tau})| > \tilde{R}. \tag{46}$$

Indeed, if (46) does not hold, then we can show as in (39) that $V(x(\bar{\tau})) \leq \frac{1}{2}P(R)$ which contradicts (45). Notice that by virtue of (34) and (32), (37) is fulfilled. Hence, we get from (46) that $|\Delta x(\bar{\tau})|_\infty > \tilde{R}$ and thus from (38) it follows that $\dot{V}(x(\bar{\tau})) = DV(x(\bar{\tau}))u(\bar{\tau}) \leq 0$, which contradicts (44). We conclude that (40) holds and the proof is complete. ∎

In the following corollary, we apply the result of Proposition 3 in order to provide two explicit feedback laws of the form (7), a linear and a nonlinear one and compare their performance in the subsequent remark.

Corollary 4. *For the multi-agent system* (1)*, assume that (ICH) is fulfilled and consider the control law* (2) *as given by* (7)*. By imposing the additional requirement* $r(0) = r(\tilde{R}) = 1$ *and defining*

$$\delta := \frac{\tilde{R}}{K} \tag{47}$$

with $\tilde{R}$ *and* K *as given in* (3) *and* (31)*, respectively, the system remains connected for all positive times, provided that the function* $r(\cdot)$ *and the constant* $\tilde{R}$ *are selected as in the following two cases (L) and (NL) (providing a linear and a nonlinear feedback, respectively).*

Case (L). *We select*

$$r(s) := 1, s \geq 0 \tag{48}$$

and

$$\tilde{R} \leq \frac{1}{\sqrt{M}} R. \tag{49}$$

(recall that $M = \mathrm{card}(\mathcal{E})$*).*

Case (NL). *We select*

$$r(s) := \begin{cases} 1, & s \in [0, \tilde{R}] \\ \frac{s}{\tilde{R}}, & s \in (\tilde{R}, R] \\ \frac{R}{\tilde{R}}, & s \in (R, \infty) \end{cases} \tag{50}$$

and

$$\tilde{R} \leq \left(\frac{2}{3M - 1} \right)^{\frac{1}{3}} R. \tag{51}$$

Proof. For the proof we apply the result of Proposition (3). In particular, it suffices to show that for both cases (L) and (NL) the selection of the function $r(\cdot)$ and the initial maximum distance $\tilde{R}$ satisfy (32) and (33), with δ as given by (47).

Case (L). Indeed, it follows from (47) and (48) that (32) is fulfilled. Furthermore, it follows from (48) and (4) that (49) is equivalent to (33).

Case (NL). Also in this case, it follows from (47) and (50) that (32) is again fulfilled. In addition, it follows from (50) and (4) that is (51) is equivalent to (33). The proof is now complete. ∎

REMARK 5. *At this point we derive the advantage of using the nonlinear controller over the linear one by comparing the ratio of the maximal initial relative distance that maintains connectivity for these two cases. In both cases we have the same bound on the input terms* v_i *and the same feedback law up to some distance between neighboring agents, which allows us to compare their performance under the criterion of maximizing the largest initial distance between two interconnected agents. In particular, this ratio depends on the number of edges in the system's graph and is given as* $\frac{1}{\sqrt{M}} / \left(\frac{2}{3M-1} \right)^{\frac{1}{3}}$. *By differentiating the latter expression, it follows that it is a strictly decreasing function of* M *with values less than 1 for* $M > 1$*, as also depicted in Figure 1.*

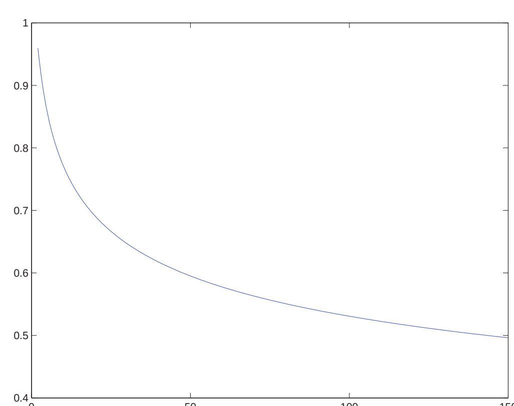

FIG. 1. *This figure shows the ratio* $\frac{1}{\sqrt{M}}$ / $\left(\frac{2}{3M-1} \right)^{\frac{1}{3}}$ *for the number of edges ranging from 2 to 150.*

4. Invariance Analysis. In what follows, we assume that the agents' initial states belong to a given **bounded** domain $\Omega \subset \mathbb{R}^n$. We aim at designing an appropriate modification of the feedback law (7) which additionally guarantees that the trajectories of the agents remain in Ω for all future times. We assume that Ω is convex and that its boundary $\partial\Omega$ is a smooth $n-1$-dimensional (embedded) submanifold of $\mathbb{R}^n$. We denote by η the smooth mapping that assigns to each $x \in \partial\Omega$ the unit outward pointing normal vector $\eta(x)$ (see Figure 2, top left). By additionally exploiting that $\partial\Omega$ is compact, i.e., a closed and bounded subset of $\mathbb{R}^n$, it follows from the tubular neighborhood theorem (see [21, Theorem 10.19]) that there exists an $\bar{\varepsilon} > 0$ such that

$$N_{\bar{\varepsilon}} := \{x - t\eta(x) : x \in \partial\Omega, |t| < \bar{\varepsilon}\} \tag{52}$$

is a tubular neighborhood of $\partial\Omega$ (see e.g, [21, page 255] for the definition of a tubular neighborhood). In addition, the following properties are fulfilled (see [21, Proposition 10.20 & Problem 10-2]):

(P1) For each $y \in N_{\bar{\varepsilon}}$ there exist unique $x \in \partial\Omega$ and $t \in (-\bar{\varepsilon}, \bar{\varepsilon})$ such that $y = x - t\eta(x)$, defining a **smooth** mapping $H : N_{\bar{\varepsilon}} \to \partial\Omega$ with $H(y) = x$, implying that $|t| = |H(y) - y|$.

(P2) For each $y \in N_{\bar{\varepsilon}}$, $H(y)$ is the closest point to the boundary of Ω, namely, $|H(y) - y| = d(y, \partial\Omega) := \inf\{|y - z| : z \in \partial\Omega\}$. Conversely, for each $y \in \mathbb{R}^n$ with $d(y, \partial\Omega) < \bar{\varepsilon}$, it holds $y \in N_{\bar{\varepsilon}}$.

From (P1), it follows that

$$H(y) - y = |H(y) - y|\eta(H(y)), \forall y \in N_{\bar{\varepsilon}} \cap \Omega. \tag{53}$$

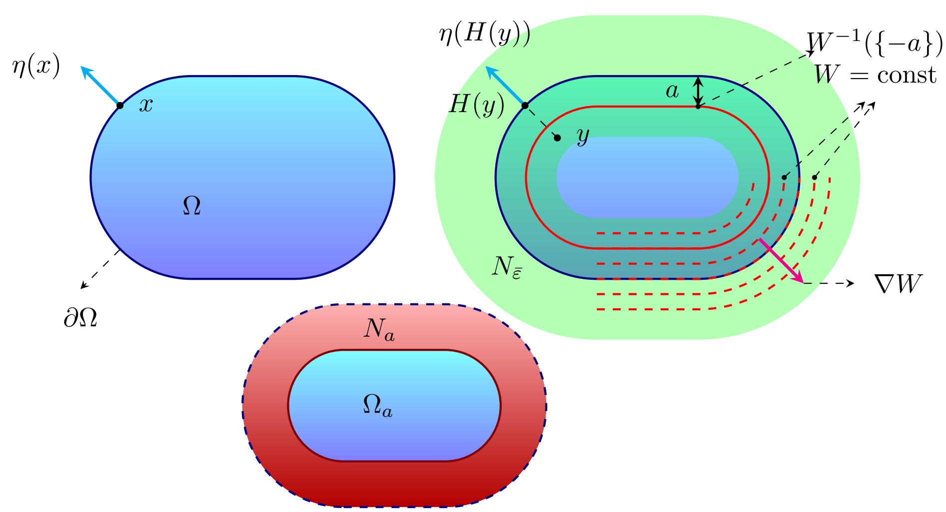

FIG. 2. *Illustration of the domain Ω, the tubular neighborhood $N_{\bar{\varepsilon}}$ and the partition of Ω into N_a and Ω_a.*

Next, for each $a \in (0, \bar{\varepsilon})$ we define

$$N_a := \{x - t\eta(x) : x \in \partial\Omega, t \in (0, a]\}, \tag{54}$$

which by virtue of (P2) is the region with distance up to a from $\partial\Omega$ towards the interior of Ω. Thus, it follows that

$$N_a = \{x \in \Omega : d(x, \partial\Omega) \leq a\}. \tag{55}$$

Also, let

$$\Omega_a := \Omega \setminus N_a \tag{56}$$

(see also Figure 2, bottom, for an illustration of N_a and Ω_a). From (P1), (P2) and (53)-(56), we obtain the following property.

