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Opinion consensus under external influences[☆]

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ABSTRACT

As a means to regulate the continuous-time bounded confidence opinion dynamics, an exo-system to the original Hegselmann–Krause model is added. Some analysis is made about the properties of the combined system. Two theorems are provided in this article in terms of sufficient conditions of the exo-system that can guarantee opinion consensus for any initial conditions. Two more corollaries are given to describe the resulting synchronized opinions.

1. Introduction

Self-organized group behavior can be observed in animal flocks as well as human society. By simple communication, a whole group of birds can react to danger rapidly, or can migrate in a certain formation. A herd of predator can hunt with special tactics by following certain self-organized rules. As higher intelligent and relatively more independent human beings, we think and behave based on our own will while influenced by many other human individuals and by the society. Sociologists, psychologists, and even engineers and mathematicians want to model and study this interaction among human societies nowadays. While psychologists focus more on how people handle and react from social input, mathematicians analyze relatively simple models and the resulting group/global behaviors.

Among the different models about human opinion dynamics in the literature, there is a type of model called bounded confidence model who allows opinion influence to happen only when the two opinions are close enough. This type of modes also has the name "Hegselmann-Krause models" from its initiators Rainer Hegselmann and Ulrich Krause [1]. These discrete-time, deterministic models force the opinion of an individual to reach the opinion average among its close-by neighbors at every time step. The models lead to a well known clustering phenomenon that the opinions will converge to one or a few certain constant values. For a given model, the number and position of those values are completely determined by the starting opinion distributions.

Despite the simple expression of the Hegselmann–Krause models, the analysis of the opinion evolution is complicated. Besides the original article by Hegselmann and Krause, there are papers such as [2–4] that study the convergence, stability and steady state of the opinions under the Hegselmann–Krause model. By applying the theory of differential equations, the model is also extended to continuous-time opinions; [5,6] provide a detailed analysis on the existence and uniqueness of the solution to the continuous-time model, which is non-trivial due to the discontinuous right-hand side.

A frequently asked question is: for what initial opinion distribution the system will have only one cluster when time evolves. Reaching only one cluster is also called reaching consensus. The sufficient and necessary condition for reaching consensus has not been given in the literature. There are a few results for sufficient conditions such as those in [7]. Other researchers test some modified version of the Hegselmann–Krause model for different purposes. Those models can be found in [7,8]. In [9], a bounded confidence model with antagonistic interactions was discussed based on the analysis of the original Altafini's model in [10–12].

Instead of the standard homogeneous Hegselmann–Krause models, heterogeneous models that consider agents as different individuals are studied in the literature [13,14], which can improve the chance of reaching consensus for random initial conditions. An example of the heterogeneous models is by introducing "stubborn agents" that are unwillingly to change their opinions [15]. The extreme case of those stubborn agents can be considered as the state of a special type of exo-system that has zero dynamics. In this paper, we combine the Hegselmann–Krause model with a general exo-system. If the exo-system satisfies certain sufficient conditions, the consensus behavior can be guaranteed for any random initial opinion distribution.

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The Hegselmann–Krause model with exo-systems is introduced in Section 2 with some properties of the model. Two theorems are provided that guarantee consensus in Section 3 together with two corollaries. In Section 4, numerical experiments are carried out to test and illustrate the results of the theorems and corollaries. A short summary is included in Section 5 as well as a forecast of possible future study.

2. Problem formulation

We study a system of N agents with their time-dependent opinions denoted as $x_i(t) \in \mathbb{R}$ for $i = 1, 2, \ldots, N$. Agent i is influenced by agent j only if the opinions of both are close enough. The dynamics of the opinions can be modeled as the following system:

$$\dot{x}_i = \sum_{j:|x_j - x_i| < d} (x_j - x_i), \tag{1}$$

where the distance $d > 0$ is called the confidence range. The system (1) is also referred as continuous-time Hegselmann–Krause (H–K) model initially introduced in [5] based on the original discrete-time "bounded confidence" models. There are both theoretical analysis and numerical simulations about this model in the literature, showing the cluster behavior of the opinions. Asymptotically, each opinion will converge to one of the several certain values called opinion clusters. The distance between any pair of opinion clusters is proven to be larger than d in [5].

If the opinions converge to a single cluster, we say the opinions reach consensus. There are a few results about consensus behavior for both system (1) and modified versions of (1) with given initial conditions. For an arbitrary random initial opinion distribution, consensus is however not guaranteed even if the neighbor graph is initially connected. In this paper we introduce certain exo-systems in addition to the H–K model so that opinion consensus can be reached for a broader range of initial conditions. The new system is modeled as follows:

$$\begin{aligned} \dot{x}_i &= \sum_{j:|x_j - x_i| < d} (x_j - x_i) + \sum_{k:|y_k - x_i| < d} (y_k - x_i) \\ \dot{y}_k &= g_k(t, y), \end{aligned} \tag{2}$$

for $i = 1, 2, \ldots, N$ and $k = N+1, N+2, \ldots, N+M$, where y_k's are the state variables for the exo-system and g_k's are bounded continuous functions that we design later. M is dimension of the exo-system. For simplifying later use, $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ will denote the stack vectors of x_i's and y_k's, respectively. In the context, we also call the state of the original H–K model x_i's normal opinions in order to separate them from the external opinion y_k's.

Because of the discontinuity of the right-hand side of the continuous-time H–K model, the classic analysis of the existence and uniqueness of the solution does not apply. There are many discussions in the literature about different types of solutions to this differential equation. In [5], the authors introduced the concept of "proper solutions" as a subset of Carathéodory solutions, which guarantee existence and uniqueness for "proper initial conditions" that are almost sure in measure. In [6], the more general Krasovskii solutions are discussed for H–K models, which exist for any initial condition without a guarantee of uniqueness. There is a detailed discussion about the difference between these two types of solutions at the end of [6]. In our case, as long as the g function in the exo-system is locally bounded and well defined in an open neighborhood around the initial time, the existence of Krasovskii solution will be guaranteed (see [16]). Since g is given for design purposes, we assume that in this paper $g_k(t, y)$ is continuous and is defined for all $t \in \mathbb{R}$, and thus we consider that $(x(t), y(t))$ is a Krasovskii solution to (2) for the rest of the paper. Similar to the approach introduced in [6], we use the following way to describe a solution.

