Explicit computation of sampling period in periodic event-triggered multi-agent control under limited data rate

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***Abstract*—This paper investigates the coordination of nonlinear sampled-data multi-agent systems subject to data rate constraint. The purpose is to design resource-efficient communication and control strategies that guarantee exponential synchronization. Two implementation scenarios are considered, the period time-triggered control and the period event-triggered control. One of the main difficulties of the problem is to obtain an explicit formula for the maximum allowable sampling period (MASP). To this end, an approach on finding the MASP for period time-triggered control is proposed first. Then, an asynchronous period event-triggered control strategy is formulated, a communication function and a control function are designed for each agent to determine respectively whether or not the sampled state and control input should be transmitted at each sampling instant. Finally, the constraint of limited data rate is considered. An observer-based encoder-decoder and a finite-level quantizer are designed respectively for the Sensor-Controller communication and the Controller-Actuator communication such that certain constraint on the data rate is satisfied. It is shown that exponential synchronization can still be achieved in the presence of data rate constraint. A simulation example is given to illustrate the effectiveness of the theoretical results.**

***Index Terms*—Sampled-data; nonlinear multi-agent systems; periodic event-triggered control; limited data rate**

I. Introduction

THE emerging Internet of Things technology enables the interconnection of numerous smart devices for the realization of the cyber-physical systems vision. To improve efficiency, flexibility and reliability of such large-scale applications, the current technological trend is to integrate sensing, computation, communication, and control into different levels of machine/factory operations and information processes with shared computational, communication and control resources [1], [2]. Efficient usage of these resources is therefore a central issue in cyber-physical systems design.

In such applications, the communication and control actions are carried out in digital platforms, where sampling operation is one of the indispensable steps to implement the digital signal. Design of controllers for sampled-data systems is often carried out by using the emulation approach, in which a continuous-time controller is designed for a continuous-time plant ignoring sampling and then the controller is discretized and implemented digitally [3]. It is obvious that this approach can be successful only if the sampling period is sufficiently small. However, a small sampling period may result in unnecessary high workloads in both Sensor-Controller communication and Controller-Actuator communication when communication resources could be more usefully assigned to some other tasks. These limitations have resulted in a recent interest on event-triggered communication (ETCm) and event-triggered control (ETCt)[1] [4], [6]–[10], [27]. However, most of the existing ETCm/ETCt paradigms require the triggering condition to be monitored continuously [5]–[8] or partially continuously [9], [10], which may result in excessive use of sensing and computational resources. Moreover, different from periodic sampled-data control (which will be called time-triggered control (TTC) in the following), in which the devices (such as sensors, controllers and actuators) are activated only at the discrete sampling instants, in the ETCm/ETCt mechanism, it is necessary for all devices to be activated all the time, which increases the energy consumption and thus reduces the lifespan of those devices. To address these problems, periodic event-triggered control (PETC) has been proposed as a solution [11]–[24]. It combines the idea of TTC and ETC, in which the triggering condition is monitored periodically at the predefined discrete sampling time sequence, which allows to achieve a balance between TTC and ETC. Several problems are still open in the area of PETC, and one of them lies in finding the minimum inter-event time (sampling period) [26].

In the area of PETC, most of the effort has been devoted to the stabilization of a single agent system [12]–[19], while the cooperation in the multiple-agent case, the so-called multi-agent systems (MAS) has not been considered to the same extent. Results on the design of PETC strategy for MAS with single-integrator dynamics are presented in [4], [20], [21]. For MAS with general linear dynamics or Lipchitz nonlinear dynamics, some recent results are reported in [22], [23] and [24], respectively. However, to the best of our knowledge, except for single-integrator MAS, the computation of the maximum allowable sampling period (MASP) for MAS is typically not carried out [23] or given by solving a set of LMIs [22], [24] without resulting in an explicit formula. We note that it is hard in general to find the explicit formula of MASP for PETC of general linear and nonlinear MAS. In addition, most of the previous proposed PETC strategies are

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[1] In most literatures, both ETCm and ETCt are called ETC. In this paper, we try to distinguish between the two. By ETCm, we refer to Sensor-Controller communication and the agent-to-agent communication. By ETCt, we refer to the Controller-Actuator communication.

synchronous in the sense that the controller of each agent has to be updated whenever a new state of itself as well as all its neighbors' is transmitted [22]–[24], which may result in very frequent Controller-Actuator communication as the number of neighbors is increasing. Moreover, ETC/PETC for distributed MAS is largely unexplored when in presence of communication constraint. As well known, in a digital communication channel, the data is first quantized, encoded into a binary sequence and then sent out. Any real network has limited channel capacity, therefore, only a finite number of bits of data can be transmitted [28], [29]. An overview of recent advances in ETC/PETC of MAS can be referred to the survey paper [25].

Motivated by the above considerations, this paper investigates the synchronization of nonlinear sampled-data MAS under limited data rate. Firstly, a TTC strategy is proposed for nonlinear MAS that guarantees exponential synchronization, where an explicit computation of the MASP is provided. Then, a PETC strategy is formulated, where communication and control functions are designed for each agent such that both Sensor-Controller communication and Controller-Actuator communication are reduced. Finally, the constraint of limited data rate is taken into account. Our approach on finding the MASP is motivated by the results in [3], [37], in which the stabilization of nonlinear networked control system was considered. However, it is far from trivial when the cooperation of MAS and the event-triggered control strategy are considered. The organization and contributions of this paper are summarized below.

In Section II, notation and some necessary preliminaries on graph theory and uniform quantizer are provided. In Section III, the class of nonlinear MAS is introduced. An explicit formula of the MASP that guarantees exponential synchronization of the sampled-data nonlinear MAS is presented. Then, in Section IV, a PETC strategy is formulated. Based on the approach developed in Section III, the MASP for PETC is obtained, and communication and control functions are designed for each agent such that exponential synchronization of the nonlinear MAS is guaranteed. Finally, in Section V, PETC of nonlinear MAS is investigated under limited data rate. An observer-based encoder-decoder with finite-level quantization is designed for the Sensor-Controller communication and a finite-level quantizer with scaling is designed for the Controller-Actuator communication, such that certain constraint on the data rate is satisfied. It is shown that exponential synchronization of the nonlinear MAS can still be achieved in the presence of data rate constraint. A simulation example is provided in Section VI for illustration of the theoretical developments and Section VII concludes the paper.

II. PRELIMINARIES

A. Notation

Let $\mathbb{R} := (-\infty,\infty)$, $\mathbb{R}_{\geq 0} := [0,\infty)$, $\mathbb{R}_{>0} := (0,\infty)$, $\mathbb{Z}_{>0} := \{1,2,\ldots\}$ and $\mathbb{Z}_{\geq 0} := \{0,1,2,\ldots\}$. Denote $\mathbb{R}^n$ as the $n-$dimension real vector space, $\mathbb{R}^{n\times m}$ as the $n\times m$ real matrix space. I_n is the identity matrix of order n and 1_n is the column vector of order n with all entries equal to one. For $(x_1,x_2,\ldots,x_m)\in\mathbb{R}^{n_1+n_2+\cdots+n_m}$, the notation $(x_1,x_2,\ldots,x_m)$ stands for $[x_1^T,x_2^T,\ldots,x_m^T]^T$. Let $\|x\|$ and $\|A\|$ be the Euclidean norm of vector x and matrix A, respectively. In addition, we use $\cup$ to denote the logical operator AND and $\cap$ the logical operator OR.

For Lipchitz functions (that are not necessarily differentiable everywhere), we will use the Clarke derivative which is defined as follows. For a Lipchitz function $R:\mathbb{R}^n\to\mathbb{R}^n$ and a vector $v\in\mathbb{R}^n$, $R^\circ(x,v) := \limsup_{h\to 0^+,y\to x}\frac{R(y+hv)-R(y)}{h}$, which corresponds to the derivative when R is continuously differentiable. We define the generalized gradient of $f:\mathbb{R}^n\to\mathbb{R}^n$ at x along v as: $\nabla_v f(x) := \{\zeta\in\mathbb{R}^n : f^\circ(x,v)\geq\langle\zeta,v\rangle,\forall v\in\mathbb{R}^n\}$, that matches the classical notion of gradient when f is differentiable.

B. Graph Theory

Let $\mathscr{G}=\{\mathscr{V},\mathscr{E}\}$ be a graph of order n with the set of nodes $\mathscr{V}=1,2,\ldots,N$, and $\mathscr{E}\subseteq\{(i,j):i,j\in\mathscr{V},j\neq i\}$ being the set of edges. If $(j,i)\in\mathscr{E}$, then node j is called a neighbor of node i and node j can receive information from node i. The neighboring set of node i is denoted by $\mathscr{N}_i=\{j\in\mathscr{E}|(j,i)\in\mathscr{E}\}$ and $\mathscr{N}_i^+=\mathscr{N}_i\cup\{i\}$. The adjacency matrix is denoted by $\mathscr{A}=(a_{ij})_{N\times N}$ and is given by $a_{ij}=1$, if $(j,i)\in\mathscr{E}$, otherwise $a_{ij}=0$. A graph is undirected if $(i,j)\in\mathscr{E}\Leftrightarrow(j,i)\in\mathscr{E}$, and a graph is connected if for every pair of nodes i,j, there exists a path which connects i and j, where a path is an ordered list of edges such that the head of each edge is equal to the tail of the following edge. Let $\mathscr{D}=(d_{ij})_{N\times N}$ represent the degree matrix which is a diagonal matrix with entries $d_{ii}=\sum_{j=1,j\neq i}^N a_{ij}$. Then the Laplacian matrix of the graph $\mathscr{G}$ is defined as $L=(l_{ij})_{N\times N}=\mathscr{D}-\mathscr{A}$.

Assumption 1: The graph $\mathscr{G}$ considered in this paper is undirected and connected.

C. Uniform Quantizer

A finite-level uniform quantizer is a map $Q_{\Delta,M}:\mathbb{R}\to\mathbb{R}$ such that

$$Q_{\Delta,M}(y)=\begin{cases}\Delta\left\lfloor\dfrac{y}{\Delta}+\dfrac{1}{2}\right\rfloor, & 0\leq y\leq M\Delta\\ M\Delta, & y>M\Delta\\ -Q_{\Delta,M}(-y), & y<0\end{cases}\tag{1}$$

where $\Delta>0$ is the quantization interval, and M is the number of quantization levels. Note that as long as the quantizer is not saturated ($|y|\leq M\Delta$), the quantization error is always bounded by $\Delta/2$, namely, $\left|Q_{\Delta,M}(y)-y\right|\leq\Delta/2$.

The above definition of the scalar-valued uniform quantizer can be easily extended to its counterpart of the vector-valued case. For any $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, define the vector uniform quantizer $Q_{\Delta,M}(x):\mathbb{R}^n\to\mathbb{R}^n$ to be $Q_{\Delta,M}(x)\triangleq(Q_{\Delta,M}(x_1),\ldots,Q_{\Delta,M}(x_n))$.