(P3) Given $a \in (0, \bar{\varepsilon})$, for any $y \in N_{\bar{\varepsilon}} \cap \Omega$, which according to (P1) can be written as $y = x - t\eta(x)$ for unique $x \in \partial\Omega$ and $t \in (0, \bar{\varepsilon})$, it holds: (i) $t \le a \iff y \in N_a$; (ii) $t = a \iff y \in \partial\Omega_a$; $t > a \iff y \in \Omega_a$.

We establish certain useful properties of the sets Ω_a in Lemma 6 below.

LEMMA 6. ***(A)*** *For any $a \in (0, \bar{\varepsilon})$ the set Ω_a is convex.*
(B) *For each $x \in \partial\Omega_a$, it holds*

$$\langle \eta(H(x)), x \rangle \ge \langle \eta(H(x)), y \rangle, \forall y \in \Omega_a,$$

with $H(\cdot)$ as defined in (P1), namely, $\{y \in \mathbb{R}^n : \langle \eta(H(x)), x \rangle = \langle \eta(H(x)), y \rangle\}$ is a supporting hyperplane of Ω_a at x.

Proof. **(A)** Indeed, let $x_1, x_2 \in \Omega_a$. We will show that also $\lambda x_1 + (1-\lambda)x_2 \in \Omega_a$, for each $\lambda \in (0,1)$. Notice first, that by virtue of (55) and (56), for both x_1 and x_2 it holds

$$d(x_i, \partial\Omega) > a, i = 1, 2. \tag{57}$$

We prove the assertion by assuming on the contrary that there exists $\bar{x} \in \{\lambda x_1 + (1-\lambda)x_2 : \lambda \in (0,1)\}$ such that $\bar{x} \notin \Omega_a$. From (56) and convexity of Ω, it follows that $\bar{x} \in N_a$, thus, we get from (P1) and (54) that $|H(\bar{x}) - \bar{x}| \le a$. Hence, we may pick

$$\tilde{x} \in \arg\min\{|H(x) - x| : x = \lambda x_1 + (1-\lambda)x_2, \lambda \in (0,1), x \in N_a\}. \tag{58}$$

The latter selection implies that

$$|H(\tilde{x}) - \tilde{x}| \le a. \tag{59}$$

Also, due to (53) and (58), which implies that $x_1 - \tilde{x} = (1-\lambda)(x_1 - x_2)$, for certain $\lambda \in (0,1)$, we get that

$$\langle H(\tilde{x}) - \tilde{x}, x_1 - x_2 \rangle = 0 \Longrightarrow \langle \eta(H(\tilde{x})), x_1 - \tilde{x} \rangle = 0, \tag{60}$$

where the left hand side of the implication is justified by the fact that the function $t \to |H(\tilde{x}) - \tilde{x} + t(x_1 - x_2)|$ has a minimum in a neighborhood of zero (otherwise there would be points on the line segment joining x_1 and x_2 with distance less than $|H(\tilde{x}) - \tilde{x}|$). In addition, by convexity of Ω and smoothness of $\partial\Omega$, the fact that $H(\tilde{x}) \in \partial\Omega$ implies that $\{y \in \mathbb{R}^n : \langle \eta(H(\tilde{x})), H(\tilde{x}) \rangle = \langle \eta(H(\tilde{x})), y \rangle\}$ is a supporting hyperplane of Ω at $H(\tilde{x})$, namely, it holds

$$\langle \eta(H(\tilde{x})), H(\tilde{x}) \rangle \ge \langle \eta(H(\tilde{x})), y \rangle, \forall y \in \mathrm{cl}(\Omega). \tag{61}$$

Next, pick $y = x_1 + \lambda(H(\tilde{x}) - \tilde{x})$, where

$$\lambda = \sup\{\bar{\lambda} > 0 : x_1 + \bar{\lambda}(H(\tilde{x}) - \tilde{x}) \in \Omega\}. \tag{62}$$

which is well defined, since Ω is bounded. Then, it follows from (57), (59) and (62) that

$$\lambda |H(\tilde{x}) - \tilde{x}| \ge d(x_1, \partial\Omega) > a \Longrightarrow \lambda > 1. \tag{63}$$

Thus, we obtain from (53), (60) and (63) that

$$\begin{aligned}\langle \eta(H(\tilde{x})), x_1 + \lambda(H(\tilde{x}) - \tilde{x})\rangle \\ &= \langle \eta(H(\tilde{x})), \tilde{x} + (H(\tilde{x}) - \tilde{x}) + x_1 - \tilde{x} + (\lambda - 1)(H(\tilde{x}) - \tilde{x})\rangle \\ &= \langle \eta(H(\tilde{x})), H(\tilde{x})\rangle + 0 + \langle \eta(H(\tilde{x})), (\lambda - 1)(H(\tilde{x}) - \tilde{x})\rangle \\ &> \langle \eta(H(\tilde{x})), H(\tilde{x})\rangle,\end{aligned}$$

which contradicts (61), since from (62) we have that $x_1 + \lambda(H(\tilde{x}) - \tilde{x}) \in \mathrm{cl}(\Omega)$.
(B) For the proof of (B), we will exploit the convexity result of Part (A), in conjunction with the fact that

$$\langle \eta(H(x)), x - y\rangle \geq -C|x - y|^2, \forall y \in \mathrm{cl}(\Omega_a), x \in \partial\Omega_a, \tag{64}$$

for certain $C > 0$. In order to show (64), notice that by virtue of (P2), (55) and (56), for any $y \in \mathrm{cl}(\Omega_a)$ and $x \in \partial\Omega_a$ it holds

$$\begin{aligned}|H(x) - x| \leq |H(x) - y| \Longrightarrow \\ |H(x) - x|^2 \leq |H(x) - x + x - y|^2 \Longrightarrow \\ |H(x) - x|^2 \leq |H(x) - x|^2 + 2\langle H(x) - x, x - y\rangle + |x - y|^2 \Longrightarrow \\ -|x - y|^2 \leq 2\langle H(x) - x, x - y\rangle.\end{aligned}$$

From the latter and (53), it follows that (64) holds with $C = \frac{1}{2|H(x)-x|}$. In order to complete the proof assume on the contrary that there exist $\tilde{y} \in \mathrm{cl}(\Omega_a)$, $\tilde{x} \in \partial\Omega_a$ and a constant $\tilde{C} > 0$, such that

$$\langle \eta(H(\tilde{x})), \tilde{x} - \tilde{y}\rangle = -\tilde{C}(< 0). \tag{65}$$

Then, it follows from convexity of $\mathrm{cl}(\Omega_a)$ that $\tilde{x} - \lambda(\tilde{x} - \tilde{y}) \in \mathrm{cl}(\Omega_a)$ for any $\lambda \in (0, 1)$ and by virtue of (64) we get that

$$\langle \eta(H(\tilde{x})), \tilde{x} - (\tilde{x} - \lambda(\tilde{x} - \tilde{y}))\rangle \geq -C|\lambda(\tilde{x} - \tilde{y})|^2. \tag{66}$$

Also, from (65), we obtain that

$$\langle \eta(H(\tilde{x})), \lambda(\tilde{x} - \tilde{y})\rangle = -\lambda\tilde{C}. \tag{67}$$

Equality of the left hand sides of (66) and (67) implies that for each $\lambda \in (0, 1)$ it holds

$$-\lambda\tilde{C} \geq -C\lambda^2|\tilde{x} - \tilde{y}|^2 \Longrightarrow \tilde{C} \leq C\lambda|\tilde{x} - \tilde{y}|^2,$$

which is violated for $\lambda < \frac{C}{\tilde{C}}|\tilde{x} - \tilde{y}|^2$. The proof is now complete. ∎

We proceed by defining a repulsion from the boundary of Ω vector field, which when added to the dynamics of each agent in (7), will ensure the desired invariance of the closed loop system and simultaneously guarantee the same robust connectivity result established above. First, define the function $W : N_{\bar{\varepsilon}} \to \mathbb{R}$ by

$$W(x) := -\langle H(x) - x, \eta(H(x))\rangle.$$

The following lemma includes certain properties of $W(\cdot)$ that will be exploited in the subsequent analysis (see also Figure 2, top right).