Given a Krasovskii Solution $(x(t), y(t))$, define the joint graph $G(x, y) = (V_x \cup V_y, E(x, y))$, where $V_x = \{1, 2, \ldots, N\}$ and $V_y = \{N+1, N+2, N+M\}$ are the sets of vertices, and

$$\begin{aligned} E(x, y) = &\{(i, j) : i, j \in V_x, |x_i - x_j| < d\} \\ &\cup \{(i, k) : i \in V_x, k \in V_y, |x_i - y_k| < d\} \end{aligned}$$

is the set of edges.[1] Slightly different from the standard graph theory definition, in this paper, we call two opinions *connected* if and only if their distance is less than d, i.e., i, j are connected if and only if $(i, j) \in E(x, y)$ for $i, j \in V_x \cup V_y$. We also need to define the concept of boundary of the edge set, which is

$$\begin{aligned} \partial E(x, y) = &\{(i, j) : i, j \in V_x, |x_i - x_j| = d\} \\ &\cup \{(i, k) : i \in V_x, k \in V_y, |x_i - y_k| = d\}. \end{aligned}$$

$\partial E(x, y)$ includes those pairs (i, j) that have the exact distance d for the corresponding opinions. The pair should not be in the set of edges but with any arbitrary small perturbation in the negative direction, it will form an edge between them. These pairs are the locations of the potential discontinuities in the system, where the Krasovskii solution allows small perturbation around them. Note that the dynamics of the exo-system is always continuous as we assumed, we only focus on the normal opinions $x_i(t)$.

By definition, for any given Krasovskii solution $(x(t), y(t))$ and for almost any time t, it holds that $\frac{d}{dt}(x(t), y(t))$ belongs to the closed convex hull of intersecting the right-hand-side of (2) with arbitrary small perturbation of the states. Since we have already defined the boundary of the edge set, we can always find a set of normalized weights $\alpha_H^{(x,y)}$, depending on both the solution (x, y) and the time t, such that

$$\begin{aligned} \frac{d}{dt} x_i(t) = &\sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \\ &\times \left(\sum_{j \in \mathcal{N}_i^H} (x_j(t) - x_i(t)) + \sum_{k \in \tilde{\mathcal{N}}_i^H} (y_k(t) - x_i(t)) \right), \end{aligned} \tag{3}$$

where $\alpha_H^{(x,y)} > 0$, $\sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} = 1$, and $\mathcal{N}_i^H$ and $\tilde{\mathcal{N}}_i^H$ are defined as follows, respectively:

$$\mathcal{N}_i^H = \{j \in V_x : (i, j) \in E(x, y) \cup H\};$$

$$\tilde{\mathcal{N}}_i^H = \{k \in V_y : (i, k) \in E(x, y) \cup H\}.$$

H denotes the possible set of edges that can be added to the graph, where the length of each of those edges is exactly d. Namely, the two opinions that have difference d can sometimes be considered as directly connected. We also define $\mathcal{N}_{ij}^H = \mathcal{N}_i^H \cap \mathcal{N}_j^H$, and $\tilde{\mathcal{N}}_{ij}^H = \tilde{\mathcal{N}}_i^H \cap \tilde{\mathcal{N}}_j^H$ for the simplification of later uses.

We start our analysis of the model by introducing the following order preservation property (Proposition 2.1) first.

Remark. It is intuitive to consider the order preservation property in the following way: when $x_i(t) = x_j(t)$ for some t, they should retain the same opinion for $t' > t$ since the derivative is only state related. However, this property only holds for "proper solutions" as shown in [5]. For general Krasovskii solutions, the equality for x_i and x_j may be lost at discontinuities even after time t [6]. Therefore, the following order preservation property only covers the cases of strict inequality.

[1] The graph is defined at time t and is time dependent. To simply the notation, we do not add t explicitly to the definitions. The edge definition here makes the graph a directed graph. However, the interaction between normal opinions is still symmetric.

Proposition 2.1 (*Order Preservation*). *In system (2), for any $i, j \in \{1, 2, \dots, N\}$, if $x_i(0) < x_j(0)$, then $x_i(t) < x_j(t)$ for any $t \geq 0$.*

Proof. We will prove the claim by contradiction. Given the continuous Krasovskii solution $(x(t), y(t))$, if the statement is false, then there must be $i, j \in \{1, 2, \dots, N\}$ and an interval $[T-\tau, T]$ such that $x_i(t) < x_j(t)$ for all $t \in [T-\tau, T)$ and $x_i(T) = x_j(T)$ for some positive τ. Due to the continuity of the opinions, we can further assume that $x_i(t) - x_j(t) > -d$ for $t \in [T-\tau, T]$ if τ is small enough. If we calculate the derivative of the opinion difference, we get that for almost all $t \in [T-\tau, T)$:

$$\begin{aligned}
&\frac{d}{dt}(x_i(t) - x_j(t)) \\
&\quad = \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Bigg(\sum_{l \in \mathcal{N}_i^H} (x_l(t) - x_i(t)) + \sum_{k \in \tilde{\mathcal{N}}_i^H} (y_k(t) - x_i(t)) \\
&\qquad - \Bigg(\sum_{m \in \mathcal{N}_j^H} (x_m(t) - x_j(t)) + \sum_{s \in \tilde{\mathcal{N}}_j^H} (y_s(t) - x_j(t)) \Bigg) \Bigg) \\
&\quad = - \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Big(|\mathcal{N}_{ij}^H| + |\tilde{\mathcal{N}}_{ij}^H| (x_i(t) - x_j(t)) \Big) \\
&\qquad + \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Bigg(\sum_{l \in \mathcal{N}_i^H \backslash \mathcal{N}_j^H} (x_l(t) - x_i(t)) \\
&\qquad + \sum_{k \in \tilde{\mathcal{N}}_i^H \backslash \tilde{\mathcal{N}}_j^H} (y_k(t) - x_i(t)) \Bigg) \\
&\qquad - \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Bigg(\sum_{m \in \mathcal{N}_j^H \backslash \mathcal{N}_i^H} (x_m(t) - x_j(t)) \\
&\qquad + \sum_{s \in \tilde{\mathcal{N}}_j^H \backslash \tilde{\mathcal{N}}_i^H} (y_s(t) - x_j(t)) \Bigg).
\end{aligned}$$