III. TTC OF NONLINEAR MAS

A. Model description

Consider nonlinear sampled-data MAS with N agents, and the dynamics of each agent can be described in the following:

$$\dot{x}_i(t)=f(x_i(t))+\hat{u}_i(t),\quad i=1,2,\ldots,N,\tag{2}$$

with

$$\hat{u}_i(t) = \sum_{j\in\mathcal{N}_i} \psi_{ij}(x_j(t_l) - x_i(t_l)), \quad t \in [t_l, t_{l+1}), \tag{3}$$

where $x_i \in \mathbb{R}^n, \hat{u}_i \in \mathbb{R}^n$ are respectively the state and the control input of the ith agent, $t_l = lh, l \in \mathbb{Z}_{\geq 0}$ is the increasing sampling sequence and $h > 0$ is the common sampling period.

Remark 1: For the sake of simplicity we analyze the case of periodic sampling, i.e., $t_{l+1} - t_l \equiv h$. Note that it is possible to consider the more general problem of an aperiodic sampling satisfying $t_{l+1} - t_l \leq h, \forall l$.

Assumption 2: The function f is Lipchitz continuous with Lipchitz constant $\rho_1 > 0$ and $f(0) = 0$.

Definition 1: The function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be in sector $[l_1, l_2]$ if for all $q \in \mathbb{R}^n$, one has

$$(q^T\phi(q) - l_1 q^T q)(q^T\phi(q) - l_2 q^T q) \leq 0.$$

Assumption 3: The functions $\psi_{ij} : \mathbb{R}^n \to \mathbb{R}^n, (i, j) \in \mathcal{E}$ are required to satisfy the following conditions:

a) for any $x_i, x_j \in \mathbb{R}^n$, one has $\psi_{ij}(x_{ji}) = -\psi_{ji}(x_{ij})$, where $x_{ij} = x_i - x_j$;

b) there exists a constant $K > 0$ such that for any $x_{ij} \in \mathbb{R}^n$, one has $x_{ij}^T \psi_{ji}(x_{ij}) \geq K x_{ij}^T x_{ij}$;

c) ψ_{ij} are globally Lipchitz continuous functions with Lipchitz constant $\rho_2 > 0$, that is, $\|\psi_{ij}(x') - \psi_{ij}(x'')\| \leq \rho_2 \|x' - x''\|$ for any $x', x'' \in \mathbb{R}^n$, and $\psi_{ij}(0) = 0, \forall (i, j) \in \mathcal{E}$.

According to item c), one has $\|\psi_{ji}(x_{ij})\| \leq \rho_2 \|x_{ij}\|$ and thus $x_{ij}^T \psi_{ji}(x_{ij}) \leq \rho_2 \|x_{ij}\|^2$. Combining with item b), one can further have $K\|x_{ij}\|^2 \leq x_{ij}^T \psi_{ji}(x_{ij}) \leq \rho_2 \|x_{ij}\|^2$. That is to say, the function ψ_{ji} is in sector $[K, \rho_2], \forall (j, i) \in \mathcal{E}$.

B. Convergence result

Before proceeding, the following notations are introduced. Define

$$\begin{aligned} y_i(t_l) &= x_i(t_l), \\ y_i(t) &= y_i(t_l) + \int_{t_l}^{t} f(y_i(s))ds, \quad t \in (t_l, t_{l+1}) \end{aligned} \tag{4}$$

and $u_i(t) = \sum_{j\in\mathcal{N}_i} \psi_{ij}(y_j(t) - y_i(t))$. The sampling-induced errors of agent i are defined as

$$e_{x_i}(t) = y_i(t) - x_i(t) \quad e_{u_i}(t) = \hat{u}_i(t) - u_i(t), \quad \forall i, \tag{5}$$

and then one has $\|e_{x_i}(t_l)\| = 0$ and $\|e_{u_i}(t_l)\| = 0$ for all t_l.

The control input (3) can then be rewritten as

$$\hat{u}_i(t) = \sum_{j\in\mathcal{N}_i} \psi_{ij}(x_{ji}(t) + e_{x_{ji}}(t)) + e_{u_i}(t),$$

where $x_{ji}(t) = x_j(t) - x_i(t)$ and $e_{x_{ji}}(t) = e_{x_j}(t) - e_{x_i}(t)$.

Let the average state of all agents be $\bar{x}(t) = \sum_{i=1}^N x_i(t)/N$. Define the state error between agent i and the average as $\xi_i(t) = x_i(t) - \bar{x}(t), \forall i$. Since the graph $\mathcal{G}$ is undirected and $\psi_{ij}(x_{ji}) = -\psi_{ij}(x_{ij})$, one has

$$\dot{\xi}_i(t) = f(x_i(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t)) + \hat{u}_i(t), \quad i = 1, 2, \ldots, N, \tag{6}$$

where $\hat{u}_i(t)$ can be equivalently rewritten as

$$\hat{u}_i(t) = \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_{ji}(t) + e_{x_{ji}}(t)) + e_{u_i}(t), \tag{7}$$

where $\xi_{ji}(t) = \xi_j(t) - \xi_i(t)$.

Definition 2: Synchronization of the nonlinear MAS (2) is said to be achieved exponentially, if there exist positive constants κ, μ such that the error vector satisfies

$$\|\xi_i(t)\| \leq \kappa e^{-\mu t}, \quad \forall i$$

and for all $t \geq 0$. The constant μ is called the *convergence rate* and the constant κ is called the *convergence coefficient*.

One can see that the exponential synchronization of the nonlinear MAS (2) is achieved if and only if the stability of the error system (6) is achieved exponentially. Therefore, in the following, the stability of the error system (6) is investigated.

Let ξ, e_x, e_u be the concatenated vectors of ξ_i, e_{x_i}, e_{u_i}, respectively, where we dropped the time argument t for notational convenience. Define $\|z\|' := d\|z\|/dt, \forall z$. Then, we get the following Propositions.

Proposition 1: Let the function $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be $V(z) = z^2$, then the derivative of $V(\|\xi\|)$ along the trajectories of (6) satisfies

$$\dot{V}(\|\xi\|) \leq -\hat{L}\|\xi\|^2 + \frac{1}{a_1}\gamma^2\|e_x\|^2 + \frac{1}{a_1}\|e_u\|^2, \tag{8}$$

where $\hat{L} = 2K\lambda_2(L) - 4\rho_1 - 2a_1, \gamma = \rho_2\sqrt{2\hat{l}^2 + 2N\hat{l}}, \hat{l} = \max_i\{d_{ii}\} \leq N - 1$ for some $a_1 > 0$, where $\lambda_2(L)$ is the algebraic connectivity of the Laplacian matrix L.

Proposition 2: For all $t \in [t_l, t_{l+1})$, the following inequalities hold

$$\|e_x\|' \leq 2\gamma\|\xi\| + (\rho_1 + 2\gamma)\|e_x\| + \sqrt{2}\|e_u\| \tag{9a}$$

$$\|e_u\|' \leq \sqrt{2}\rho_1\gamma\|\xi\| + \sqrt{2}\rho_1\gamma\|e_x\|. \tag{9b}$$

Now, we provide an explicit computation of the MASP h^* for the nonlinear sampled-data MAS (2) such that synchronization can be achieved exponentially. Firstly, we introduce two auxiliary functions ϕ_1, ϕ_2. Let $\phi_1(\tau) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\phi_2(\tau) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be such that

$$\frac{d\phi_1(\tau)}{d\tau} = -m_1\phi_1^2(\tau) - m_2\phi_1(\tau) - m_3 \tag{10a}$$

$$\frac{d\phi_2(\tau)}{d\tau} = -n_1\phi_2^2(\tau) - n_2\phi_2(\tau) - n_3 \tag{10b}$$

where

$$\begin{aligned} m_1 &= \frac{4\gamma^3}{{a_1}^2} + a_2, \quad m_2 = 2\rho_1 + 4\gamma + (\hat{L} - 2a_1), \\ m_3 &= \gamma + a_2, \quad n_1 = \frac{\gamma^2\rho_1^2}{a_1^2} + \frac{2\gamma\rho_1^2}{a_2}, \\ n_2 &= \hat{L} - 2a_1, \quad n_3 = \frac{2\gamma}{a_2} + 1 \end{aligned} \tag{11}$$

for some $a_2 > 0$, and the constants $\hat{L}, a_1, \gamma$ given in Proposition 1. To state our main results, we further introduce the following function:

$$\mathcal{T}(x,y,z)=\begin{cases}\dfrac{2}{r}\arctan\left(\dfrac{r}{y}\right) & 4xz>y^2\\ \dfrac{2}{y} & 4xz=y^2\\ \dfrac{2}{r}\mathrm{arctanh}\left(\dfrac{r}{y}\right) & 4xz<y^2\end{cases}$$

where $r:=\sqrt{|4xz-y^2|}$. Then, the MASP h^* is selected as

$$h^*=\min\{\mathcal{T}(m_1,m_2,m_3),\mathcal{T}(n_1,n_2,n_3)\}. \tag{12}$$

Theorem 1: Consider the nonlinear MAS (2) with the control input (3), where the sampling period h is chosen from $h\in(0,h^*)$. Suppose Assumptions 1-3 hold with $K>2\rho_1/\lambda_2(L)$. Then, the synchronization of the nonlinear MAS (2) is achieved exponentially with convergence rate $K\lambda_2(L)-2\rho_1-2a_1$ and convergence coefficient $\|\xi(0)\|$, where $a_1\in(0,(K\lambda_2(L)-2\rho_1)/2)$.

Remark 2: In this paper, the communication graph is assumed to be undirected. Nevertheless, it is worth to point out that the results obtained in this paper are applicable to the case of a directed graph with a spanning tree.

Remark 3: Two constants, i.e., a_1,a_2 are introduced for the computation of MASP. It can be seen that both the MASP and the convergence rate are related to the constant a_1. Moreover, a small value of a_1 means faster convergence, however, the MASP will be smaller. Therefore, a_1 can be used as a trade-off between the rate of communication and the rate of convergence. In addition, from (10a) and (10b), one can see that a small value of a_2 means a bigger $\mathcal{T}(m_1,m_2,m_3)$ and a smaller $\mathcal{T}(n_1,n_2,n_3)$, while a big value of a_2 means smaller $\mathcal{T}(m_1,m_2,m_3)$ and bigger $\mathcal{T}(n_1,n_2,n_3)$. Therefore, a_2 can be used as a trade-off between $\mathcal{T}(m_1,m_2,m_3)$ and $\mathcal{T}(n_1,n_2,n_3)$ such that the maximum MASP can be achieved. A way of tuning parameters a_1,a_2 is given as follows: i) choose a_1 such that the requirement of convergence performance is satisfied; ii) find a_2 such that $\mathcal{T}(m_1,m_2,m_3)=\mathcal{T}(n_1,n_2,n_3)$.