LEMMA 7. ***(A)*** *It holds*

$$\nabla W(x) = \eta(H(x)), \forall x \in N_{\bar{\varepsilon}}.$$

(B) *For each $a \in (0, \bar{\varepsilon})$, it holds $\partial\Omega_a = W^{-1}(\{-a\})$, namely, $W(\cdot)$ is a globally defining function for $\partial\Omega_a$ (see [21, page 184]).*
(C) *Given any $a \in (0, \bar{\varepsilon})$ and $x \in N_{\bar{\varepsilon}}$, let $\tilde{x} := x - (a - |H(x) - x|)\nabla W(x)$. Then,*

$$\tilde{x} \in \partial\Omega_a \tag{68}$$

and it holds

$$\nabla W(\tilde{x}) = \nabla W(x). \tag{69}$$

(D) *For any $a \in (0, \bar{\varepsilon})$, $x \in N_a$ and $y \in \mathrm{cl}(\Omega_a)$ it holds*

$$\langle x - y, \nabla W(x)\rangle \geq 0.$$

Proof. **(A)** In order to prove this part, we need to show that for each $x \in N_{\bar{\varepsilon}}$ it holds

$$W(x + \delta x) - W(x) - \langle \eta(H(x)), \delta x\rangle = o(\delta x). \tag{70}$$

Notice first, that

$$H(x + \delta x) = H(x) + DH(x)\delta x + o(\delta x), \tag{71}$$

where $DH(\cdot)$ in (71) stands for the derivative of $H(\cdot)$. Also, since $H(y) \in \partial\Omega$ for each $y \in N_\varepsilon$, it holds $DH(y)z \in T_{H(y)}\partial\Omega$ for all $z \in \mathbb{R}^n$, with $T_{H(y)}\partial\Omega$ denoting the tangent space of $\partial\Omega$ at $H(y)$. The latter implies that

$$\langle DH(y)z, \eta(H(y))\rangle = 0. \tag{72}$$

Similarly, we obtain that

$$\eta(H(x + \delta x)) = \eta(H(x)) + D(\eta \circ H)(x)\delta x + o(\delta x). \tag{73}$$

where $\eta \circ H$ stands for the composition of η and H. In addition, for all $y \in \partial\Omega$ it holds $|\eta(y)|^2 = 1$, which by direct differentiation implies that $\eta(y)^T D\eta(y) = 0$. Thus, it follows that $D\eta(y)z \in T_y\partial\Omega$ for all $z \in \mathbb{R}^n$, or equivalently

$$\langle D\eta(y)z, \eta(y)\rangle = 0. \tag{74}$$

Next, by picking $x \in N_{\bar{\varepsilon}}$ and exploiting (71) and (73), we evaluate

$$\begin{aligned}
-W(x + \delta x) + W(x) + \langle \eta(H(x)), \delta x\rangle &= \langle H(x + \delta x) - (x + \delta x), \eta(H(x + \delta x))\rangle \\
&- \langle H(x) - x, \eta(H(x))\rangle + \langle \eta(H(x)), \delta x\rangle \\
&= \langle H(x) + DH(x)\delta x + o(\delta x) - (x + \delta x), \eta(H(x)) + D(\eta \circ H)(x)\delta x + o(\delta x)\rangle \\
&- \langle H(x) - x, \eta(H(x))\rangle + \langle \eta(H(x)), \delta x\rangle \\
&= \langle H(x) - x, D(\eta \circ H)(x)\delta x\rangle - \langle \delta x, \eta(H(x))\rangle \\
&+ \langle DH(x)\delta x, \eta(H(x))\rangle + \langle \eta(H(x)), \delta x\rangle + o(\delta x) \\
&= \langle H(x) - x, D\eta(H(x))DH(x)\delta x\rangle + \langle DH(x)\delta x, \eta(H(x))\rangle + o(\delta x). \qquad (75)
\end{aligned}$$

From (74) and (53), we deduce that the first term in (75) is zero. Likewise, it follows from (72) that the second term in (75) is zero as well and we conclude that (70) is satisfied.

(B) In order to prove part (B), we need to show that $W(x) = -a \Longrightarrow x \in \partial\Omega_a$. By taking into account (P3)(ii), it suffices to show that $x = H(x) - a\eta(H(x))$. Note first, that since $W(x) = -a$, namely, $-\langle H(x) - x, \eta(H(x))\rangle = -a$, it follows from (53) and the fact that $|\eta(H(x))| = 1$, that

$$-\langle |H(x) - x|\eta(H(x)), \eta(H(x))\rangle = -a \Longrightarrow |H(x) - x| = a. \tag{76}$$

In addition, from (53) we get that

$$x = H(x) - |H(x) - x|\eta(H(x)), \tag{77}$$

which by virtue of (76) implies that $x = H(x) - a\eta(H(x))$ as desired.

(C) In order to prove (68) in part (C) of the lemma, it suffices by virtue of (P3)(ii) to show that $\tilde{x} = H(x) - a\eta(H(x)) \in \partial\Omega_a$. Hence, we get from (77) and part (A) of the lemma that

$$\begin{aligned} \tilde{x} &= H(x) - |H(x) - x|\eta(H(x)) - (a - |H(x) - x|)\eta(H(x)) \\ &= H(x) - a\eta(H(x)), \end{aligned} \tag{78}$$

which provides validity of (68). In addition, from (78) and (P1) we get that $H(\tilde{x}) = H(x)$. Thus, it follows from part (A) of the lemma that (69) is satisfied as well.

(D) For the proof of part (D), notice first that $x \in \partial\Omega_{\bar{a}}$ and $y \in \mathrm{cl}(\Omega_{\bar{a}})$ for certain $\bar{a} \in (0, a]$. Thus, by applying Lemma 6(B) with $a := \bar{a}$, we get that $\langle x - y, \eta(H(x))\rangle \geq 0$. From the latter and Lemma 7(A), namely, the fact that $\nabla W(x) = \eta(H(x))$, we obtain the desired result. ∎

Next, pick $\varepsilon \in (0, \bar{\varepsilon})$, select a Lipschitz continuous function $h : [0, 1] \to [0, 1]$ that satisfies

$$h(0) = 0;\ h(1) = 1;\ h(\cdot) \text{ strictly increasing} \tag{79}$$

and consider the vector field $g : \Omega \to \mathbb{R}^n$ defined as

$$g(x) := \begin{cases} -c\delta h\left(\frac{\varepsilon - |H(x) - x|}{\varepsilon}\right)\nabla W(x), & \text{if } x \in N_\varepsilon, \\ 0, & \text{if } x \in \Omega_\varepsilon, \end{cases} \tag{80}$$

with $h(\cdot)$ as given above and appropriate positive constants c, δ which serve as design parameters. Then, it follows from (79), (80), the Lipschitz property for $h(\cdot)$ and smoothness of $H(\cdot)$, $W(\cdot)$, that the vector field $g(\cdot)$ is Lipschitz continuous on Ω.