Note that by the assumption of $0 > x_i(t) - x_j(t) > -d$, we always have:

- for any element $l \in \mathcal{N}_i^H \setminus \mathcal{N}_j^H$, $x_l(t) < x_i(t)$
- for any element $k \in \tilde{\mathcal{N}}_i^H \setminus \tilde{\mathcal{N}}_j^H$, $y_k(t) < x_i(t)$
- for any element $m \in \mathcal{N}_j^H \setminus \mathcal{N}_i^H$, $x_m(t) > x_j(t)$
- for any element $s \in \tilde{\mathcal{N}}_j^H \setminus \tilde{\mathcal{N}}_i^H$, $y_s(t) > x_j(t)$

Hence, we can derive that

$$\begin{aligned}
\frac{d}{dt}(x_i(t) - x_j(t)) &\leq - \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Big(|\mathcal{N}_{ij}^H| + |\tilde{\mathcal{N}}_{ij}^H| (x_i(t) - x_j(t)) \Big) \\
&\leq - \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Big((N+M)(x_i(t) - x_j(t)) \Big) \\
&= -(N+M)(x_i(t) - x_j(t)),
\end{aligned}$$

since $|\mathcal{N}_{ij}^H| \leq N$ and $|\tilde{\mathcal{N}}_{ij}^H| \leq M$ always hold. By the comparison lemma[2],

$$x_i(T) - x_j(T) \leq e^{-(N+M)\tau}(x_i(T-\tau) - x_j(T-\tau)) < 0,$$

which contradict with the assumption that $x_i(T) = x_j(T)$ (see Fig. 1). □

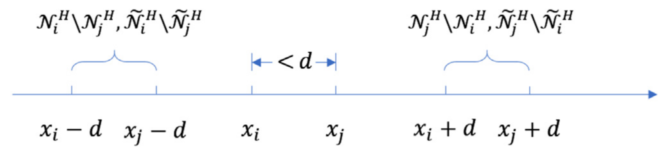

Fig. 1. Plot describing the above statements.

We introduce the following notations for later use:

$$x_{max}(t) = \max_i \{x_i(t)\}, \ x_{min}(t) = \min_i \{x_i(t)\}$$

$$y_{max}(t) = \max_k \{y_k(t)\}, \ y_{min}(t) = \min_k \{y_k(t)\}$$

Proposition 2.1 guarantees that the state $x(t)$ will preserve orders when time evolves. x_{min} will coincide with one of state variables that has the minimal initial value, and x_{max} will be one of the state variables with the maximal initial value. Thus, in the later use, we always consider that the notation *min* and *max* indicate specific vertices in the graph $G(x, y)$ for $x(t)$ variables. Note that this may not hold for $y(t)$.

3. Main results

Intuitively, if the signals $y_k(t)$'s generated by the exo-system, called exo-opinions later, change slow enough, the opinions of normal agents $x_i(t)$'s should keep following at least one of them. We can actually give a bound for the speed of the exo-system. Consider the problem (2) with a certain initial condition. Suppose that $|g_k(t, y)| \leq v < d$ holds for all t, all y and all k, then the agent with the smallest opinion should not be "left behind" by all the exo-opinions if it is connected to at least one of them initially. Namely, if $x_{min}(0) - y_{min}(0) > -d$ initially, then for any $t \geq 0$, we can find some $k \in V_y$ (depending on t) such that

$$x_{min}(t) - y_k(t) > -d.$$

This can be shown by contradiction. Assume that $x_{min}(0) - y_s(0) > -d$ for some $s \in V_y$. Suppose that at $t^* > 0$, it is the first time that $x_{min}(t^*) - y_s(t^*) \leq -d$ for all $s \in V_y$. Because of the continuity, there must be some k such that $x_{min}(t^*) - y_k(t^*) = -d$. For $\epsilon = \frac{d-v}{2} > 0$, we can find $\tau > 0$ small enough so that there exists a fixed $k \in V_y$ such that

$$\begin{aligned}
&-d < x_{min}(t) - y_k(t) \leq -d + \epsilon \text{ and} \\
&x_{min}(t) - y_s(t) \leq 0, \text{ for all } s \in V_y,
\end{aligned}$$

for all $t \in [t^* - \tau, t^*)$ and $x_{min}(t^*) - y_k(t^*) = -d$. If we keep the notation introduced in the previous section that $\tilde{\mathcal{N}}_{min}^H = \{k \in V_y : (min, k) \in E(x, y) \cup H\}$, we can derive that $k \in \tilde{\mathcal{N}}_{min}^H$ for all $t \in [t^* - \tau, t^*)$ and for all $H \subset \partial E(x, y)$.

For almost all $t \in [t^* - \tau, t^*)$, we have

$$\begin{aligned}
\frac{d}{dt}(x_{min}(t) - y_k(t)) = &\sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Bigg(\sum_{j \in \mathcal{N}_{min}^H} (x_j(t) - x_{min}(t)) \\
&+ \sum_{s \in \tilde{\mathcal{N}}_{min}^H} (y_s(t) - x_{min}(t)) \Bigg) \\
&- g_k(t^*, y(t)) \\
\geq &\sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \Big(y_k(t) - x_{min}(t) \Big) - v \\
\geq &d - \epsilon - v = \frac{d-v}{2} > 0,
\end{aligned}$$

[2] The integral form of Grönwall's inequality is needed here.

This implies that $x_{min} - y_k$ will not decrease during the interval $[t^* - \tau, t^*)$, which contradicts the assumption. We can derive a similar result for the maximum opinion x_{max}. Since it will be used frequently in the remaining of the paper, let us introduce the concept of "cover".