IV. PETC of Nonlinear MAS

In Section III, the TTC is considered for nonlinear MAS. To reduce the communication frequency as well as the controller update frequency, in this section, a distributed PETC strategy is developed. Different from TTC, at each sampling instant, a communication function and a control function are designed for each agent to determine whether or not the sampled data or the control input should be transmitted through the network, respectively. By introducing such a PETC strategy, a substantial reduction of resources may be achieved.

A. Periodic Event-triggered Communication

For each agent i, let $t^i_{\sigma_i},\sigma_i\in\mathbb{Z}_{\geq 0}$ be the increasing sequence of communication time instants at which x_i is transmitted and $\{t^i_{\sigma_i}\}$ be the set of communication instants. On the sensor side, each agent implements an estimator of itself using the most recently transmitted data $x_i(t^i_{\sigma_i})$, that is:

$$\begin{aligned}&\dot{\hat{y}}_i(t)=f(\hat{y}_i(t)), \quad t\in[t^i_{\sigma_i},t^i_{\sigma_i+1})\\ &\hat{y}_i(t^i_{\sigma_i})=x_i(t^i_{\sigma_i}).\end{aligned} \tag{13}$$

For agent i, the communication error at the sampling instant t_l is defined as $e_{y_i}(t_l)=\hat{y}_i(t_l)-x_i(t_l)$, and the communication time sequence $t^i_{\sigma_i}$ is generated by

$$t^i_{\sigma_i+1}=\inf_{t_l}\left\{t_l>t^i_{\sigma_i}:h_1\left(t_l,e_{y_i}(t_l)\right)\geq 0\right\}, \tag{14}$$

where

$$h_1\left(t_l,e_{y_i}(t_l)\right)=\left\|e_{y_i}(t_l)\right\|-c_1e^{-\alpha t_l} \tag{15}$$

with constants $c_1>0,\alpha>0$. The function $h_1\left(t_l,e_{y_i}(t_l)\right)\geq 0$ is called the *communication function*. Without loss of generality, we assume $t^i_0=0,\forall i$.

From the definition of $e_{y_i}(t_l)$, one can see that only the sampled state $x_i(t_l)$ is required to implement the communication function (15) for each agent i. At each t_l, $\|e_{y_i}(t_l)\|$ is compared to a certain threshold, i.e., $c_1e^{-\alpha t_l}$, and the sampled state $x_i(t_l)$ is transmitted if and only if $\|e_{y_i}(t_l)\|$ is bigger than or equal to that threshold.

B. Periodic Event-triggered Control

On the control side, each agent implements estimators of itself $\hat{y}_i(t)$ as well as its neighbors $\hat{y}_j(t)$, that is,

$$\begin{aligned}&\dot{\hat{y}}_j(t)=f(\hat{y}_j(t)), \quad t\in[t^j_{\sigma_j},t^j_{\sigma_j+1})\\ &\hat{y}_j(t^j_{\sigma_j})=x_j(t^j_{\sigma_j}), \quad j\in\mathcal{N}_i^+.\end{aligned} \tag{16}$$

The event-triggered controller for agent i is designed as

$$\hat{u}_i(t)=\sum_{j\in\mathcal{N}_i}\psi_{ij}(\hat{y}_j(T^i_k)-\hat{y}_i(T^i_k)), \quad t\in[T^i_k,T^i_{k+1}), \tag{17}$$

where $T^i_k,k\in\mathbb{Z}_{\geq 0}$ is the controller update time sequence.

Define $q_i(t)=\sum_{j\in\mathcal{N}_i}\psi_{ij}(\hat{y}_j(t)-\hat{y}_i(t))$. Then the control error at sampling instant t_l is defined as $\hat{e}^l_{u_i}(t_l)=q_i(T^i_k)-q_i(t_l)$. Let $\{T^i_k\}$ be the set of controller update instants, in which T^i_{k+1} is generated by

$$T^i_{k+1}=\inf_{t_l}\left\{t_l>T^i_k:h_2\left(t_l,\hat{e}^l_{u_i}(t_l)\right)\geq 0\right\}, \tag{18}$$

where

$$h_2\left(t_l,\hat{e}^l_{u_i}(t_l)\right)=\left\|\hat{e}^l_{u_i}(t_l)\right\|-c_2e^{-\alpha t_l} \tag{19}$$

with constant $c_2>0$. The function $h_2\left(t_l,\hat{e}^l_{u_i}(t_l)\right)\geq 0$ is called the *control function*. Without loss of generality, we assume $T^i_0=0,\forall i$.

From the definition of $q_i(t)$, $e^l_{u_i}(t_l)$ and (16), one can see that only the transmitted states of agent i itself and its neighbors, i.e., $x_j(t^j_{\sigma_j}),j\in\mathcal{N}_i^+$ are required to implement the control function (19) for each agent i.

C. Convergence result

Define $\hat{e}_{u_i}(t)=q_i(t_l)-q_i(t),t\in[t_l,t_{l+1})$ and $e_{y_i}(t)=\hat{y}_i(t)-y_i(t)$, where y_i is defined in (4). Then, one has $e_{y_i}(t_l)=\hat{y}_i(t_l)-y_i(t_l)=\hat{y}_i(t_l)-x_i(t_l)$. For convenience, we extend the control error as piecewise constant signals as $\hat{e}^l_{u_i}(t)=\hat{e}^l_{u_i}(t_l),t\in[t_l,t_{l+1})$. Combining the definition of $e_{x_i}(t)$ given in (5) and $\hat{e}^l_{u_i}(t),\hat{e}_{u_i}(t)$ and $e_{y_i}(t)$, (17) can be rewritten as

$$\begin{aligned}\hat{u}_i&=\sum_{j\in\mathcal{N}_i}\psi_{ij}(x_{ji}+e_{x_{ji}}+e_{y_{ji}})+\hat{e}^l_{u_i}+\hat{e}_{u_i},\\ &=\sum_{j\in\mathcal{N}_i}\psi_{ij}(\xi_{ji}+e_{x_{ji}}+e_{y_{ji}})+\hat{e}^l_{u_i}+\hat{e}_{u_i},\end{aligned} \tag{20}$$

where $e_{y_{ji}} = e_{y_j} - e_{y_i}$. Since the graph $\mathscr{G}$ is undirected and $\psi_{ij}(x_{ji}) = -\psi_{ij}(x_{ij})$, one has $\sum_{i=1}^{N} \hat{u}_i = \sum_{i=1}^{N} \hat{e}_{u_i}^l$. Then, one can further have

$$\dot{\xi}_i = f(x_i) - \frac{1}{N}\sum_{i=1}^{N} f(x_i) + \hat{u}_i - \frac{1}{N}\sum_{i=1}^{N} \hat{e}_{u_i}^l. \tag{21}$$

Let $\xi, e_x, e_y, \hat{e}_u^l, \hat{e}_u$ be the concatenated vectors of $\xi_i, e_{x_i}, e_{y_i}, \hat{e}_{u_i}^l, \hat{e}_{u_i}$, respectively. Then, we get the following Proposition.

Proposition 3: The derivative of $V(\|\xi\|)$ along the trajectories of (21) satisfies

$$\begin{aligned} \dot{V}(\|\xi\|) \leq & -\hat{L}\|\xi\|^2 + \frac{4}{a_1}\gamma^2\|e_x\|^2 + \frac{4}{a_1}\gamma^2\left\|e_y\right\|^2 \\ & + \frac{4}{a_1}\left\|\hat{e}_u^l\right\|^2 + \frac{2}{a_1}\|\hat{e}_u\|^2, \end{aligned}$$

and the inequalities

$$\begin{aligned} \|e_x\|' \leq & 3\gamma\|\xi\| + (\rho_1 + 3\gamma)\|e_x\| + 3\gamma\left\|e_y\right\| \\ & + \sqrt{3}\|\hat{e}_u\| + \sqrt{3}\left\|\hat{e}_u^l\right\|, \\ \|e_u\|' \leq & \sqrt{3}\rho_1\gamma(\|\xi\| + \|e_x\| + \left\|e_y\right\|) \end{aligned}$$

hold for all $t \in [t_l, t_{l+1})$.

Similar to TTC, one can derive that the MASP for PETC is given by

$$\hat{h}^* = \min\{\mathscr{T}(\hat{m}_1, \hat{m}_2, \hat{m}_3), \mathscr{T}(\hat{n}_1, \hat{n}_2, \hat{n}_3)\}, \tag{23}$$

where

$$\begin{aligned} & \hat{m}_1 = \frac{9\gamma^3 + 2a_1}{a_1{}^2} + a_2, \quad \hat{m}_2 = 2\rho_1 + 6\gamma + (\hat{L} - 2a_1), \\ & \hat{m}_3 = 4\gamma + a_2, \quad \hat{n}_1 = \frac{3\gamma^2\rho_1^2 + a_1}{a_1^2} + \frac{3\gamma\rho_1^2}{a_2}, \\ & \hat{n}_2 = \hat{L} - 2a_1, \quad \hat{n}_3 = \frac{3\gamma}{a_2} + 2. \end{aligned} \tag{24}$$

Then, we can get the following result.

Theorem 2: Consider the nonlinear MAS (2) with the control input (17), where the sampling period h is chosen from $h \in (0, \hat{h}^*)$. Suppose Assumptions 1-3 hold with $K > 2\rho_1/\lambda_2(L)$. The communication time sequence and the controller update time sequence are given respectively by (14) and (18) with $\alpha > K\lambda_2(L) - 2\rho_1 - 2a_1$, synchronization of the nonlinear MAS (2) is achieved exponentially with convergence rate $K\lambda_2(L) - 2\rho_1 - 2a_1$ and convergence coefficient $\sqrt{d_1}$ if $a_1 \in (0, (K\lambda_2(L) - 2\rho_1)/2)$, where $d_1 = \|\xi(0)\|^2 + (\beta_1 + \beta_2)/(2\alpha - 2K\lambda_2(L) + 4\rho_1 + 4a_1)$, $\beta_1 = Nc_1^2\gamma^2\left(4/a_1 + 3(3\gamma + \rho_1^2)\right)e^{2(\rho_1+\alpha)h}$ and $\beta_2 = Nc_2^2(4/a_1 + 3\gamma)e^{2\alpha h}$.

Remark 4: In this paper, the approach on finding the MASP is related to [3]. In [3], a sufficient condition is provided on the existence of an explicit formula for MASP that guarantees the stabilization of a networked control system. In this paper, we consider the synchronization of MAS in a networked control system framework. We note that the technical difficulty of our approach lies in the design of the asynchronous event-triggered communication and control strategy such that the sufficient condition provided in [3] is satisfied. This problem is nontrivial since we do not assume a passivity property on the nonlinear dynamics f. Instead, we overcome this problem by introducing estimators and suitably-designed communication and control functions for Sensor-Controller communication and Controller-Actuator communication, respectively.