Having defined the mappings for the extra term in the dynamics of the candidate controller for each agent, we now state our first main result which guarantees the desired forward invariance property for the trajectories of the closed loop system.

THEOREM 8. *Consider the multi-agent system (1) and assume that for the initial states of the agents it holds $x(0) \in \Omega^N$, where Ω is a convex and bounded domain of $\mathbb{R}^n$. Also, let $\varepsilon \in (0, \bar{\varepsilon})$, with $\bar{\varepsilon}$ as given in (52), N_ε, Ω_ε as defined by (54), (56), respectively and select the control law*

$$u_i = g(x_i) - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i, \tag{81}$$

with $r(s) \geq 0$ for all $s \geq 0$ and $g(\cdot)$ given in (80) for certain $c > 1$ and $\delta > 0$. Then, assuming that the input terms $v_i(\cdot)$, $i \in \mathcal{N}$ satisfy (34) with the selected constant δ, it follows that Ω^N is forward invariant for the solution of the closed loop system (1), (81), namely, it holds $x(t) \in \Omega^N$ for all $t \geq 0$.

Proof. Given that the stack vector of the agents' initial states satisfies $x(0) \in \Omega^N$, let $[0, T_{\max})$ be the maximal forward interval for which the solution $x(\cdot)$ of (1), (81) exists and remains inside Ω^N. We claim that for all $t \in [0, T_{\max})$ the solution remains inside $\mathrm{cl}(\Omega_{\tilde{\varepsilon}})^N$ with

$$\tilde{\varepsilon} := \min\left\{ \min\{|H(x_i(0)) - x_i(0)|, i \in \mathcal{N}_0^{\varepsilon}\}, \varepsilon\left(1 - h^{-1}\left(\frac{1}{c}\right)\right)\right\}, \tag{82}$$

$$\mathcal{N}_0^{\varepsilon} := \{i \in \mathcal{N} : x_i(0) \in N_{\varepsilon}\}$$

and where $c > 1$ and $h(\cdot)$ are given in the statement of the proposition and (79), respectively. From (82), we get that

$$\tilde{\varepsilon} \leq \varepsilon - \varepsilon h^{-1}\left(\frac{1}{c}\right) \Longrightarrow h\left(\frac{\varepsilon - \tilde{\varepsilon}}{\varepsilon}\right) \geq \frac{1}{c}. \tag{83}$$

In addition, it follows from the fact that $x(t)$ remains in the compact subset $\mathrm{cl}(\Omega_{\tilde{\varepsilon}})^N$ of Ω^N for all $t \in [0, T_{\max})$, that $T_{\max} = \infty$, which provides the desired result. In order to prove our claim, define for each $i \in \mathcal{N}$ the function

$$m_i(t) := \begin{cases} \varepsilon - |H(x_i(t)) - x_i(t)|, & \text{if } x_i(t) \in N_{\varepsilon}, \\ 0, & \text{if } x_i(t) \in \Omega_{\varepsilon}, \end{cases} \quad , \quad t \in [0, T_{\max}) \tag{84}$$

and let

$$m(t) := \max\{m_i(t) : i \in \mathcal{N}\}, t \in [0, T_{\max}), \tag{85}$$

where $m_i(t)$ denotes the distance of agent i from Ω_{ε} at time t and $m(t)$ is the maximum over those distances for all agents. Hence, for all $t \in [0, T_{\max})$ and all $\hat{\varepsilon} \in (0, \varepsilon]$ we obtain from (84), (85) and (P3) the following equivalences

$$x_i(t) \in N_{\hat{\varepsilon}} \iff m_i(t) \in [\varepsilon - \hat{\varepsilon}, \varepsilon), \tag{86}$$

$$x_i(t) \in \partial\Omega_{\hat{\varepsilon}} \iff m_i(t) = \varepsilon - \hat{\varepsilon}, \tag{87}$$

$$x_i(t) \in \mathrm{cl}(\Omega_{\hat{\varepsilon}}), \forall i \in \mathcal{N} \iff m(t) \in [0, \varepsilon - \hat{\varepsilon}]. \tag{88}$$

Notice that the functions $m_i(\cdot)$, $i \in \mathcal{N}$ and $m(\cdot)$ are continuous and due to (82), it holds

$$m(0) \leq \varepsilon - \tilde{\varepsilon}. \tag{89}$$

We claim that

$$m(t) \leq \varepsilon - \tilde{\varepsilon}, \forall t \in [0, T_{\max}), \tag{90}$$

with $\tilde{\varepsilon}$ as given in (82). Indeed, suppose on the contrary that there exists $T \in (0, T_{\max})$ such that

$$m(T) = \varepsilon - \tilde{\varepsilon} + 2\Delta\varepsilon, \Delta\varepsilon \in \left(0, \frac{\tilde{\varepsilon}}{2}\right) \tag{91}$$

and define

$$\tau := \min\left\{\tilde{\tau} \in [0,T] : m(t) \geq \varepsilon - \tilde{\varepsilon} + \Delta\varepsilon, \forall t \in [\tilde{\tau}, T]\right\}. \tag{92}$$

Then, it follows from (91) that τ is well defined and from (89), (92) and the continuity of $m(\cdot)$ that

$$m(\tau) = \varepsilon - \tilde{\varepsilon} + \Delta\varepsilon \tag{93}$$

and that there exists a sequence $(t_\nu)_{\nu\in\mathbb{N}}$ with

$$t_\nu \searrow \tau \text{ and } m(t_\nu) \geq \varepsilon - \tilde{\varepsilon} + \Delta\varepsilon, \forall \nu \in \mathbb{N}. \tag{94}$$

From (85), (93), (94) and the infinite pigeonhole principle, namely, that in each finite partition of an infinite set there exists a set with infinite cardinality, we deduce that there exists $i \in \mathcal{N}$ and a subsequence $(t_{\nu_k})_{k\in\mathbb{N}}$ of $(t_\nu)_{\nu\in\mathbb{N}}$ such that

$$m_i(t_{\nu_k}) \geq \varepsilon - \tilde{\varepsilon} + \Delta\varepsilon, \forall k \in \mathbb{N}; m_i(\tau) = \varepsilon - \tilde{\varepsilon} + \Delta\varepsilon. \tag{95}$$

Thus, it follows by virtue of (86) and (87) that

$$x_i(t_{\nu_k}) \in N_{\tilde{\varepsilon}-\Delta\varepsilon}, \forall k \in \mathbb{N}; x_i(\tau) \in \partial\Omega_{\tilde{\varepsilon}-\Delta\varepsilon}. \tag{96}$$

Notice, that according to Lemma 7(B), $W(x)$ is a global defining function for $\partial\Omega_{\tilde{\varepsilon}-\Delta\varepsilon}$ with larger values outside $\Omega_{\tilde{\varepsilon}-\Delta\varepsilon}$. Thus, we deduce that

$$\left.\frac{d}{dt}W(x_i(t))\right|_{t=\tau} = \lim_{k\to\infty} \frac{W(x_i(t_{\nu_k})) - W(x_i(\tau))}{t_{\nu_k} - \tau} \geq 0. \tag{97}$$

On the other hand, we have that

$$\begin{aligned} \left.\frac{d}{dt}W(x_i(t))\right|_{t=\tau} &= \langle \nabla W(x_i(\tau)), \dot{x}_i(\tau)\rangle \\ &= \langle \nabla W(x_i(\tau)), g(x_i(\tau)) + v_i(\tau) - \sum_{j\in\mathcal{N}_i} r(|x_i(\tau) - x_j(\tau)|)(x_i(\tau) - x_j(\tau))\rangle. \end{aligned} \tag{98}$$

By taking into account (96) we get from (79), (80) and (83) that

$$|g(x_i(\tau))| = c\delta h\left(\frac{\varepsilon - \tilde{\varepsilon} + \Delta\varepsilon}{\varepsilon}\right) > c\delta h\left(\frac{\varepsilon - \tilde{\varepsilon}}{\varepsilon}\right) = c\delta\frac{1}{c} = \delta. \tag{99}$$