Definition 3.1. We say the opinion x is *covered* by the exo-opinions y if $x_{min} > y_k - d$ for some $k \in \{1, 2, \ldots, M\}$ and $x_{max} < y_s + d$ for some $s \in \{1, 2, \ldots, M\}$.

We can then summarize the derivation above in the following lemma:

Lemma 3.1. *For the system (2), if we assume that the initial normal opinions are covered by the initial exo-opinions and that the function g satisfies $|g_k(t, y)| \le v < d$, then for any $t \ge 0$ the opinions $x(t)$ remain covered by the exo-opinions $y(t)$.*

Remark. Lemma 3.1 states that if the exo-system evolves slow enough, the normal opinions will keep being covered by the exo-opinions, namely $x_i(t) \in (y_{min} - d, y_{max} + d)$ for all i. Note that the index of y_{min} and y_{max} may change over time.

Based on this result, the question that arises is whether we can design the exo-system so that the agents will reach consensus asymptotically. Since the coverage maintains if the exo-opinions move slowly enough, a natural hypothesis is that if the exo-opinions converge, i.e. $|y_k(t) - y_s(t)| \to 0$ as $t \to \infty$ for any k and s, we will have $x_{max}(t) - x_{min}(t) \to 0$ as $t \to \infty$. This hypothesis turns out to be true.

Theorem 3.2. *For the system (2), if the following hold:*

1. *x(0) is covered by y(0);*
2. *the function g satisfies $|g_k(t, y)| \le v < d$;*
3. *$y_k(t) - y_s(t) \to 0$ as $t \to \infty$ for any k and s,*

then $x_{max}(t) - x_{min}(t) \to 0$ as $t \to \infty$.

Proof. The proof will be divided into two phases:

(i). Firstly, we show that for $x_{min}(t)$ and $x_{max}(t)$, there exists a time such that both opinions will be and remain connected to all the exo-opinions after that time.

(ii). Secondly, we show that if both $x_{min}(t)$ and $x_{max}(t)$ remain connected to all the exo-opinions, they will reach consensus.

We will prove them in a reverse order due to their complexity. Before that, we can assume without loss of generality that the exo-opinions have already converged to a small difference between them, meaning that there exists a small $\epsilon > 0$ such that $|y_k(t) - y_s(t)| < \epsilon$ for all $t \ge 0$. The value of ϵ should be very small (at least smaller than d) and will be determined later. Lemma 3.1 guarantees that $x(t)$ is always covered by $y(t)$ for any $t \ge 0$, meaning that we can rewrite the system by introducing a new time variable that starts from a positive t.

Proof of (ii): If there exists $T_F \ge 0$ such that for $t \ge T_F$, both $|x_{min}(t) - y_k(t)| < d$ and $|x_{max}(t) - y_k(t)| < d$ hold for all k, and then for almost all $t \ge T_F$ we have

$$\begin{aligned}
&\frac{d}{dt}(x_{max}(t) - x_{min}(t)) \\
&= \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \left(\sum_{j \in \mathcal{N}_{max}^H} (x_j(t) - x_{max}(t)) + \sum_{k \in V_y} (y_k(t) - x_{max}(t)) \right) \\
&\quad - \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \left(\sum_{j \in \mathcal{N}_{min}^H} (x_j(t) - x_{min}(t)) - \sum_{k \in V_y} (y_k(t) - x_{min}(t)) \right) \\
&\le \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \left(\sum_{k \in V_y} (y_k(t) - x_{max}(t)) - \sum_{k \in V_y} (y_k(t) - x_{min}(t)) \right) \\
&= -M(x_{max}(t) - x_{min}(t))
\end{aligned}$$

By the comparison lemma, $x_{max}(t) - x_{min}(t) \le e^{-M(t-T_F)}(x_{max}(T_F) - x_{min}(T_F)) \to 0$ as $t \to \infty$. $\diamond$

Proof of (i): Since all the exo-opinions have already converged within a small distance $\epsilon < d$, an opinion $x_i(t)$ is connected to all of them if and only if $x_i(t) \in (y_{max} - d, y_{min} + d)$. We will show the statement (i) by the following argument:

Claim: we claim that for any $t > 0$, if $x_{min}(t) \le y_{max} - d + \epsilon$, then $\frac{d}{dt}(x_{min}(t) - y_k(t))$ is strictly larger than a positive constant for any k (if the derivative exists, which should be the case almost surely).

This implies that there must be a T_1 such that for any $t > T_1$, $x_{min}(t) > y_{max}(t) - d$. A similar argument can be made for x_{max} that there exists T_2 such that for any $t > T_2$, $x_{max}(t) < y_{min}(t) + d$. Therefore, we have

$$y_{max}(t) - d < x_{min}(t) < x_{max}(t) < y_{min}(t) + d$$

holds for any $t > \max\{T_1, T_2\}$, and hence both $x_{min}(t)$ and $x_{max}(t)$ will keep connected to all the exo-opinions. The remaining of the proof is to show the **Claim**.

Proof of the **Claim**: Suppose that at time $t \ge 0$,

$$x_{min}(t) \le y_{max}(t) - d + \epsilon. \tag{4}$$

Since the exo-opinions are clustered within distance ϵ, for any k we have

$$y_k(t) \in (y_{max}(t) - \epsilon, y_{max}(t)]. \tag{5}$$

If $\epsilon < \frac{d}{2}$, we additionally have $x_{min}(t) < y_k(t)$ for all $k \in V_y$. By Lemma 3.1, there always exists κ (depending on t) such that $x_{min} > y_\kappa - d$, meaning that $\kappa \in \tilde{\mathcal{N}}_{min}^H$. Moreover, by combining (4) and (5), we get

$$y_\kappa(t) - x_{min}(t) > d - 2\epsilon.$$

Now for all $k \in V_y$ we have that

$$\begin{aligned}
\frac{d}{dt}(x_{min}(t) - y_k(t)) &= \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \left(\sum_{j \in \mathcal{N}_{min}^H} (x_j(t) - x_{min}(t)) \right. \\
&\quad \left. + \sum_{s \in \tilde{\mathcal{N}}_{min}^H} (y_s(t) - x_{min}(t)) \right) - \dot{y}_k(T_1) \\
&> \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} (y_\kappa(t) - x_{min}(t)) - v \\
&> d - 2\epsilon - v
\end{aligned}$$