V. PETC OF NONLINEAR MAS UNDER LIMITED DATA RATE

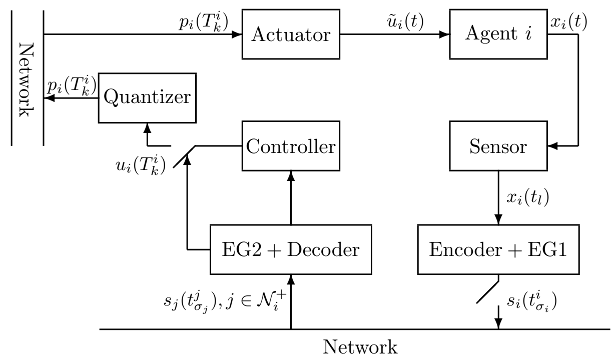

Fig. 1. Communication and control structure of agent i.

The results obtained in Section III and IV are based on perfect communication. However, any real network has limited channel capacity, which means that only a finite number of bits of information can be transmitted. Based on this observation, in this section, we further investigate PETC of nonlinear MAS under limited data rate.

A. Communication and control structure

It is assumed that both Sensor-Controller communication and Controller-Actuator communication are conducted via a network. Therefore, both state quantization and input quantization are considered. The desired communication and control structure of agent i is shown in Fig. 1. Under input quantization, the dynamics of agent i can be rewritten as:

$$\dot{x}_i(t) = f(x_i(t)) + \tilde{u}_i(t), \quad i = 1, 2, \ldots, N, \tag{25}$$

where $\tilde{u}_i$ is the quantized control input with scaling, which will be specified later.

Without loss of generality, we consider agent i and its neighbors $j \in \mathcal{N}_i$ to illustrate the idea, and only show how information flows within agent i and how agent i obtains information from agent $j, j \in \mathcal{N}_i$.

Step 1: The sensor samples the state of agent i at the sampling instants t_l, and then the sampled state $x_i(t_l)$ is sent to the event generator 1 (EG1), which determines whether or not the sampled state should be transmitted. On the sensor side, agent i implements an estimator of itself, which is given by (13). The communication time instants $t_{\sigma_i}^i$ are determined by

$$t_{\sigma_i+1}^i = \inf_{t_l}\left\{t_l > t_{\sigma_i}^i : h_1\left(t_l, e_{y_i}\left(t_l\right)\right) \geq 0 \cup t_l - t_{\sigma_i}^i \geq Rh\right\}, \tag{26}$$

where R is a positive integer, and $h_1(\cdot,\cdot)$ is defined in (15). Compared to (14), a constraint on the maximum time interval between two successive communications is enforced. It means that each agent can endure at most Rh units of time without communication. The reason for this constraint comes from the quantization error induced by the dynamic encoder-decoder, which is defined below. The good news is that, in general, R can be chosen arbitrarily large as long as it has a finite value.

Step 2: When the transmitted state $x_i(t^i_{\sigma_i})$ is generated, the encoder s_i of agent i is activated:

$$\begin{cases} \hat{x}_i(t^i_{-1}) =0 \\ s_i(t^i_{\sigma_i}) =Q_{\Delta,M^i_1}\left(\dfrac{x_i(t^i_{\sigma_i})-\hat{x}_i(t^i_{\sigma_i-1})-\int_{t^i_{\sigma_i-1}}^{t^i_{\sigma_i}} f(\hat{x}_i(s))ds}{g_1(t^i_{\sigma_i})}\right), \\ \hat{x}_i(t^i_{\sigma_i}) =\hat{x}_i(t^i_{\sigma_i-1})+\int_{t^i_{\sigma_i-1}}^{t^i_{\sigma_i}} f(\hat{x}_i(s))ds+g_1(t^i_{\sigma_i})s_i(t^i_{\sigma_i}), \end{cases} \tag{27}$$

where $x_i(t^i_{\sigma_i})$, $s_i(t^i_{\sigma_i})$ are the input and the output of the encoder. The function $g_1(t)>0$ is a scaling function which will be defined later. In the above, $\hat{x}_i(t^i_{\sigma_i})$ is the internal state of the encoder and $Q_{\Delta,M^i_1}(\cdot)$ is a finite-level uniform quantizer defined in (1) with quantization interval Δ and quantization level M^i_1.

Step 3: After the signal $s_j(t^j_{\sigma_j}), j\in\mathcal{N}_i^+$ of agent i itself or its neighbors $j\in\mathcal{N}_i$ is received on the control side, the decoder ς^i_j will be activated:

$$\begin{cases} \varsigma^i_j(t^j_{-1}) =0 \\ \varsigma^i_j(t^j_{\sigma_j}) =\varsigma^i_j(t^j_{\sigma_j-1})+\int_{t^j_{\sigma_j-1}}^{t^j_{\sigma_j}} f\left(\varsigma^i_j(s)\right)ds \\ \qquad\qquad +g_1(t^j_{\sigma_j})s_j(t^j_{\sigma_j}), \quad j\in\mathcal{N}_i^+, \end{cases} \tag{28}$$

where $\varsigma^i_j(t^j_{\sigma_j})$ is the output of the decoder. From (27) and (28), one has $\hat{x}_i(t^i_{\sigma_i})=\varsigma^j_i(t^i_{\sigma_i}),\forall j\in\mathcal{N}_i^+$.

Step 4: The decoded states $\varsigma^i_j(t^j_{\sigma_j}), j\in\mathcal{N}_i^+$ are sent to event generator 2 (EG2) to determine whether or not the controller is updated. Similar to Section IV, each agent implements estimators of itself $\tilde{y}_i(t)$ as well as its neighbors $\tilde{y}_j(t)$ based on decoded states, that is,

$$\begin{aligned} \dot{\tilde{y}}_j(t) &= f(\tilde{y}_j(t)), \quad t\in[t^j_{\sigma_j},t^j_{\sigma_j+1}) \\ \tilde{y}_j(t^j_{\sigma_j}) &= \varsigma^i_j(t^j_{\sigma_j}), \quad j\in\mathcal{N}_i^+. \end{aligned} \tag{29}$$

Then, the controller for agent i is designed as

$$\hat{u}_i(t)=\sum_{j\in\mathcal{N}_i}\psi_{ij}(\tilde{y}_j(T^i_k)-\tilde{y}_i(T^i_k)), t\in[T^i_k,T^i_{k+1}), \tag{30}$$

where T^i_k is the controller update time sequence. Let $\tilde{q}_i(t)=\sum_{j\in\mathcal{N}_i}\psi_{ij}(\tilde{y}_j(t)-\tilde{y}_i(t))$. Redefine in this section $\hat{e}^l_{u_i}(t_l)=\tilde{q}_i(T^i_k)-\tilde{q}_i(t_l)$ and $\hat{e}_{u_i}(t)=\tilde{q}_i(t_l)-\tilde{q}_i(t)$. Then, the controller update time sequence T^i_k is determined by

$$T^i_{k+1}=\inf_{t_l}\left\{t_l>T^i_k: h_2\left(t_l,\hat{e}^l_{u_i}(t_l)\right)\geq 0\cup t_l-t^i_{\sigma_i}\geq Rh\right\}, \tag{31}$$

where $h_2(\cdot,\cdot)$ is defined in (19). Similar to (26), an additional constraint on the maximum time interval between two successive controller updates is enforced.

Step 5: When the transmitted control input $\hat{u}_i(T^i_k)$ is generated, it is quantized with scaling:

$$p_i(T^i_k) =Q_{\Delta,M_2}\left(\frac{\hat{u}_i(T^i_k)}{g_2(T^i_k)}\right). \tag{32}$$

The function $Q_{\Delta,M_2}(\cdot)$ is a finite-level uniform quantizer defined in (1) with quantization interval Δ and quantization level M_2, and $g_2(t)>0$ is a scaling function.

Now, the quantized control input in (25) is given by

$$\tilde{u}_i(t)=g_2(T^i_k)p_i(T^i_k), \quad t\in[T^i_k,T^i_{k+1}). \tag{33}$$

B. Convergence result

Define the state quantization induced errors of agent i as $e^x_{q_i}(t^i_{\sigma_i})=\varsigma^i_i(t^i_{\sigma_i})-x_i(t^i_{\sigma_i})$. Let $e^y_{q_i}(t)=\tilde{y}_i(t)-\hat{y}_i(t)$. Then one can get $e^y_{q_i}(t^i_{\sigma_i})=e^x_{q_i}(t^i_{\sigma_i})$ since $\tilde{y}_i(t^i_{\sigma_i})=\varsigma^i_i(t^i_{\sigma_i})$ and $\hat{y}_i(t^i_{\sigma_i})=x_i(t^i_{\sigma_i})$. Define the control quantization induced errors of agent i as $e^u_{q_i}(T^i_k)=\tilde{u}_i(T^i_k)-\hat{u}_i(T^i_k)$. For convenience, we extend the control error and the control quantization induced error respectively as piecewise constant signals as $\hat{e}^l_{u_i}(t)=\hat{e}^l_{u_i}(t_l),t\in[t_l,t_{l+1})$ and $e^u_{q_i}(t)=\tilde{u}_i(T^i_k)-\hat{u}_i(T^i_k),t\in[T^i_k,T^i_{k+1})$.

Recall that $e_{x_i}(t)=y_i(t)-x_i(t), e_{y_i}(t)=\hat{y}_i(t)-y_i(t)$. Combining the definition of $e_{x_i}(t),e_{y_i}(t),\hat{e}_{u_i}(t),\hat{e}^l_{u_i}(t_l),e^y_{q_i}(t)$ and $e^u_{q_i}(t)$, (33) can be rewritten as

$$\begin{aligned} \tilde{u}_i(t)=&\sum_{j\in\mathcal{N}_i}\psi_{ij}(\xi_{ji}(t)+e_{x_{ji}}(t)+e_{y_{ji}}(t)+e^y_{q_{ji}}(t)) \\ &+\hat{e}^l_{u_i}(t)+\hat{e}_{u_i}(t)+e^u_{q_i}(t). \end{aligned} \tag{34}$$

Since the graph $\mathscr{G}$ is undirected and $\psi_{ij}(x_{ji})=-\psi_{ij}(x_{ij})$, one has $\sum_{i=1}^N\tilde{u}_i=\sum_{i=1}^N(\hat{e}^l_{u_i}+e^u_{q_i})$. Then, one can further have

$$\dot{\xi}_i =f(x_i)-\frac{1}{N}\sum_{i=1}^N f(x_i)+\tilde{u}_i-\frac{1}{N}\sum_{i=1}^N(\hat{e}^l_{u_i}+e^u_{q_i}). \tag{35}$$

Let e^y_q,e^u_q be the concatenated vectors of $e^y_{q_i},e^u_{q_i}$, respectively. Then, we get the following Proposition.