Also, due to (80) it holds

$$\nabla W(x_i(\tau)) = -ag(x_i(\tau)), \tag{100}$$

for certain $a > 0$. Then, we get from (99), (100) and the fact that $|v_i(\tau)| \leq \delta$ that

$$\begin{aligned} \langle \nabla W(x_i(\tau)), g(x_i(\tau)) + v_i(\tau)\rangle &\leq \langle \nabla W(x_i(\tau)), g(x_i(\tau))\rangle + |v_i(\tau)| \\ &= -|g(x_i(\tau))| + |v_i(\tau)| < 0. \end{aligned} \tag{101}$$

Furthermore, we have from (93) and (88) that $x_j(\tau) \in \mathrm{cl}(\Omega_{\tilde{\varepsilon}-\Delta\varepsilon})$ for all $j \in \mathcal{N}_i$ and from (96) that $x_i(\tau) \in N_{\tilde{\varepsilon}-\Delta\varepsilon}$. Thus, it follows from Lemma 7(D) applied with $a := \tilde{\varepsilon} - \Delta\varepsilon$, $x := x_i(\tau)$ and $y := x_j(\tau)$, that

$$\langle \nabla W(x_i(\tau)), x_i(\tau) - x_j(\tau)\rangle \geq 0. \tag{102}$$

From (101), (102) and the fact that $r(s) \geq 0$ for all $s \geq 0$, we obtain that (98) is negative, which contradicts (97). Hence, (90) holds, which implies that $x(t)$ remains in the compact subset $\mathrm{cl}(\Omega_{\bar{\varepsilon}})^N$ of Ω^N for all $t \in [0, T_{\max})$. Thus, $T_{\max} = \infty$ and we conclude that the solution $x(\cdot)$ of the system remains in Ω^N for all $t \geq 0$.

Hence, we have shown that for each initial condition in Ω^N the solution of the closed loop system is well defined and remains in a compact subset of Ω^N for all positive times. The proof is now complete. ∎

Having shown that the control law in (81) establishes forward invariance of the closed loop system within Ω^N, we proceed by proving that the connectivity result of Proposition 3 remains valid with the same bounds for the input terms v_i and the relative initial distances between the agents, when the initial condition of each agent lies in Ω. In particular, we obtain the following result.

THEOREM 9. *For the multi-agent system* (1)*, assume that the hypotheses of Theorem 8 are fulfilled and that the function $r(\cdot)$ in* (81) *satisfies Property (P). In addition, assume that the (ICH)* (3) *holds for certain $\tilde{R} \in (0, R)$, and that the constant δ in* (34)*,* (80) *the distance $\tilde{R}$ and the function $r(\cdot)$ satisfy* (32) *and* (33)*. Then, in addition to forward invariance of Ω^N with respect to the solution of the closed loop system* (1)*,* (81)*, the topology of the multi-agent network remains connected for all positive times.*

Proof. Notice first, that by the result of Theorem 8, the solution of the closed loop system (1), (81), is well defined and remains inside Ω^N for all positive times. In order to prove that the network topology will also remain connected, we will appropriately modify the corresponding proof of Proposition 3. In particular, we exploit the energy function $V(\cdot)$ as given by (8) and show that when $|\Delta x|_\infty \geq \tilde{R}$, namely, when the maximum distance between two agents exceeds $\tilde{R}$ then its derivative along the vector field defined by the closed loop system is non-positive. Thus, by using the same arguments with those in proof of Proposition 3 we can deduce that the system remains connected. Indeed, by evaluating the derivative of $V(\cdot)$ along the vector field $u = (u_1, \ldots, u_N)$ as specified by the control laws u_i, $i \in \mathcal{N}$ in (81) we obtain

$$
\begin{aligned}
DV(x)u &= \sum_{i \in \mathcal{N}} D_{x_i} V(x) u_i \\
&= \sum_{i \in \mathcal{N}} D_{x_i} V(x) g(x_i) - \sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \\
&\leq \sum_{i \in \mathcal{N}} D_{x_i} V(x) g(x_i) - \sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right|. \qquad (103)
\end{aligned}
$$

By taking into account (18) and using precisely the same arguments with those in the proof of Proposition 3, it suffices to show that the first term of inequality (103), which by virtue of (9) is equal to

$$\sum_{i \in \mathcal{N}} D_{x_i} V(x) g(x_i) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|) \langle (x_i - x_j), g(x_i) \rangle,$$

is nonpositive for all $x \in \Omega$. Given the partition Ω_ε, N_ε of Ω, we consider for each agent $i \in \mathcal{N}$ the partition $\mathcal{N}_i^{\Omega_\varepsilon}$, $\mathcal{N}_i^{N_\varepsilon}$ of its neighbors' set, corresponding to its neighbors that belong to Ω_ε and N_ε, respectively. Also, we denote by $\mathcal{E}^{N_\varepsilon}$ the set of edges $\{i, j\}$ with both $x_i, x_j \in N_\varepsilon$. Then, by taking into account that due to (80), $g(x_i) = 0$ for

$x_i \in \Omega_\varepsilon$, it follows that

$$\begin{aligned}
&\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i} r(|x_i-x_j|)\langle(x_i-x_j),g(x_i)\rangle \\
&= \sum_{\{i\in\mathcal{N}:x_i\in N_\varepsilon\}}\sum_{j\in\mathcal{N}_i} r(|x_i-x_j|)\langle(x_i-x_j),g(x_i)\rangle \\
&= \sum_{\{i\in\mathcal{N}:x_i\in N_\varepsilon\}}\sum_{j\in\mathcal{N}_i^{\Omega_\varepsilon}\cup\mathcal{N}_i^{N_\varepsilon}} r(|x_i-x_j|)\langle(x_i-x_j),g(x_i)\rangle \\
&= \sum_{\{i\in\mathcal{N}:x_i\in N_\varepsilon\}}\sum_{j\in\mathcal{N}_i^{\Omega_\varepsilon}} r(|x_i-x_j|)\langle(x_i-x_j),g(x_i)\rangle \\
&+ \sum_{\{i,j\}\in\mathcal{E}^{N_\varepsilon}} r(|x_i-x_j|)[\langle(x_i-x_j),g(x_i)\rangle+\langle(x_j-x_i),g(x_j)\rangle]. \qquad (104)
\end{aligned}$$

In order to prove that both terms in (104) are less than or equal to zero and hence derive our desired result on the sign of $DV(x)u$, we exploit the following fact.

Fact V. Consider the vectors $\alpha,\beta,\gamma\in\mathbb{R}^n$ with the following properties:

$$|\alpha|=1,\ |\beta|=1, \qquad (105)$$

$$\langle\alpha,\gamma\rangle\geq 0,\ \langle\beta,\gamma\rangle\leq 0 \qquad (106)$$

Then for every quadruple $\lambda_\alpha,\lambda_\beta,\mu_\alpha,\mu_\beta\in\mathbb{R}_{\geq 0}$ satisfying

$$\lambda_\alpha\geq\lambda_\beta,\mu_\alpha\geq\mu_\beta, \qquad (107)$$

it holds

$$\langle(\mu_\alpha\alpha-\mu_\beta\beta),\tilde{\delta}\rangle\geq 0, \qquad (108)$$

where

$$\tilde{\delta}:=\lambda_\alpha\alpha+\gamma-\lambda_\beta\beta. \qquad (109)$$

We provide the proof of Fact V in the Appendix.

We are now in position to show that both terms in the right hand side of (104) are nonpositive, which according to our previous discussion establishes the desired connectivity maintenance result.

Proof of the fact that the first term in (104) is nonpositive. For each i,j in the first term in (104) we get by applying Lemma 7(D) with $a:=\varepsilon$, $x:=x_i\in N_\varepsilon$ and $y:=x_j\in\Omega_\varepsilon$ that

$$\begin{aligned}
&r(|x_i-x_j|)\langle x_i-x_j,g(x_i)\rangle \\
&=-r(|x_i-x_j|)c\delta h\left(\frac{\varepsilon-|H(x_i)-x_i|}{\varepsilon}\right)\langle x_i-x_j,\nabla W(x_i)\rangle\leq 0
\end{aligned}$$

and hence, that the first term is nonpositive.