Now we let $\epsilon = \frac{d-v}{4} < \frac{d}{2}$, and hence get $\frac{d}{dt}(x_{min}(t) - y_k(t)) > \frac{d-v}{2} > 0$. $\diamond$

To summarize, if we start with the choice of $\epsilon = \frac{d-v}{4}$, then there is $T = \max\{T_1, T_2\} > 0$ such that for any $t > T$ it holds that $x_{max}(t) - x_{min}(t)$ converge to zero exponentially. Therefore, we can conclude that $x_{max} - x_{min} \to 0$ as $t \to \infty$. $\square$

Remark. Although Theorem 3.2 guarantees consensus of the opinions, it does not provide any description of the limit. Especially, we do not have that $x_i(t) \to y_k(t)$ in general. Nevertheless, we can give a bound for the synchronized opinions from the following corollary.

Corollary 3.3. *Let the assumptions hold in Theorem 3.2. Then for any $\delta > 0$ there exists $T > 0$ such that $|x_i(t) - y_k(t)| < \frac{v}{M} + \delta$ for all $i \in V_x$, $k \in V_y$, and all $t > T$.*

Proof. We will prove the corollary for x_{min} and x_{max}, and the rest opinions will be between these two and therefore also satisfy the statement. In fact, it is enough to show that for any $\delta > 0$, there is $T > 0$ such that $x_{min}(t) > y_k(t) - \frac{v}{M} - \delta$ and $x_{max}(t) < y_k(t) + \frac{v}{M} + \delta$ for any k and any $t > T$.

We can again assume, without loss of generality, that $|y_s(t) - y_k(y)| < \epsilon$ for all $t \geq 0$ for a given positive ϵ. From the proof Theorem 3.2, if $\epsilon \leq \frac{d-v}{4}$, there exists $T_1 > 0$ such that $x_{min}(t)$ keeps connected to all the exo-signals for $t > T_1$. For any specific $k \in V_y$ and almost all $t > T_1$,

$$\begin{aligned}\frac{d}{dt}(x_{min}(t) - y_k(t)) &= \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)}\Bigg(\sum_{j\in\mathcal{N}_{min}^H}(x_j(t) - x_{min}(t)) \\ &\quad + \sum_{s\in V_y}(y_s(t) - x_{min}(t))\Bigg) - \dot{y}_k(t) \\ &\geq \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)} \sum_{s\in V_y}\Big((y_s(t) - y_k(t)) \\ &\quad + (y_k(t) - x_{min}(t))\Big) - \dot{y}_k(t) \\ &\geq -M(x_{min}(t) - y_k(t)) - M\epsilon - v\end{aligned}$$

If we denote $z^*(t, \tau, z_0)$ the solution to the system

$$\begin{aligned}\dot{z}(t) &= -Mz(t) - (M\epsilon + v) \\ z(\tau) &= z_0,\end{aligned} \tag{6}$$

then it holds that

$$z^*(t, \tau, z_0) = e^{-M(t-\tau)}(z_0 + \epsilon + \frac{v}{M}) - \epsilon - \frac{v}{M} \to -\epsilon - \frac{v}{M}$$
as $t \to \infty$.

We can thus always find $T_{2,k} > 0$ such that $x_{min}(t) - y_k(t) \geq -2\epsilon - \frac{v}{M}$ for all $t > T_{2,k}$ due to the comparison lemma.[3] Therefore, if we set $\epsilon = \min\{\frac{d-v}{4}, \frac{\delta}{4}\}$, and $T_{min} = \max\{T_1, \max_k\{T_{2,k}\}\}$, we have

$$x_{min}(t) \geq y_k(t) - \frac{v}{M} - \frac{\delta}{2} > y_k(t) - \frac{v}{M} - \delta,$$

for all k, and for all $t > T_{min}$. A similar approach can be used to prove that $x_{max}(t) < y_k(t) + \frac{v}{M} + \delta$ for all k, and for all $t > T_{max}$ with the corresponding T_{max}. The corollary is proven by letting $T = \max\{T_{min}, T_{max}\}$. $\square$

In general, there exists a gap between $x(t)$ and $y(t)$ with width less, equal, or converging from above to $\frac{v}{M}$. Additionally, if the exo-signals converge to a constant value, then the gap disappears.

Corollary 3.4. *For the system (2), if we assume that*

1. *x(0) is covered by y(0);*
2. *the function g satisfies $|g_k(t, y)| \leq v < d$;*
3. *there exists $y^* \in \mathbb{R}$ such that $y_k(t) \to y^*$ as $t \to \infty$ for all k,*

then $x_i(t) \to y^*$ as $t \to \infty$ for all i.

Proof. From Theorem 3.2 we know that $x_i(t) \to x^*(t)$ as $t \to \infty$ for some $x^*(t)$. The only thing we need to show is that $x^*(t) \to y^*$ as $t \to \infty$. If we consider the two steps in the proof of Theorem 3.2, all of them will still hold since the assumptions of the theorem are satisfied. In (ii), for almost all $t \geq T_F$, we additionally have

$$\begin{aligned}\frac{d}{dt}(x_{min}(t) - y^*) &= \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)}\Bigg(\sum_{j\in\mathcal{N}_{min}^H}(x_j(t) - x_{min}(t)) \\ &\quad + \sum_{k\in V_y}(y_k(t) - x_{min}(t))\Bigg) \\ &\geq \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)} \sum_{k\in V_y}(y_k(t) - x_{min}(t)) \\ &= -M(x_{min}(t) - y^*) + \sum_{k\in V_y}(y_k(t) - y^*)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(x_{max}(t) - y^*) &= \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)}\Bigg(\sum_{j\in\mathcal{N}_{max}^H}(x_j(t) - x_{max}(t)) \\ &\quad + \sum_{k\in V_y}(y_k(t) - x_{max}(t))\Bigg) \\ &\leq \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)} \sum_{k\in V_y}(y_k(t) - x_{max}(t)) \\ &= -M(x_{max}(t) - y^*) + \sum_{k\in V_y}(y_k(t) - y^*)\end{aligned}$$