Proposition 4: The derivative of $V(\|\xi\|)$ along the trajectories of (35) satisfies

$$\begin{aligned} \dot{V}(\|\xi\|)\leq& -\hat{L}\|\xi\|^2+\frac{9}{a_1}\gamma^2\|e_x\|^2+\frac{9}{a_1}\gamma^2\left\|e_y\right\|^2+\frac{9}{a_1}\gamma^2\left\|e^y_q\right\|^2 \\ &+\frac{6}{a_1}\left\|\hat{e}^l_u\right\|^2+\frac{3}{a_1}\|\hat{e}_u\|^2+\frac{6}{a_1}\left\|e^u_q\right\|^2, \end{aligned}$$

and the inequalities

$$\begin{aligned} \|e_x\|' \leq& 4\gamma\|\xi\|+(\rho_1+4\gamma)\|e_x\|+4\gamma\left\|e_y\right\|+4\gamma\left\|e^y_q\right\| \\ &+2\|\hat{e}_u\|+2\left\|\hat{e}^l_u\right\|+2\left\|e^u_q\right\|, \\ \|e_u\|' \leq& 2\rho_1\gamma(\|\xi\|+\|e_x\|+\left\|e_y\right\|+\left\|e^y_q\right\|) \end{aligned}$$

hold for all $t\in[t_l,t_{l+1})$.

The MASP for PETC under limited data rate is given by

$$\tilde{h}^*=\min\{\mathscr{T}(\tilde{m}_1,\tilde{m}_2,\tilde{m}_3),\mathscr{T}(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)\}, \tag{37}$$

where

$$\begin{aligned} &\tilde{m}_1=\frac{16\gamma^3+4a_1}{{a_1}^2}+a_2, \quad \tilde{m}_2=2\rho_1+8\gamma+(\hat{L}-2a_1), \\ &\tilde{m}_3=9\gamma+a_2, \quad \tilde{n}_1=\frac{4\gamma^2\rho_1^2+2a_1}{a_1^2}+\frac{4\gamma\rho_1^2}{a_2}, \\ &\tilde{n}_2=\hat{L}-2a_1, \quad \tilde{n}_3=\frac{4\gamma}{a_2}+3. \end{aligned} \tag{38}$$

With the designed encoder-decoder (27), (28) and the control quantizer (32), the following result is obtained.

Theorem 3: Consider the nonlinear MAS (25) with the control input (33), where the sampling period h is chosen from $h \in (0, \tilde{h}^*)$. Suppose Assumptions 1-3 hold with $K > 2\rho_1/\lambda_2(L)$. The communication time sequence and the controller update time sequence are given by (26) and (31) respectively with $\alpha > 0$. Let the encoder-decoder be given by (27), (28) and the scaling functions $g_1(t) = c_3 e^{-\eta t}$ and $g_2(t) = c_4 e^{-\eta t}$ with constants $c_3 > 0, c_4 > 0$ and $0 < \eta < \min\{K\lambda_2(L) - 2\rho_1 - 2a_1, \alpha\}$, where $a_1 \in (0, (K\lambda_2(L) - 2\rho_1)/2)$. The quantization levels M_1^i, M_2 satisfy

$$M_1^i \geq \max\left\{\frac{c_1 e^{(\rho_1+\alpha)h} + hCe^{\eta h}}{c_3\Delta} + \frac{\sqrt{n}}{2} e^{(\rho_1+\eta)Rh}, \frac{\|x_i(0)\|}{c_3\Delta}\right\} \quad (39)$$

and

$$M_2 \geq \frac{C}{c_4\Delta}, \quad (40)$$

where $C = \gamma(\sqrt{d_2} + \sqrt{N}c_1 + \sqrt{Nn}c_3\Delta e^{(\eta+\rho_1)Rh}/2)e^{\eta Rh} + c_2 + c_4\sqrt{n}\Delta/2$, and γ is defined in Proposition 1. Then, synchronization of the nonlinear MAS (25) is achieved exponentially with convergence rate η and convergence coefficient $\sqrt{d_2}$, where $d_2 = \|\xi(0)\|^2 + (\beta_3 + \beta_4)/|2\alpha - 2K\lambda_2(L) + 4\rho_1 + 4a_1| + k_1/(2K\lambda_2(L) - 4\rho_1 - 4a_1 - 2\eta)$, $\beta_3 = Nc_1^2\gamma^2\left(9/a_1 + 4(4\gamma + \rho_1^2)\right)e^{2(\rho_1+\alpha)h}$, $\beta_4 = Nc_2^2(6/a_1 + 4\gamma)e^{2\alpha h}$ and $k_1 = \beta_3 n c_3^2\Delta^2 e^{2(\eta+\rho_1)Rh}/4c_1^2 + \beta_4 n c_4^2 \Delta^2 e^{2\eta Rh}/4c_2^2$.

Remark 5: There is existing literature considering ETC of MAS under limited data rate [32]–[36]. In [32]–[34], MAS with single-integrator dynamics are considered, and ETC, self-triggered control and PETC are proposed, respectively. Discrete-time MAS with general linear dynamics are studied in [35], while continuous-time MAS with general linear dynamics are investigated in [36]. In both [35] and [36], synchronous ETC strategy is designed. However, in this work, nonlinear MAS with PETC strategy is instead considered. Moreover, to the best of our knowledge, an asynchronous PETC strategy under limited date rate is proposed for the first time.

Remark 6: In this paper, three different communication and control strategies, i.e., TTC, PETC and PETC under limited data rate are considered and different triggering conditions are proposed, respectively. In TTC, a universal clock is used to trigger both state and control input transmission. In PETC, despite of a universal clock, communication and control conditions are further designed to determine whether or not the state and the control input should be transmitted, respectively. When limited data rate is considered for PETC, an additional clock is required for both state transmission and controller update. The reason for introducing this additional clock is to ensure that the quantization error is limited.

Remark 7: For TTC, the rate of communication (for both state and control input) is determined by the sampling period h. However, for PETC, we note that the rate of communication is not mainly determined by the sampling period h, but the communication and control conditions (14) and (18). For PETC under limited data rate, the rate of communication is determined by both the sampling period h and the communication and control functions (26) and (31), since the period of the additional clock is a constant times of h.

VI. SIMULATION RESULTS

In this section, a simulation example is provided to validate the effectiveness of the theoretical results.

Consider a MAS consisting of 4 single-link robot arms, the communication graph is characterized by the adjacency matrix $A = [0,1,0,1;1,0,1,0;0,1,0,1;1,0,1,0]$. The initial state $x_i(0)$ of each robot arm i is chosen randomly from the box $[-5;5] \times [-5;5]$. The dynamics of the $i-$th robot arm is described by (2), where

$$f(x_{i_1}(t), x_{i_2}(t)) = \begin{pmatrix} x_{i_2} \\ -\sin(x_{i_1}) \end{pmatrix}.$$

and $x_i = (x_{i1}, x_{i2})^T$. Clearly, $f(x_i)$ is Lipchitz with a Lipschitz constant $\rho_1 = 1$. The control input for agent i is designed as $\hat{u}_i(t) = 2\sum_{j\in\mathcal{N}_i}(x_j(t_l) - x_i(t_l)), \quad t \in [t_l, t_{l+1})$. One can verify that Assumption 3 holds with $K = 2$ and $\rho_2 = 2$. According to Proposition 1, one can calculate $\hat{L} = 4 - 2a_1, \gamma = 4\sqrt{2}$. Choosing $a_1 = 0.8$ such that $\hat{L} - 2a_1 = 4 - 4a_1 = 0.8 > 0$, then one can get $h^* = 3.18 \times 10^{-3}$. The sampling period h is chosen as $h = 3 \times 10^{-3}$. The simulation results for TTC are shown in Fig. 2, where the evolution of the states x_{i1} and x_{i2} is plotted.

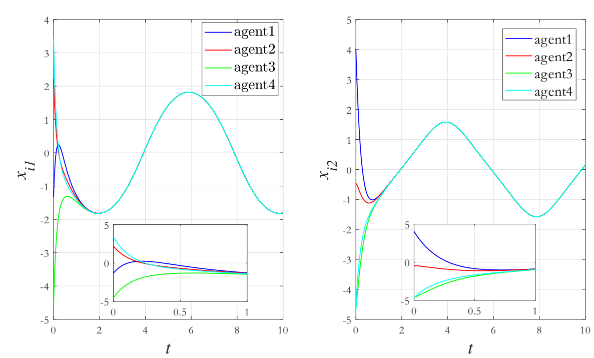

Fig. 2. The evolution of x_{i1}, x_{i2} under TTC

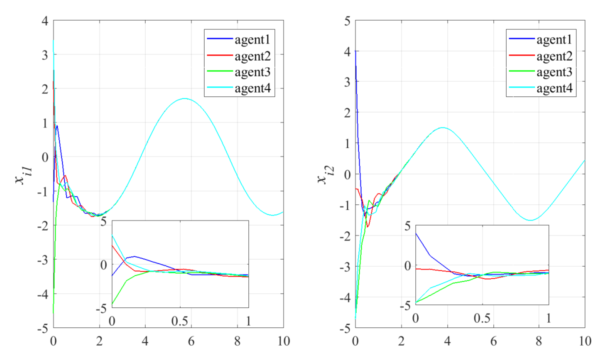

Fig. 3. The evolution of x_{i1}, x_{i2} under PETC

For PETC and PETC under limited data rate (PETC*), the sampling period $h = 1 \times 10^{-3}$. Given $c_1 = c_2 = 4, \alpha = 2.01$. The simulation results for PETC are shown in Fig. 3. Choosing $R = 100$, and the parameters for the scaling functions g_1, g_2 as $c_3 = c_4 = 4$ and $\eta = 1.8 < \min\{K\lambda_2(L) - 2\rho_1 - 2a_1, \alpha\}$, the simulation results for PETC* are plotted in Fig. 4.

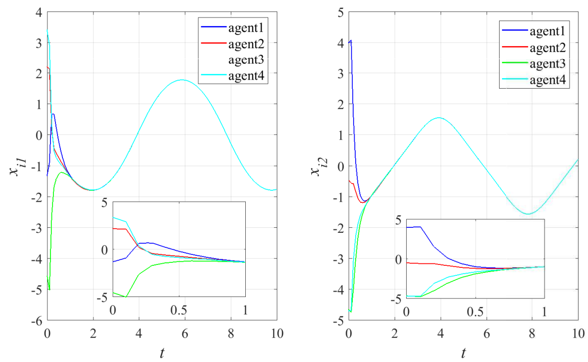

Fig. 4. The evolution of x_{i1}, x_{i2} under PETC*

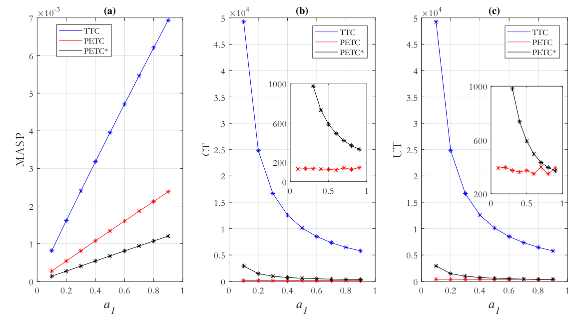

Fig. 5. The effects of tuning a_1 on the MASP, the communication times (CT) and the controller update times (UT).