Proof of the fact that the second term in (104) is nonpositive. We exploit Fact V in order to prove that for each $\{i,j\}\in\mathcal{E}^{N_\varepsilon}$ the quantity

$$\langle(x_i-x_j),g(x_i)\rangle+\langle(x_j-x_i),g(x_j)\rangle \qquad (110)$$

in the second term of (104) is nonpositive as well. Notice that both $x_i, x_j \in N_\varepsilon$ and without loss of generality we may assume that

$$|H(x_i) - x_i| \leq |H(x_j) - x_j|, \tag{111}$$

namely, that x_i is farther from the boundary of Ω_ε than x_j. Then, by setting

$$\alpha := \nabla W(x_i); \beta := \nabla W(x_j), \tag{112}$$
$$\gamma := \tilde{x}_i - \tilde{x}_j, \tag{113}$$

with

$$\tilde{x}_i := x_i - (\varepsilon - |H(x_i) - x_i|)\nabla W(x_i), \tag{114}$$
$$\tilde{x}_j := x_j - (\varepsilon - |H(x_j) - x_j|)\nabla W(x_j) \tag{115}$$

and

$$\lambda_\alpha := \varepsilon - |H(x_i) - x_i|; \lambda_\beta := \varepsilon - |H(x_j) - x_j|, \tag{116}$$
$$\mu_\alpha := c\delta h\left(\frac{\varepsilon - |H(x_i) - x_i|}{\varepsilon}\right); \mu_\beta := c\delta h\left(\frac{\varepsilon - |H(x_j) - x_j|}{\varepsilon}\right), \tag{117}$$

it follows from (112) that $|\alpha| = |\beta| = 1$, and from (79), (111), (116) and (117) that $\lambda_\alpha \geq \lambda_\beta \geq 0, \mu_\alpha \geq \mu_\beta \geq 0$. Furthermore, we get from (114), (115) and Lemma 7(C) with $a := \varepsilon$, $x := x_i, x_j$, that $\tilde{x}_i, \tilde{x}_j \in \partial\Omega_\varepsilon$ and $\nabla W(x_i) = \nabla W(\tilde{x}_i)$, $\nabla W(x_j) = \nabla W(\tilde{x}_j)$. Thus, it follows from (112), (113) and application of Lemma 7(D) with $a := \varepsilon$, $x = \tilde{x}_i$ and $y = \tilde{x}_j$ that $\langle \alpha, \gamma \rangle \geq 0$ and similarly, that $\langle \beta, \gamma \rangle \leq 0$. It thus follows that all requirements of Fact V are fulfilled. Furthermore, by taking into account (112)-(116), we get that

$$\tilde{\delta} = \lambda_\alpha \alpha + \gamma - \lambda_\beta \beta = x_i - x_j. \tag{118}$$

Hence, we establish by virtue of (80), (108), (109), (112), (117) and (118) that

$$\begin{aligned} \langle (\mu_\alpha \alpha - \mu_\beta \beta), \tilde{\delta} \rangle = -\langle (g(x_i) - g(x_j)), (x_i - x_j) \rangle \geq 0 \iff \\ \langle (x_i - x_j), g(x_i) \rangle + \langle (x_j - x_i), g(x_j) \rangle \leq 0, \end{aligned}$$

as desired. We conclude that the network topology remains connected during the evolution of the system and the proof is now complete. ∎

Remark 10. *The result of Theorem 9 remains valid under the hypotheses of Theorem 8 for the closed loop system (1), (81), if the (ICH) (3) holds for certain $\tilde{R} \in (0, R)$, the function $r(\cdot)$ in (81) satisfies $r(s) \geq 0$ for all $s \geq 0$ (not necessarily Property (P)), (33) holds, and the following condition is fulfilled. There exists a constant $\delta > 0$, such that (38) holds with $u = (u_1, \ldots, u_N)$ and u_i, $i \in \mathcal{N}$ as given by (7), $V(\cdot)$ as given by (8) and all v_i, $i \in \mathcal{N}$ with $|v_i| \leq \delta$. This observation follows from Theorem 8 and the arguments applied for the proofs of Proposition 3 and Theorem 9, and can provide improved bounds on the additional input terms v_i for certain networks where the verification of condition (38) does not necessarily require tools from algebraic graph theory.*

5. Example and Simulation Results. We consider a system of four agents with states $x_1, x_2, x_3, x_4 \in \mathbb{R}^2$, whose initial conditions lie inside the open planar circular domain $\Omega := \{x \in \mathbb{R}^2 : |x| < \rho\}$. For the example we select the agents' neighbors' sets as $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{1,3\}$, $\mathcal{N}_3 = \{2,4\}$, $\mathcal{N}_4 = \{3\}$, i.e., such that the network topology is given by a path graph. Their dynamics are given by (1), (81) with $r(\cdot)$ satisfying $r(s) \geq 0$ for all $s \geq 0$. Therefore, the connectivity and invariance analysis will be based on the establishment of the conditions provided in Remark 10.

Notice first, that in the case of a circle the tubular neighborhood $N_{\bar{\varepsilon}}$ in (52) is well defined for $\bar{\varepsilon} = \rho$. In addition, the maps $H(\cdot)$ and $\nabla W(x)$ in (P2) and Lemma 7(A), are given as $H(x) = \frac{\rho x}{|x|}$ and $\nabla W(x) = \frac{x}{|x|}$, respectively, for all $x \in N_{\bar{\varepsilon}}$. Thus given $\varepsilon \in (0, \rho)$ and the partition of Ω in $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : |x| < \rho - \varepsilon\}$ and $N_\varepsilon = \{x \in \mathbb{R}^2 : \rho - \varepsilon \leq |x| < \rho\}$ we obtain the function $g(\cdot)$ in (80) as

$$g(x) := \begin{cases} 0, & \text{if } |x| < \rho - \varepsilon, \\ c\delta \frac{\rho - \varepsilon - |x|}{\varepsilon} \frac{x}{|x|}, & \text{if } \rho - \varepsilon \leq |x| < \rho, \end{cases} \tag{119}$$

where $\delta > 0$, $c > 1$ and $h(\cdot)$ has been selected as $h(s) = s$, $s \in [0,1]$. The repulsion vector field $g(\cdot)$ is illustrated together with the agents and their network topology in Fig. 3, below.

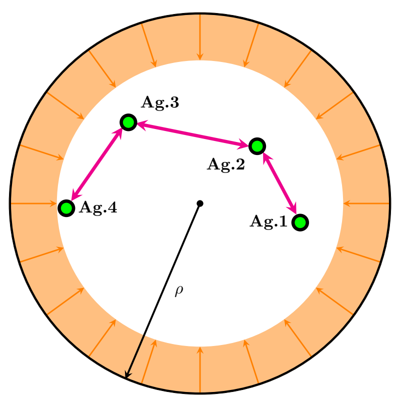

FIG. 3. *4-Agent Example in a Circular Domain.*

We proceed to determine a bound $\delta > 0$ such that (38) is fulfilled with $u = (u_1, \dots, u_N)$ and u_i, $i \in \mathcal{N}$ as given by (7), $V(\cdot)$ as given by (8) and all v_i, $i \in \mathcal{N}$ with $|v_i| \leq \delta$. For notational convenience we denote as $\ell_i := |x_{i+1} - x_i|$ and $r_i := r(|x_{i+1} - x_i|) = r(\ell_i)$, $i = 1, 2, 3$. Given $u = (u_1, \dots, u_N)$ with u_i, $i \in \mathcal{N}$ as in (7), the derivative of the energy function $V(\cdot)$ along u is given by virtue of (9) as

$$\begin{aligned} DV(x)u &= \sum_{i=1}^{4} D_{x_i} V(x) u_i = \sum_{i=1}^{4} \left(\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j)^T u_i \right) \\ &= r_1(x_1 - x_2)^T [r_1(x_2 - x_1) + v_1 - (r_1(x_1 - x_2) + r_2(x_3 - x_2) + v_2)] \\ &+ r_2(x_2 - x_3)^T [r_1(x_1 - x_2) + r_2(x_3 - x_2) + v_2 - (r_2(x_2 - x_3) + r_3(x_4 - x_3) + v_3)] \\ &+ r_3(x_3 - x_4)^T [r_2(x_2 - x_3) + r_3(x_4 - x_3) + v_3 - (r_3(x_3 - x_4) + v_4)]. \end{aligned}$$