If we denote $z^*(t, z_0)$ as the solution to the system

$$\dot{z}(t) = -Mz(t) + u(t), \tag{7}$$

with the initial condition $z(0) = z_0$, where $u(t) = \sum_{k=1}^{m}(y_k(t) - y^*) \to 0$ as $t \to \infty$, then we claim that $z^*(t, z_0) \to 0$ as $t \to \infty$ for any initial condition. The claim will be proven in Appendix. Since both $|x_{min}(t) - y^*|$ and $|x_{max}(t) - y^*|$ are bounded, thanks to Lemma 3.1, by the comparison lemma we have

$$\begin{aligned}&z^*(t, x_{min}(0) - y^*) \leq x_{min}(t) - y^* \leq x_{max}(t) - y^* \\ &\quad \leq z^*(t, x_{max}(0) - y^*)\end{aligned}$$

Both z^* on the two sides of the inequality converge to zero. Hence, we have both $x_{min}(t)$ and $x_{max}(t)$ converge to y^*. $\square$

Although Theorem 3.2 requires the exo-opinions to reach consensus in order to synchronize the normal opinions, we could achieve the same goal with non-converging exo-opinions. In fact, as long as the exo-opinions are constrained in a bounded region after a certain time, we will still have a consensus result, which can be formulated by the following theorem.

Theorem 3.5. *For the system (2), if we assume that*

1. *x(0) is covered by y(0);*
2. *the function g satisfies $|g_k(t, y)| \leq v < d$;*
3. *there exist $a \in \mathbb{R}$, $r \in (0, d)$, and $T > 0$ such that $y_k(t) \in [a, a + r]$ for $t > T$, and for all k,*

then $x_{max}(t) - x_{min}(t) \to 0$ as $t \to \infty$.

Proof. The theorem can be proven by the two phase approach again:

(i). Firstly, we show that for $x_{min}(t)$ and $x_{max}(t)$, there exists a time such that both opinions will be and remain connected to all the exo-opinions after that time.

(ii). Secondly, we show that if both $x_{min}(t)$ and $x_{max}(t)$ remain connected to all the exo-opinions, they will reach consensus.

[3] We can use the comparison lemma or Grönwall's inequality here because $|x_{min} - y_k| < d$ for any k.

We only prove (i) here, and the proof of (ii) will be identical to that in the proof of Theorem 3.2.

We only consider $t > T$. $x_{min}(t)$ is connected to all the exo-opinions if it is in the interval $(a+r-d, a+d)$ since $y_k(t) \in [a, a+r]$ for all k. We will show that if $x_{min}(t) \leq a+r-d+\epsilon$ for some small $\epsilon > 0$, then $\frac{d}{dt}x_{min}(t) \geq c > 0$ for some constant c (if the derivative exists), meaning that if t is large enough, $x_{min}(t)$ will keep being larger than $a+r-d$.

Suppose $x_{min}(t) \leq a+r-d+\epsilon$. If we further assume that $\epsilon \leq d-r$, we also have $x_{min}(t) \leq a \leq y_k(t)$ for all k. Because $x(t)$ is covered by $y(t)$ due to Lemma 3.1, there exists at least one exo-opinion $y_\kappa(t)$ connected to it, meaning that $\kappa \in \tilde{\mathcal{N}}^H_{min}$. If the derivative exists at t, we have

$$\begin{aligned}\frac{d}{dt}x_{min}(t) &= \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)} \Bigg(\sum_{j\in\mathcal{N}^H_{min}} (x_j(t)-x_{min}(t)) \\ &\quad + \sum_{k\in\tilde{\mathcal{N}}^H_{min}} (y_k(t)-x_{min}(t)) \Bigg) \\ &\geq \sum_{H\subset\partial E(x,y)} \alpha_H^{(x,y)} \Big(y_\kappa(t) - x_{min}(t) \Big) \\ &\geq a-(a+r-d+\epsilon) = d-r-\epsilon\end{aligned}$$

If we choose $\epsilon = \frac{d-r}{2}$, then we have

$$\frac{d}{dt}x_{min}(t) > \frac{d-r}{2} > 0. \tag{8}$$

There thus must exist a T_1 such that $x_{min}(t) > a+r-d$ for all $t > T_1$. We can derive a similar result for $x_{max}(t)$ so that there exists a T_2 such that $x_{max}(t) \leq a+d$ for all $t > T_2$. Combining these two we get that

$$x_i(t) \in [x_{min}(t), x_{max}(t)] \subset (a+r-d, a+d)$$

for all $t > \max\{T_1, T_2\}$ and thus stays connected to all the exo-opinions for all i. □

Remark. In Theorem 3.5, the bound for r is tight, meaning that we cannot draw the same conclusion for $r = d$. If $r = d$, we can only derive that $\frac{d}{dt}x_{min}(t) \geq 0$ from (8) and there may not exist such T_1 in the proof. There is also a counter example that we do not have a consensus result for $r = d$. For instance two static exo-opinions with distance d and two normal opinions converge to them separately from outside. On the other hand, if we only have the condition that $|y_k(t)-y_s(t)| \leq r < d$ for t large enough without the constraint of a fixed bound, there can be situations that the normal opinions $x(t)$ keep moving on a certain formation together with the exo-opinions while keeping fixed distance among each other.

4. Numerical examples

There are three numerical experiments carried on in this section to test and illustrate the results of Section 3. In all three simulations, the following settings are shared. We let $N = 200$ initial opinions randomly distributed by a uniform distribution on the interval $[0, 1]$. The confidence range d among the agents is set to be 0.05. In order to reduce the computational burden, there are only $M = 2$ exo-opinions, which is the smallest amount of exo-opinions that can cover the initial opinions. In each of the three experiments, we choose different g functions so that the exo-opinions will satisfy the other assumptions and test the evolution of all the normal opinions. We use ode45 in Matlab as the integrator. In general, Matlab does not have a guarantee for the performance in terms of integrating discontinuous functions. We lower the tolerance to 10^{-6} and the integrator performs well enough comparing with a fixed step-size integrator. All the following simulations are carried out in this setting.