TABLE I
COMMUNICATION TIMES AND CONTROLLER UPDATE TIMES

CT/UT	TTC	PETC	PETC*
agent1	3333/3333	31/90	99/106
agent2	3333/3333	30/87	99/107
agent3	3333/3333	29/87	100/105
agent4	3333/3333	32/90	99/108

Table I summarises the communication times (CT) and controller update times (UT) for each agent under the three cases. One can see that the introduction of communication and control functions significantly reduces the rate of communication and controller update. One can also see that synchronization of the nonlinear MAS can still be achieved under data rate constraint.

Next, the effects of tuning parameter a_1 on the MASP and the total communication/controller update times are shown in Fig. 5. From Fig. 5(a), one can see that the MASP increases as a_1 increases for all the three cases and $h^* > \hat{h}^* > \tilde{h}^*$ when choosing the same a_1. From Fig. 5(b) and (c), one can see that for both TTC and PETC*, the communication/controller update times are decreasing as the sampling period is increasing. However, the communication/controller update times for PETC remain almost the same no matter what the sampling period is. We note that the relationship of the MASP for the three cases is $h^* > \hat{h}^* > \tilde{h}^*$. However, the relationship of the total communication/controller update times is PETC $<$ PETC* $<$ TTC, which again verifies that the introduction of communication and control functions reduces significantly the rate of communication.

VII. CONCLUSION

In this paper, we have presented an approach on finding the MASP that guarantees exponential synchronization of nonlinear sampled-data MAS. Two different implementation scenarios were considered, the TTC and PETC. Firstly, TTC of the nonlinear MAS was considered and the approach on finding the MASP was established. Then, a PETC strategy was formulated and the explicit formula for MASP was obtained based on the approach. Finally, the PETC strategy was investigated under the constraint of limited data rate. It was proved that exponential synchronization of the nonlinear MAS can still be achieved under limited data rate.

APPENDIX

A. Proof of Proposition 1

From (6), one has

$$\|\xi_i\|' = \frac{d\|\xi_i\|}{dt} = \frac{\xi_i^T\left(f(x_i) - f(\bar{x}) + f(\bar{x}) - \frac{1}{N}\sum_{i=1}^{N} f(x_i) + \hat{u}_i\right)}{\|\xi_i\|}$$

$$\leq \rho_1\|\xi_i\| + \frac{1}{N}\sum_{i=1}^{N}\rho_1\|\xi_i\| + \frac{\xi_i^T\hat{u}_i}{\|\xi_i\|},$$

and thus $\xi_i^T\dot{\xi}_i \leq \rho_1\|\xi_i\|^2 + \frac{\rho_1}{N}\|\xi_i\|\sum_{i=1}^N\|\xi_i\| + \xi_i^T\hat{u}_i$. Then,

$$\|\xi\|' = \frac{\sum_{i=1}^N \xi_i^T(t)\dot{\xi}_i}{\|\xi\|} \leq 2\rho_1\|\xi\| + \frac{\xi^T\hat{u}}{\|\xi\|}. \tag{41}$$

Differentiating $V(\|\xi\|)$ along the trajectories of (41), one has

$$\dot{V}(\|\xi\|) \leq 4\rho_1\|\xi\|^2 + 2\sum_{i=1}^{N}\xi_i^T\left(\delta_i + \hat{\delta}_i + e_{u_i}\right), \tag{42}$$

where $\delta_i = \sum_{j\in\mathcal{N}_i}\psi_{ij}(\xi_{ji})$ and $\hat{\delta}_i = \sum_{j\in\mathcal{N}_i}\psi_{ij}(\xi_{ji} + e_{x_{ji}}) - \delta_i$.

From Assumption 3, one has $x_{ij}^T\psi_{ji}(x_{ij}) \geq Kx_{ij}^Tx_{ij} = K\|x_{ij}\|^2, \forall x_{ij} \in \mathbb{R}^n$, then $\sum_{(i,j)\in\mathscr{E}}\xi_{ij}^T\psi_{ji}(\xi_{ij}) \geq K\sum_{(i,j)\in\mathscr{E}}\|\xi_{ij}\|^2 = 2K\xi^T(L\otimes I_n)\xi$.

According to the definition of ξ, one has $1_{Nn}^T\xi = 0$, and if the graph $\mathscr{G}$ is connected, one has $\xi^T(L\otimes I_n)\xi \geq \lambda_2(L)\|\xi\|^2$ and $\lambda_2(L) > 0$ [30]. Then, $\sum_{i=1}^N\xi_i^T\delta_i = \sum_{i=1}^N\xi_i^T\sum_{j\in\mathcal{N}_i}\psi_{ij}(\xi_{ji}) = -\frac{1}{2}\sum_{i=1}^N\xi_{ij}^T\sum_{j\in\mathcal{N}_i}\psi_{ji}(\xi_{ij}) \leq -K\lambda_2(L)\|\xi\|^2$. In addition, the function $\psi_{ij}, \forall(i,j) \in \mathscr{E}$ is globally Lipchitz, and then one has $\sum_{i=1}^N \left\|\hat{\delta}_i\right\|^2 \leq \sum_{i=1}^N \left\|\sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_{ji}+e_{x_{ji}}) - \sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_{ji})\right\|^2 \leq \rho_2^2 \sum_{i=1}^N \left(\sum_{j\in\mathcal{N}_i} \|e_{x_{ji}}\|\right)^2 \leq \rho_2^2(2\hat{l}^2+2N\hat{l})\|e_x\|^2$.

Using the inequality $2xy \leq ax^2 + 1/ay^2, \forall a > 0$, (42) can be rewritten as

$$\begin{aligned}\dot{V}(\|\xi\|) \leq &-(2K\lambda_2(L) - 4\rho_1 - 2a_1)\|\xi\|^2 \\ &+ \frac{1}{a_1}\left(\rho_2^2(2\hat{l}^2+2N\hat{l})\|e_x\|^2 + \|e_u\|^2\right)\end{aligned} \tag{43}$$

for some $a_1 > 0$. ■

B. Proof of Proposition 2

According to (5), one has $\dot{e}_{x_i}(t) = \dot{y}_i(t) - \dot{x}_i(t) = f(y_i(t)) - f(x_i(t)) - \hat{u}_i(t)$, which jumps at $t = t_l$. Thus, for $\forall t \in [t_l, t_{l+1})$, one has that e_{x_i} is continuous. Similar to Proposition 1, one can get

$$\|e_x(t)\|' = \frac{\sum_{i=1}^N e_{x_i}^T(t)\dot{e}_{x_i}(t)}{\|e(t)\|} \leq \rho_1\|e_x(t)\| + \|\hat{u}(t)\|. \tag{44}$$

Since $\|\hat{u}(t)\|^2 = \sum_{i=1}^N \left\|\sum_{j\in\mathcal{N}_i} \psi_{ij}(\xi_{ji}(t)+e_{x_{ji}}(t)) + e_{u_i}(t)\right\|^2 \leq 4\rho_2^2(2\hat{l}^2+2N\hat{l})\left(\|\xi(t)\|^2 + \|e_x(t)\|^2\right) + 2\|e_u(t)\|^2$, then one has

$$\|\hat{u}(t)\| \leq 2\gamma(\|\xi(t)\| + \|e_x(t)\|) + \sqrt{2}\|e_u(t)\|. \tag{45}$$

Substituting (45) into (44) yields (9a).

The proof of (9b) is similar to that of (9a) and hence omitted. ■

C. Proof of Theorem 1

To prove Theorem 1, we first prove the following result.

Claim 1: There exist $\phi_1(0) > 0, \phi_2(0) > 0$, such that $\phi_1(\tau) \geq 0$ and $\phi_2(\tau) \geq 0$ for all $\tau \in [0, h^*)$, where h^* is defined in (12).

Proof: Since $K > (2\rho_1 + 2a_1)/\lambda_2(L)$, one has $\hat{L} - 2a_1 > 0$. One can see from (10a) and (10b) that both ϕ_1 and ϕ_2 are strictly decreasing.

Let $\phi_1^{-1}(0)$ be the corresponding time that ϕ_1 decreases from initial value $\phi_1(0) > 0$ to 0, then one can compute that

$$\phi_1^{-1}(0) = \begin{cases} \frac{2}{r_1}\arctan\left(\frac{\phi_1(0)\sqrt{r_1}}{\phi_1(0)m_2+2m_3}\right) & 4m_1m_3 > m_2{}^2 \\ \frac{4m_1\phi_1(0)}{2m_1m_2\phi_1(0)+m_2^2} & 4m_1m_3 = m_2{}^2 \\ \frac{2}{r_1}\text{arctanh}\left(\frac{\phi_1(0)\sqrt{r_1}}{\phi_1(0)m_2+2m_3}\right) & 4m_1m_3 < m_2{}^2, \end{cases}$$

where m_1, m_2, m_3 are defined in (11). Note that $\phi_1^{-1}(0)$ increases as $\phi_1(0)$ increases, and

$$\mathscr{T}(m_1, m_2, m_3) = \lim_{\phi_1(0)\to\infty} \phi_1^{-1}(0).$$

Similarly, one can get that $\mathscr{T}(n_1, n_2, n_3) = \lim_{\phi_2(0)\to\infty} \phi_2^{-1}(0)$. Therefore, for all $\tau \in [0, h^*)$, one can conclude that there exist $\phi_1(0) > 0, \phi_2(0) > 0$ such that $\phi_1(\tau) \geq 0$ and $\phi_2(\tau) \geq 0$. □

Define $\tau : \mathbb{R}_{\geq 0} \to [0, h]$ as $\tau(t) = t - kh, t \in [kh, (k+1)h), k = 0, 1, \ldots,$ where $h \in (0, h^*)$. Let $z := (\xi, e_x, e_u)$. Define the Lyapunov function candidate as

$$R_1(z) = V(\|\xi\|) + \frac{1}{a_1}\gamma\phi_1(\tau)V(\|e_x\|) + \frac{1}{a_1}\phi_2(\tau)V(\|e_u\|). \tag{46}$$

Since $h < h^*$, by properly choosing $\phi_1(0), \phi_2(0) > 0$, one can conclude from Claim 1 that $\phi_1(\tau) \geq 0$ and $\phi_2(\tau) \geq 0$ for all $\tau \in [0, h]$. Therefore, $R_1(z)$ is positive definite.