For $|v_i| \leq \delta$, $i = 1, 2, 3, 4$ we obtain from the Cauchy-Schwartz inequality that

$$\begin{aligned} DV(x)u \leq & -2r_1^2\ell_1^2 - 2r_2^2\ell_2^2 - 2r_3^2\ell_3^2 + 2r_1\ell_1 r_2\ell_2 + 2r_2\ell_2 r_3\ell_3 + 2(r_1\ell_1 + r_2\ell_2 + 2_3\ell_3)\delta \\ = & -2(1-k)(r_1^2\ell_1^2 + r_2^2\ell_2^2 + r_3^2\ell_3^2) - (2kr_1^2\ell_1^2 - 2r_1\ell_1 r_2\ell_2 + kr_2^2\ell_2^2) \\ & -(2kr_2^2\ell_2^2 - 2r_2\ell_2 r_3\ell_3 + kr_3^2\ell_3^2) + 2(r_1\ell_1 + r_2\ell_2 + 2_3\ell_3)\delta, \end{aligned}$$

for any $k \in \mathbb{R}$. By additionally exploiting the elementary fact that for any $a, b > 0$ it holds $2ka^2 - 2ab + kb^2 \geq 0$ for all $k \geq \frac{\sqrt{2}}{2}$, it follows that $DV(x)u \leq -2\left(1 - \frac{\sqrt{2}}{2}\right)(r_1^2\ell_1^2 + r_2^2\ell_2^2 + r_3^2\ell_3^2) + 2(r_1\ell_1 + r_2\ell_2 + r_3\ell_3)\delta$. In order to show (38) it suffices to show that $-\left(1 - \frac{\sqrt{2}}{2}\right)(r_1^2\ell_1^2 + r_2^2\ell_2^2 + r_3^2\ell_3^2) + (r_1\ell_1 + r_2\ell_2 + r_3\ell_3)\delta \leq 0$ whenever $\max\{\ell_1, \ell_2, \ell_3\} \geq \tilde{R}$. By additionally requiring that $r(\cdot)$ is increasing and satisfies $r(\tilde{R}) > 0$, the latter is equivalent to showing that $l_1^2 + l_2^2 + l_3^2 - (l_1 + l_2 + l_3)\theta \geq 0$, whenever $\max\{l_1, l_2, l_3\} \geq 1$, where we have set $l_i := \frac{r_i\ell_i}{r(\tilde{R})\tilde{R}}$ and $\theta := \frac{\delta}{\left(1 - \frac{\sqrt{2}}{2}\right) r(\tilde{R})\tilde{R}}$. The latter follows if we assume that without any loss of generality $l_1 = 1$, $\theta \leq 1$, and specify θ such that

$$1 + l_2^2 + l_3^2 - \theta(1 + l_2 + l_3) \geq 0. \tag{120}$$

The left hand side of (120) is minimized when $2l_2 = 2l_3 = \theta$, and becomes $-\frac{\theta^2}{2} - \theta + 1$. Thus, by selecting $\theta = \sqrt{3} - 1$, we obtain the maximum value of θ for which (120) is valid for all l_2, l_3, and we get $\delta = 0.2144 r(\tilde{R})\tilde{R}$. We next select $r(s) := as$ for certain $a > 0$ and obtain the function $P(s) = a\frac{s^3}{3}$ in (4). Thus, we can specify the maximum initial distance $\tilde{R}$ between interconnected agents in such a way that (33) is fulfilled, i.e., such that $3a\tilde{R}^3 = aR^3$, by selecting $\tilde{R} = \frac{1}{\sqrt[3]{3}}R$.

For the simulation results we pick $\rho = 10$, $\varepsilon = 5$ and $c = 1.1$ in (119). In addition we select $R = 10$ and $a = 0.2$ which provide the maximum initial distance $\tilde{R} = 6.9336$ and the bound $\delta = 2.0615$ on the inputs v_i. We consider two different cases for the initial positions of the agents and their inputs v_i and depict the system's evolution for each case over the time interval $[0, 12]$ in Fig. 4, below. The inputs in the left figure have been selected as $v_1(t) = (-1, -1)$, $t \in [0, 6]$, $v_1(t) = (-2, 0)$, $t \in [9, 12]$, $v_2(t) = (0, -1)$, $t \in [0, 6]$, $v_2(t) = (0, 0)$, $t \in [9, 12]$, $v_3(t) = (0, 0)$, $t \in [0, 12]$, and $v_4(t) = (2, 0)$, $t \in [0, 6]$, $v_4(t) = (0, 0)$, $t \in [9, 12]$, respectively, and as the convex combination $\frac{(9-t)}{3}v_i(6) + \frac{(t-6)}{3}v_i(9)$ for $t \in (6, 9)$, $i = 1, 2, 4$. Thus, the network moves downward and the distance between agents 2, 3 and 4 increases over the time interval $[0, 6]$. After the transient period $(6, 9)$, namely, for $t \in [9, 12]$ the only agent with nonzero input is 1, and this results in the motion of the group to the left and convergence of the other agents towards agent 1. The corresponding inputs in the right figure are $v_1(t) = (1, 0)$, $t \in [0, 3]$, $v_1(t) = (1, 1.5)$, $t \in [6, 12]$, $v_2(t) = v_3(t) = (0, 0)$, $t \in [0, 12]$, $v_4(t) = (1, 0)$, $t \in [0, 3]$, $v_4(t) = (1, -1.5)$, $t \in [6, 12]$, respectively, and the convex combination $\frac{(6-t)}{3}v_i(3) + \frac{(t-3)}{3}v_i(6)$ for $t \in (3, 6)$, $i = 1, 4$. Thus, the network moves to the left and the agents approach each other over the time interval $[0, 3]$. After the transient period $(3, 6)$, the agents obtain a vertical distance, due to the additional upward motion of 1 and the downward motion of 4 imposed by the vertical component of their corresponding input terms. We observe that in both cases the requirements on the maximum initial distance between interconnected agents and the bounds on the input terms are satisfied, which by virtue of Remark 10 result in connectivity maintenance and invariance of the agents' trajectories inside the circular domain.

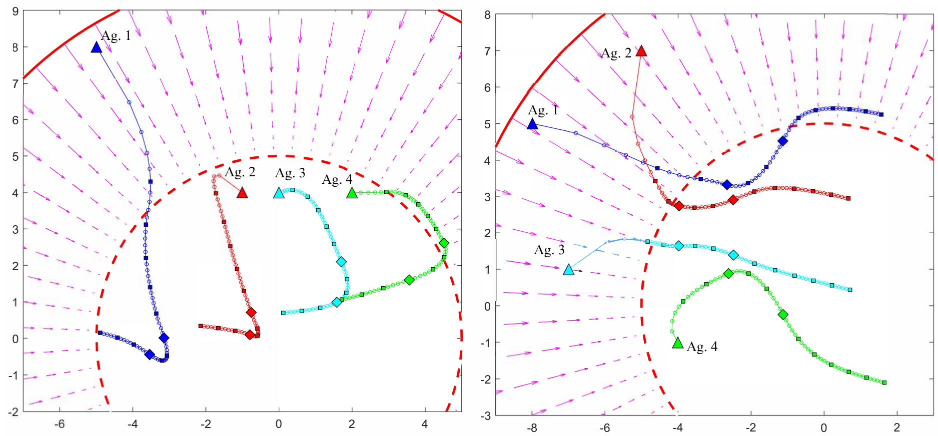

Fig. 4. *This figure shows the evolution of agents 1, 2, 3 and 4 for times* $t \in [0, 12]$*. The initial conditions are depicted by the triangles and the diamonds represent the start and endpoint of the transient time intervals* $(6, 9)$ *and* $(3, 6)$ *for the input terms* v_i *in the left and right figure, respectively.*

6. Conclusions. We have designed a decentralized control framework for single integrator multi-agent systems in order to maintain connectivity of the network during the evolution of the system and established robustness of this property with respect to additional bounded input terms. Furthermore, under the assumption that the initial conditions of the agents lie inside a bounded and convex domain, a modification of the proposed control law guarantees forward invariance of the agents' trajectories inside this domain, while simultaneously preserving the robust connectivity result.