4.1. Example one

Set the exo-system as:

$$\dot{y}_k = \alpha(\frac{1}{2} + \frac{1}{2}\sin(2dt) - y_k), \text{ for } k = 1, 2, \tag{9}$$

with the initial condition

$$y_1(0) = 0, \; y_2(0) = 1,$$

for a given constant parameter α. Then both exo-opinions will track the signal $\frac{1}{2} + \frac{1}{2}\sin(2dt)$ for positive α. Meanwhile we need to choose α small enough in order to fulfill the assumption that $|\dot{y}| \leq v < d$. If the exo-signals remain in the interval $[0, 1]$, we have

$$|\dot{y}_k| \leq \alpha(\frac{1}{2} + \frac{1}{2} + 1) = 2\alpha.$$

We can choose any $\alpha < \frac{d}{2}$. Since $d = 0.05$ in the simulation, α is chosen to be 0.024 here.

According to Theorem 3.2, since the exo-opinions will track and asymptotically converge to the same signal, the normal opinions x_i must reach consensus. Fig. 2 shows the numerical solution to the differential equation (2) for the given initial conditions. The blue solid curves represent the opinions x_i and the red dashed curves are the exo-opinions; x_i's rapidly form several clusters for the first few seconds but follow the exo-opinions when and after they get close. The normal opinions converge to the same curve after a while although the curve does not coincide with the exo-opinion. There is a small gap between them.

4.2. Example two

We test Corollary 3.4 by letting the exo-signal converge to a constant value in this experiment. Set the exo-system as:

$$\begin{aligned}\dot{y}_i &= \beta(\frac{1}{2} - y_i), \\ y_1(0) &= 0, \; y_2(0) = 1,\end{aligned} \tag{10}$$

so that the both exo-signals will converge to the value $\frac{1}{2}$. According to Corollary 3.4, x_i will also converge to $\frac{1}{2}$ for all i. Fig. 3 illustrates the consensus result.

4.3. Example three

Theorem 3.5 can guarantee consensus even for non-converging exo-opinions. We simulate the scenario that the exo-opinions keep oscillating in a small region by choosing the exo-system as:

$$\begin{aligned}\dot{y}_1 &= \frac{d}{2}(\frac{1}{2} + 2d\sin(2d(t+\frac{\pi}{3})) - y_1) \\ \dot{y}_2 &= \frac{d}{2}(\frac{1}{2} + 2d\sin(2dt) - y_2) \\ y_1(0) &= 0, \; y_2(0) = 1.\end{aligned} \tag{11}$$

The two signals generated by this system will be constrained in the interval $(\frac{1}{2} - \frac{d}{2}, \frac{1}{2} + \frac{d}{2})$ when time is large enough, and keep a phase difference between each other. Fig. 4 shows that the normal opinions converge to the same curve that is also oscillating with the same frequency but a much smaller amplitude.

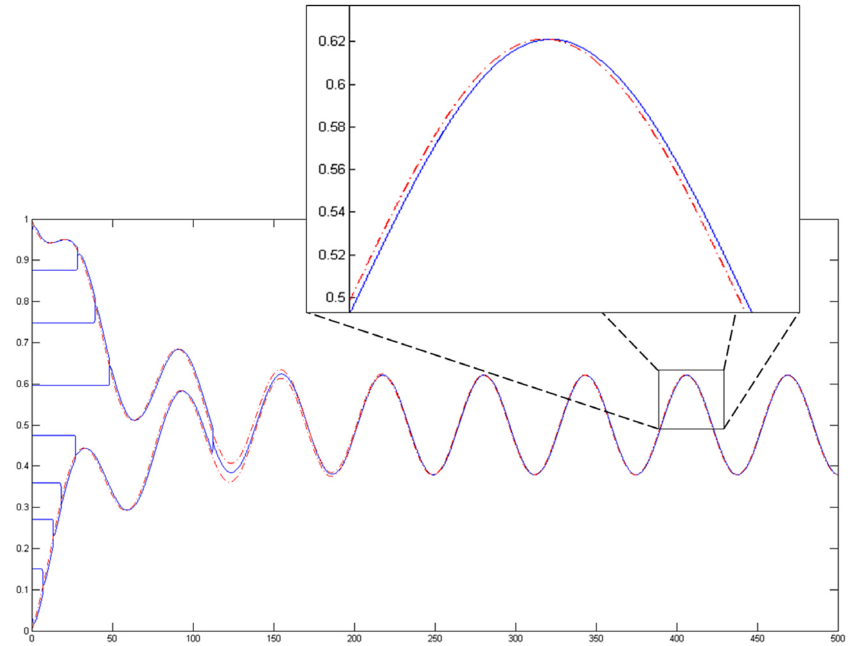

Fig. 2. Evolution of the normal opinions x_i (solid blue curves) and exo-opinions y_k (dashed red curves) with the system (2) and the choice of the exo-system (9). The normal opinions reach consensus asymptotically while keeping a small gap to the exo-opinions.

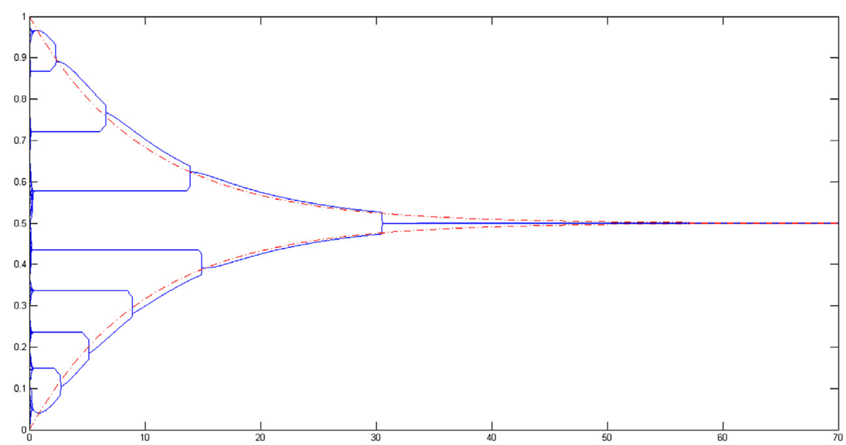

Fig. 3. Evolution of the normal opinions x_i (solid blue curves) and exo-opinions y_k (dashed red curves) with the system (2) and the choice of the exo-system (10). The normal opinions reach consensus and converge to the value where the exo-opinions converge.