Taking the derivative of $R_1(z)$ on $\forall t \in [t_l, t_{l+1})$, one has

$$\begin{aligned}\dot{R}_1(z) =&\dot{V}(\|\xi\|) + \frac{1}{a_1}\gamma\dot{\phi}_1(\tau)\|e_x\|^2 + 2\frac{1}{a_1}\gamma\phi_1(\tau)\|e_x\|\|e_x\|' \\ &+ \frac{1}{a_1}\dot{\phi}_2(\tau)\|e_u\|^2 + 2\frac{1}{a_1}\phi_2(\tau)\|e_u\|\|e_u\|'.\end{aligned} \tag{47}$$

Substituting (8), (9a), (9b) and (10a), (10b) into (47), one can further have

$$\begin{aligned}\dot{R}_1(z) \leq &-\hat{L}\|\xi\|^2 + \frac{1}{a_1}\gamma^2\|e_x\|^2 + \frac{1}{a_1}\|e_u\|^2 \\ &+ 2\frac{1}{a_1}\gamma\phi_1(\tau)\|e_x\|(\gamma\|\xi\| + (\rho_1+\gamma)\|e_x\| + \|e_u\|) \\ &+ \frac{1}{a_1}\gamma\Big(-m_1\phi_1^2(\tau) - m_2\phi_1(\tau) - m_3\Big)\|e_x\|^2 \\ &+ 2\frac{1}{a_1}\phi_2(\tau)\|e_u\|(\rho_1\gamma\|\xi\| + \rho_1\gamma\|e_x\|) \\ &+ \frac{1}{a_1}\Big(-n_1\phi_2^2(\tau) - n_2\phi_2(\tau) - n_3\Big)\|e_u\|^2 \\ =&-(\hat{L}-2a_1)\|\xi\|^2 - \frac{1}{a_1}(\hat{L}-2a_1)\gamma\phi_1(\tau)\|e_x\|^2 \\ &- \frac{1}{a_1}(\hat{L}-2a_1)\phi_2(\tau)\|e_u\|^2 \\ \leq &-(\hat{L}-2a_1)R_1(z),\end{aligned} \tag{48}$$

which implies $\dot{R}_1(z) < 0, \forall t \in [t_l, t_{l+1})$ when $\|\xi\| \neq 0$. Based on the comparison theorem in [31] and (48), one can get that the solution of $R_1(z)$ satisfies

$$R_1(z(t)) \leq e^{-(\hat{L}-2a_1)(t-t_l)}R_1(z(t_l)), \quad \forall t \in [t_l, t_{l+1}).$$

Moreover, at the jump, i.e., when $t = t_l^+$, one has $\|\xi(t_l^+)\| = \|\xi(t_l)\|$, $\|e_x(t_l^+)\| = 0 \leq \|e_x(t_l)\|$ and $\|e_u(t_l^+)\| = 0 \leq \|e_u(t_l)\|$. That is to say, $R_1(z)$ is non-increasing during the jump. Then, one can further have $R_1(z(t)) \leq e^{-(\hat{L}-2a_1)t}R_1(z(0)) =$

$e^{-(\hat{L}-2a_1)t/2}\|\xi(0)\|^2, \forall t$ and thus $\|\xi(t)\| \leq \sqrt{R_1(z(t))} \leq \|\xi(0)\|e^{-(\hat{L}-2a_1)t/2} = \|\xi(0)\|e^{-(K\lambda_2(L)-2\rho_1-2a_1)t}, \forall t$. That is, synchronization of the nonlinear MAS (2) is achieved exponentially with convergence rate $K\lambda_2(L)-2\rho_1-2a_1$ and convergence coefficient $\|\xi(0)\|$. ■

D. Proof of Theorem 2

Define $\hat{\tau}: \mathbb{R}_{\geq 0} \to [0,h]$ as $\hat{\tau}(t) = t-kh, t \in [kh,(k+1)h), k = 0,1,\ldots,$ where $h \in (0,\hat{h}^*)$. Let $\hat{z} := (\xi, e_x, \hat{e}_u)$. Define the Lyapunov function candidate

$$R_2(\hat{z}) = V(\|\xi\|) + \frac{1}{a_1}\gamma\hat{\phi}_1(\hat{\tau})V(\|e_x\|) + \frac{1}{a_1}\hat{\phi}_2(\hat{\tau})V(\|\hat{e}_u\|), \quad (49)$$

where $d\hat{\phi}_1(\hat{\tau})/d\hat{\tau} = -\hat{m}_1\hat{\phi}_1^2(\hat{\tau}) - \hat{m}_2\hat{\phi}_1(\hat{\tau}) - \hat{m}_3$ and $d\hat{\phi}_2(\hat{\tau})/d\hat{\tau} = -\hat{n}_1\hat{\phi}_2^2(\hat{\tau}) - \hat{n}_2\hat{\phi}_2(\hat{\tau}) - \hat{n}_3$, and $\hat{m}_i, \hat{n}_i, i = \{1,2,3\}$ are given in (24). Similar to Theorem 1, one can guarantee that $R_2(\hat{z})$ is positive definite by properly choosing $\hat{\phi}_1(0), \hat{\phi}_2(0) > 0$. Taking the derivative of $R_2(\hat{z})$ on $t \in [t_l, t_{l+1})$, one has

$$\dot{R}_2(\hat{z}) \leq -(\hat{L}-2a_1)R_2(\hat{z}) + \left(\frac{4\gamma^2}{a_1} + 3\gamma^2(3\gamma+\rho_1^2)\right)\|e_y\|^2 + \left(\frac{4}{a_1}+3\gamma\right)\|\hat{e}_u^l\|^2. \quad (50)$$

For $\forall t \in [t_l, t_{l+1})$ and $\forall i$, i) if $t_l \in \{t^i_{\sigma_i}\}$, one has $\|e_{y_i}(t)\| = 0$, else if ii) $t_l \notin \{t^i_{\sigma_i}\}$, one has $\|e_{y_i}(t_l)\| < c_1 e^{-\alpha_1 t_l}$ and

$$\begin{aligned}\|e_{y_i}(t)\| =& \|\hat{y}_i(t) - y_i(t)\| = \left\|\int_{t^i_{\sigma_i}}^t f(\hat{y}_i(s))ds - \int_{t_l}^t f(y_i(s))ds\right\| \\ \leq& e^{\rho_1\left(t-t^i_{\sigma_i}\right)}\|\hat{y}_i(t^i_{\sigma_i}) - e^{\rho_1(t-t_l)}y_i(t_l)\| \\ =& e^{\rho_1(t-t_l)}\|e_{y_i}(t_l)\|\end{aligned} \quad (51)$$

for $t \in (t_l, t_{l+1})$. Thus, one has $\|e_{y_i}\| < c_1 e^{\rho_1(t-t_l)}e^{-\alpha t_l} < c_1 e^{(\rho_1+\alpha)h}e^{-\alpha t}$ and

$$\|e_y\| < \sqrt{N}c_1 e^{(\rho_1+\alpha)h}e^{-\alpha t}, \quad \forall t \geq 0. \quad (52)$$

Besides, for $\forall t \in [t_l, t_{l+1})$ and $\forall i$, i) if $t_l \in \{T^i_k\}$, one has $\|e_{u_i}(t_l)\| = 0$, else if ii) $t_l \notin \{T^i_k\}$, one has $\|e_{u_i}(t_l)\| < c_2 e^{-\alpha t_l} < c_2 e^{\alpha h}e^{-\alpha t}$. Then, one can get

$$\|e^l_u\| < \sqrt{N}c_2 e^{\alpha h}e^{-\alpha t}, \quad \forall t \geq 0. \quad (53)$$

Substituting (52) and (53) into (50), one has

$$\dot{R}_2(\hat{z}) < -(\hat{L}-2a_1)R_2(\hat{z}) + (\beta_1+\beta_2)e^{-2\alpha t}, \quad (54)$$

where β_1, β_2 are given in the statement of Theorem 2. Furthermore, if one has $\alpha > K\lambda_2(L) - 2\rho_1 - 2a_1$, then based on the comparison theorem and (54), one can get that the solution of $R_2(\hat{z})$ satisfies $R_2(\hat{z}) \leq e^{-(\hat{L}-2a_1)t}R_2(\hat{z}) + \int_0^t e^{-(\hat{L}-2a_1)(t-s)}((\beta_1+\beta_2)e^{-2\alpha t})ds \leq d_1 e^{-(\hat{L}-2a_1)t}$, where $d_1 = \|\xi(0)\|^2 + (\beta_1+\beta_2)/(2\alpha - 2K\lambda_2(L) + 4\rho_1 + 4a_1)$. Then, one can further have $\|\xi(t)\| \leq \sqrt{d_1}e^{-(\hat{L}-2a_1)t/2} = \sqrt{d_1}e^{-(K\lambda_2(L)-2\rho_1-2a_1)t}$. That is, synchronization of the nonlinear MAS (2) is achieved exponentially with convergence rate $K\lambda_2(L)-2\rho_1-2a_1$ and convergence coefficient $\sqrt{d_1}$. ■

E. Proof of Theorem 3

Firstly, we assume that the communication bandwidth is unlimited, that is, the quantizer (1) will never be saturated and thus we have $\|Q_{\Delta,M_1^i}(x(t)) - x(t)\| \leq \sqrt{n}\Delta/2, \|Q_{\Delta,M_2}(u(t)) - u(t)\| \leq \sqrt{n}\Delta/2, \forall x(t), u(t) \in \mathbb{R}^n, \forall i$.

Define $\tilde{\tau}: \mathbb{R}_{\geq 0} \to [0,h]$ as $\tilde{\tau}(t) = t-kh, t \in [kh,(k+1)h), k = 0,1,\ldots,$ where $h \in (0,\tilde{h}^*)$. Consider the Lyapunov function candidate

$$R_3(\hat{z}) = V(\|\xi\|) + \frac{1}{a_1}\gamma\tilde{\phi}_1(\tilde{\tau})V(\|e_x\|) + \frac{1}{a_1}\tilde{\phi}_2(\tilde{\tau})V(\|\hat{e}_u\|), \quad (55)$$

where $d\tilde{\phi}_1(\tilde{\tau})/d\tilde{\tau} = -\tilde{m}_1\tilde{\phi}_1^2(\tilde{\tau}) - \tilde{m}_2\tilde{\phi}_1(\tilde{\tau}) - \tilde{m}_3$ and $d\tilde{\phi}_2(\tilde{\tau})/d\tilde{\tau} = -\tilde{n}_1\tilde{\phi}_2^2(\tilde{\tau}) - \tilde{n}_2\tilde{\phi}_2(\tilde{\tau}) - \tilde{n}_3$.