Future research includes the application of optimization tools in order to improve the bounds on the extra input terms and the initial relative distances of the agents, and the consideration of nonconvex domains, by additionally relating the derived bounds to curvature properties of their boundaries.

7. Appendix. In the Appendix, we provide the proofs of Facts I, II and III, IV, which were used in the proofs of Lemma 2 and Proposition 3, respectively, and of Fact V, which was used in the proof of Theorem 9. For convenience we state the elementary inequality

$$2(|w|^2 + |z|^2) \geq |w - z|^2, \forall w, z \in \mathbb{R}^n, \tag{121}$$

which is a direct consequence of the triangle inequality.

Proof of Fact I. Let $\{e_k\}_{k \in \mathcal{N}}$ be an orthonormal basis of eigenvectors corresponding to the ordered eigenvalues of $L_w(x)$. Then, for each $l = 1, \ldots, n$ we have that

$$c_l(x^\perp) = \sum_{k=2}^{N} \mu_k e_k; \mu_k \in \mathbb{R}, k = 2, \ldots, N$$

and hence, that

$$|c_l(x^\perp)| = \left(\sum_{k=2}^{N} \mu_k^2 \right)^{\frac{1}{2}}.$$

Thus, we get that

$$
\begin{aligned}
|L_w(x)c_l(x^{\perp})| &= \left|\sum_{k=2}^{N}\mu_k L_w(x)e_k\right| = \left|\sum_{k=2}^{N}\mu_k\lambda_k(x)e_k\right| \\
&= \left(\sum_{k=2}^{N}(\mu_k\lambda_k(x))^2\right)^{\frac{1}{2}} \geq \lambda_2(x)\left(\sum_{k=2}^{N}\mu_k^2\right)^{\frac{1}{2}} = \lambda_2(x)|c_l(x^{\perp})|,
\end{aligned}
$$

which establishes (22).

Proof of Fact II. By taking into account the Cauchy Schwartz inequality we obtain

$$
\begin{aligned}
\sum_{l=1}^{n}|c_l(x)||c_l(y)| &\leq \left(\sum_{l=1}^{n}|c_l(x)|^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{n}|c_l(y)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{l=1}^{n}\sum_{i=1}^{N}c_l(x_i)^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{n}\sum_{i=1}^{N}c_l(y_i)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{N}\sum_{l=1}^{n}c_l(x_i)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N}\sum_{l=1}^{n}c_l(y_i)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{N}|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{N}|y_i|^2\right)^{\frac{1}{2}} = |x||y|
\end{aligned}
$$

and hence (23) holds.

Proof of Fact III. By the definition of $x^{\perp}$ and $\bar{x}$, it follows that there exists $\tilde{x} \in \mathbb{R}^n$ such that $x - x^{\perp} = \bar{x} = (\tilde{x}, \ldots, \tilde{x}) \in \mathbb{R}^{Nn}$. Hence, we have that

$$
\begin{aligned}
|x^{\perp}| = |x-\bar{x}| = |(x_1,\ldots,x_N) - (\tilde{x},\ldots,\tilde{x})| &= \left(\sum_{i=1}^{N}|x_i-\tilde{x}|^2\right)^{\frac{1}{2}} \implies \\
\sqrt{2(N-1)}|x^{\perp}| &= \left(\sum_{i=1}^{N}2(N-1)|x_i-\tilde{x}|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{\{i,j\}\in\mathcal{E}(K(\mathcal{N}))}2(|x_i-\tilde{x}|^2+|x_j-\tilde{x}|^2)\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\mathcal{E}(K(\mathcal{N}))$ stands for the edge set of the complete graph with vertex set $\mathcal{N}$. Then, it follows from (121) that

$$
\begin{aligned}
\left(\sum_{\{i,j\}\in\mathcal{E}(K(\mathcal{N}))}2(|x_i-\tilde{x}|^2+|x_j-\tilde{x}|^2)\right)^{\frac{1}{2}} &\geq \left(\sum_{\{i,j\}\in\mathcal{E}(K(\mathcal{N}))}|x_i-x_j|^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_{\{i,j\}\in\mathcal{E}}|x_i-x_j|^2\right)^{\frac{1}{2}} = |\Delta x|,
\end{aligned}
$$

which provides the desired result.

Proof of Fact IV. Notice that (36) is equivalently written as

$$2|x^{\perp}|^2 \geq \max_{\{i,j\}\in\mathcal{E}} |x_i - x_j|^2 \iff 2\left(\sum_{i=1}^{N} |x_i - \tilde{x}|^2\right) \geq \max_{\{i,j\}\in\mathcal{E}} |x_i - x_j|^2,$$

with $\tilde{x} \in \mathbb{R}^n$ as in proof of Fact III. Let $\{\hat{i},\hat{j}\} \in \mathcal{E}$ such that $|x_{\hat{i}} - x_{\hat{j}}| = \max_{\{i,j\}\in\mathcal{E}} |x_i - x_j|$. Then, by taking into account (121) we have

$$2\left(\sum_{i=1}^{N} |x_i - \tilde{x}|^2\right) \geq 2(|x_{\hat{i}} - \tilde{x}|^2 + |x_{\hat{j}} - \tilde{x}|^2) \geq |x_{\hat{i}} - x_{\hat{j}}|^2 = \max_{\{i,j\}\in\mathcal{E}} |x_i - x_j|^2$$

and thus (36) is fulfilled.

Proof of Fact V. By taking into account (105)-(107) and (109) we evaluate

$$\begin{aligned}
\langle(\mu_\alpha\alpha - \mu_\beta\beta), \tilde{\delta}\rangle &= \langle(\mu_\alpha\alpha - \mu_\beta\beta), (\lambda_\alpha\alpha - \lambda_\beta\beta) + \gamma\rangle \\
&= \langle(\mu_\alpha\alpha - \mu_\beta\beta), (\lambda_\alpha\alpha - \lambda_\beta\beta)\rangle + \mu_\alpha\langle\alpha,\gamma\rangle - \mu_\beta\langle\beta,\gamma\rangle \\
&\geq \langle(\mu_\alpha\alpha - \mu_\beta\beta), (\lambda_\alpha\alpha - \lambda_\beta\beta)\rangle \\
&= \mu_\alpha\lambda_\alpha|\alpha|^2 - (\mu_\alpha\lambda_\beta + \mu_\beta\lambda_\alpha)\langle\alpha,\beta\rangle + \mu_\beta\lambda_\beta|\beta|^2 \\
&\geq \mu_\alpha\lambda_\alpha|\alpha|^2 - (\mu_\alpha\lambda_\beta + \mu_\beta\lambda_\alpha)|\alpha||\beta| + \mu_\beta\lambda_\beta|\beta|^2 \\
&= \mu_\alpha\lambda_\alpha - \mu_\alpha\lambda_\beta - \mu_\beta\lambda_\alpha + \mu_\beta\lambda_\beta \\
&= \mu_\alpha(\lambda_\alpha - \lambda_\beta) - \mu_\beta(\lambda_\alpha - \lambda_\beta) = (\mu_\alpha - \mu_\beta)(\lambda_\alpha - \lambda_\beta) \geq 0
\end{aligned}$$

and hence, (108) holds.

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