Remark. The normal opinions converge to different clusters very fast in the beginning of the simulations although initially they should be randomly distributed in the interval [0, 1]. Due to numerical errors from the ode solver in MATLAB, the normal opinions are actually oscillating if zoomed in. If the tolerance of the ODE solver is decreased, the amplitude of the oscillation will also be reduced, while the computational cost will be increased.

5. Conclusions and future works

With the help of an exo-system, the opinion consensus with a bounded confidence model has been analyzed and guaranteed. Briefly speaking, if the exo-opinions initially spread widely enough, move slowly enough, and converge close enough as time evolves, the normal opinions will be guided to reach consensus. The specific conditions of the exo-system for guaranteeing consensus are given by the two theorems and we provide two theorems. These conditions are sufficient but not necessary. For exo-opinions that do not converge to constant values, there are in general gaps between the synchronized opinions and exo-opinions.

The exo-system can also help to regulate the opinions not only to a single cluster but also to other formations. The inverse problem is: for a given opinion cluster formation, whether we can always find an exo-system such that the normal opinions converge to that formation with an arbitrary initial distribution. The inverse problem is equivalently important and a topic of current research.

Appendix

Proof of the claim in Corollary 3.4. For the system

$$\dot{z}(t) = -Mz(t) + u(t),\ z(t) \in \mathbb{R},\ M > 0 \tag{12}$$

with $u(t)$ being continuous, having bounded derivative and converging to zero, we want to show that the solution $z^*(t, z_0)$ also converges to zero with any initial condition $z(0) = z_0$.

Since $u(t)$ converges to zero, for any $\delta > 0$, there exists a $T > 0$ such that $|u(t)| \leq \delta$ for all $t > T$. We can also assume that

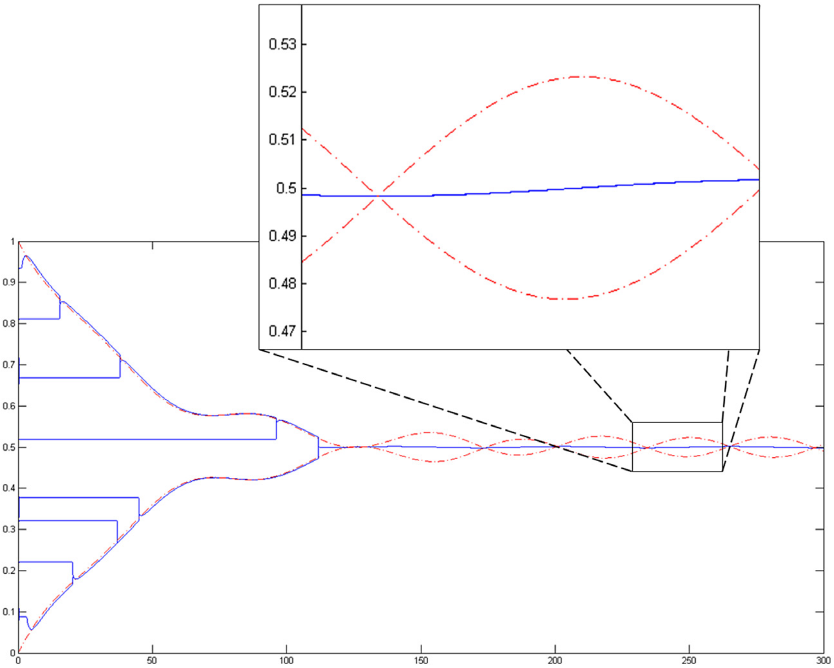

Fig. 4. Evolution of the normal opinions x_i (solid blue curves) and exo-opinions y_k (dashed red curves) with the system (2) and the choice of the exo-system (11). Although the exo-opinions do not converge to the same curve, the normal opinions reach consensus.

$|u(t)| \leq Ut$ for all t because $u(t)$ has bounded derivative. According to the solution to the linear systems we have

$$\begin{aligned}
|z^*(t, z_0)| &= \left| e^{-Mt} z_0 + \int_0^t e^{-M(t-s)} u(s) ds \right| \\
&\leq |e^{-Mt} z_0| + \left| \int_0^T e^{-M(t-s)} u(s) ds \right| + \left| \int_T^t e^{-M(t-s)} u(s) ds \right| \\
&\leq |e^{-Mt} z_0| + \left| \int_0^T e^{-M(t-s)} U s ds \right| + \left| \int_T^t e^{-M(t-s)} \delta ds \right| \\
&\leq e^{-Mt} \left(|z_0| + \frac{U}{M} \left(T e^{MT} - \frac{1}{M} e^{MT} + 1 \right) \right) \\
&\quad + \frac{1}{M} (1 - e^{-M(t-T)}) \delta \\
&\leq e^{-Mt} \left(|z_0| + \frac{1}{M} \left(T e^{MT} + 1 \right) \right) + \frac{\delta}{M}.
\end{aligned}$$

For any $\epsilon > 0$, we let $\delta = \frac{M\epsilon}{2}$ and get a corresponding T. We can then have

$$T' = \max \left\{ \frac{1}{M} \ln \frac{2(M|z_0| + Te^{MT} + 1)}{\epsilon}, 0 \right\} \geq 0$$

so that for all $t > T'$ we have $e^{Mt} \left(|z_0| + \frac{1}{M} \left(Te^{MT} + 1 \right) \right) < \frac{\epsilon}{2}$. As a result, we have

$$|z^*(t, z_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $t > T'$ and thus $z^*(t, z_0)$ converges to zero for any z_0. □

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