Similar to Theorem 2, one can get that the derivative of $R_3(\hat{z})$ on $t \in [t_l, t_{l+1})$ satisfies

$$\dot{R}_3(\hat{z}) \leq -(\hat{L}-2a_1)R_3(\hat{z}) + \left(\frac{6}{a_1}+4\gamma\right)(\|\hat{e}^l_u\|^2 + \|\hat{e}^u_q\|^2) + \left(\frac{9\gamma^2}{a_1} + 4\gamma^2(4\gamma+\rho_1^2)\right)(\|e_y\|^2 + \|e^y_q\|^2). \quad (56)$$

According to (27), one has

$$\begin{aligned}&\left\|x_i(t^i_{\sigma_i}) - \hat{x}_i(t^i_{\sigma_i})\right\| \\ =&\left\|x_i(t^i_{\sigma_i}) - \hat{x}_i(t^i_{\sigma_i-1}) - \int_{t^i_{\sigma_i-1}}^{t^i_{\sigma_i}} f(\hat{x}_i(s))ds \right. \\ &\left. - g_1(t^i_{\sigma_i})Q_{\Delta,M_1^i}\left(\frac{x_i(t^i_{\sigma_i}) - \hat{x}_i(t^i_{\sigma_i-1}) - \int_{t^i_{\sigma_i-1}}^{t^i_{\sigma_i}} f(\hat{x}_i(s))ds}{g_1(t^i_{\sigma_i})}\right)\right\| \\ \leq& \frac{\sqrt{n}\Delta g_1(t^i_{\sigma_i})}{2}.\end{aligned}$$

Combining (27) and (28), one further has $e^x_{q_i}(t^i_{\sigma_i}) = \left\|\varsigma^i_i(t^i_{\sigma_i}) - x_i(t^i_{\sigma_i})\right\| = \left\|\hat{x}_i(t^i_{\sigma_i}) - x_i(t^i_{\sigma_i})\right\| \leq \sqrt{n}\Delta g_1(t^i_{\sigma_i})/2$.

Besides, one has $t - t^i_{\sigma_i} \leq Rh, \forall t \in [t^i_{\sigma_i}, t^i_{\sigma_i+1})$, then according to (26), one can get $\left\|e^y_{q_i}(t)\right\| \leq e^{\rho_1 Rh}\left\|e^y_{q_i}(t^i_{\sigma_i})\right\| = e^{\rho_1 Rh}\left\|e^x_{q_i}(t^i_{\sigma_i})\right\| \leq e^{\rho_1 Rh}\sqrt{n}\Delta g_1(t^i_{\sigma_i})/2$.

From the definition of g_1, one has $g_1(t^i_{\sigma_i}) = c_3 e^{-\eta t^i_{\sigma_i}} = c_3 e^{\eta(t-t^i_{\sigma_i})}e^{-\eta t} \leq c_3 e^{\eta Rh}e^{-\eta t}$. Then, one further has $\left\|e^y_{q_i}(t)\right\| \leq c_3\sqrt{n}\Delta e^{(\eta+\rho_1)Rh}e^{-\eta t}/2$ and thus $\|e^y_q(t)\| \leq \sqrt{N}c_3\sqrt{n}\Delta e^{(\eta+\rho_1)Rh}e^{-\eta t}/2, \forall t$. Similarly, one has $\|e^u_{q_i}(t)\| \leq \sqrt{n}\Delta g_2(T^i_k)/2 \leq c_4\sqrt{n}\Delta e^{\eta Rh}e^{-\eta t}/2$, and thus $\|e^u_q(t)\| \leq \sqrt{Nn}c_4\Delta e^{\eta Rh}e^{-\eta t}/2, \forall t$. Then, (56) can be rewritten as $\dot{R}_3(\hat{z}) < -(\hat{L}-3a_1)R_2(\hat{z}) + (\beta_3+\beta_4)e^{-2\alpha t} + k_1 e^{-2\eta t}$, where β_3, β_4, k_1 are given in the statement of Theorem 3. Moreover, one has $\eta < \min\{K\lambda_2(L) - 2\rho_1 - 4a_1/2, \alpha\}$, thus $R_3(\hat{z}) < d_2 e^{-2\eta t}$ and $\|\xi(t)\| < \sqrt{d_2}e^{-\eta t}$ for all $t \geq 0$. Therefore, one can conclude that synchronization of the nonlinear MAS (25) is achieved exponentially with convergence rate η and convergence coefficient $\sqrt{d_2}$.

In the above, we assume that the communication bandwidth is unlimited so that the quantizer (1) will never be saturated. In the following, we will show that the state quantizer and the control quantizer will never be saturated in the case of finite quantization levels M_1^i and M_2, respectively, which implies that the above results hold for the finite-level quantizer case as well. For agent i, define

$$\chi_i^x(\sigma_i) = \frac{x_i(t_{\sigma_i}^i) - \hat{x}_i(t_{\sigma_i-1}^i) - \int_{t_{\sigma_i-1}^i}^{t_{\sigma_i}^i} f(\hat{x}_i(s))ds}{g_1(t_{\sigma_i}^i)}, \sigma_i = 0, 1, \ldots$$

and $\chi_i^u(k) = \hat{u}_i(T_k^i)/g_2(T_k^i), k = 0, 1, \ldots$.

Firstly, the state quantizer is analyzed. If $\sigma_i = 0$, one has $\|\chi_i^x(0)\| = \|x_i(0)/g_1(0)\| \leq \|x_i(0)/c_3\| \leq \Delta M_1^i$, where M_1^i is given in (39). If $\sigma_i \geq 1$, define $\varsigma_i(t_{\sigma_i}^i) = \hat{x}_i(t_{\sigma_i-1}^i) + \int_{t_{\sigma_i-1}^i}^{t_{\sigma_i}^i} f(\hat{x}_i(s))ds$, then one has $\|\chi_i^x(\sigma_i)\| = \|(x_i(t_{\sigma_i}^i) - \varsigma_i(t_{\sigma_i}^i))/g_1(t_{\sigma_i}^i)\| = \|(x_i(t_{\sigma_i}^i) - \hat{y}_i(t_{\sigma_i}^{i-}) + \hat{y}_i(t_{\sigma_i}^{i-}) - \varsigma_i(t_{\sigma_i}^i))/g_1(t_{\sigma_i}^i)\|$, where $\hat{y}_i(t_{\sigma_i}^{i-})$ is the value of $\hat{y}_i$ before jumping, and it is used to distinguish with $\hat{y}_i(t_{\sigma_i}^i)$. According to the definition of $\hat{y}_i(t)$, one has $x_i(t_{\sigma_i}^i) - \hat{y}_i(t_{\sigma_i}^{i-}) = x_i(t_{\sigma_i}^i - h) + \int_{t_{\sigma_i}^i-h}^{t_{\sigma_i}^i} f(x_i(s)) + \tilde{u}_i(s)ds - \hat{y}_i(t_{\sigma_i}^i - h) - \int_{t_{\sigma_i}^i-h}^{t_{\sigma_i}^i} f(\hat{y}_i(s))ds$.

According to the definition of $e_{y_i}(t_l)$, one has $\|x_i(t_{\sigma_i}^i - h) - \hat{y}_i(t_{\sigma_i}^i - h)\| = \|e_{y_i}(t_{\sigma_i}^i - h)\|$. If $t_{\sigma_i}^i - h \in \{t_{\sigma_i}^i\}$, $\|e_{y_i}(t_{\sigma_i}^i - h)\| = 0$, otherwise, $\|e_{y_i}(t_{\sigma_i}^i - h)\| \leq c_1 e^{-\alpha(t_{\sigma_i}^i - h)}$. Thus, $\|x_i(t_{\sigma_i}^i - h) - \hat{y}_i(t_{\sigma_i}^i - h) + \int_{t_{\sigma_i}^i-h}^{t_{\sigma_i}^i} f(x_i(s)) - f(\hat{y}_i(s))ds\| \leq c_1 e^{(\rho_1+\alpha)h} e^{-\alpha t_{\sigma_i}^i}$. Moreover, $\tilde{u}(t) = \tilde{u}(t_{\sigma_i}^i - h)$ is a constant vector for $t \in [t_{\sigma_i}^i - h, t_{\sigma_i}^i)$. Therefore,

$$\begin{aligned}\left\|x_i(t_{\sigma_i}^i) - \hat{y}_i(t_{\sigma_i}^{i-})\right\| \leq & c_1 e^{(\rho_1+\alpha)h} e^{-\alpha t_{\sigma_i}^i} \\ & + h(\left\|\hat{u}_i(t_{\sigma_i}^i - h)\right\| + \left\|e_{q_i}^u(t_{\sigma_i}^i - h)\right\|).\end{aligned} \tag{57}$$

a) If $t_{\sigma_i}^i - h \in \{T_k^i\}$, then $\hat{u}_i(t_{\sigma_i}^i - h) = \sum_{j \in \mathcal{N}_i} \psi_{ij}(x_{ji}(t_{\sigma_i}^i - h) + e_{y_{ji}}(t_{\sigma_i}^i - h) + e_{q_{ji}}^y(t_{\sigma_i}^i - h))$. Let $\hat{u}(t)$ be the column stack vector of $\hat{u}_i(t)$. Then one has

$$\begin{aligned}\left\|\hat{u}_i(t_{\sigma_i}^i - h)\right\| \leq & \gamma(\left\|\xi(t_{\sigma_i}^i - h)\right\| + \left\|e_y(t_{\sigma_i}^i - h)\right\| + \left\|e_q^y(t_{\sigma_i}^i - h)\right\|) \\ \leq & \hat{C}e^{-\eta(t_{\sigma_i}^i - h)},\end{aligned} \tag{58}$$

where $\hat{C} = \gamma(\sqrt{d_2} + \sqrt{N}c_1 + \sqrt{Nn}c_3\Delta e^{(\eta+\rho_1)Rh}/2)$.

Substituting (58) into (57), one can get

$$\begin{aligned}\|x_i(t_{\sigma_i}^i) - \hat{y}_i(t_{\sigma_i}^{i-})\| \leq & c_1 e^{(\rho_1+\alpha)h} e^{-\alpha t_{\sigma_i}^i} \\ & + h\left(\hat{C} + \frac{c_4\sqrt{n}\Delta}{2}\right) e^{-\eta(t_{\sigma_i}^i - h)}.\end{aligned} \tag{59}$$

b) If $t_{\sigma_i}^i - h \notin \{T_k^i\}$, then one has $\left\|\hat{u}_i(t_{\sigma_i}^i - h)\right\| \leq \left\|\hat{u}_i(T_{k'}^i)\right\| + c_2 e^{-\alpha(t_{\sigma_i}^i - h)}$, where $k' = \arg\min_{l \in \mathbb{Z}: T_l^i < t_{\sigma_i}^i - h}\{t_{\sigma_i}^i - h - T_l^i\}$. According to a), one can get $\left\|\hat{u}_i(T_{k'}^i)\right\| \leq \hat{C}e^{-\eta T_{k'}^i} \leq \hat{C}e^{\eta Rh} e^{-\eta(t_{\sigma_i}^i - h)}$. Thus,

$$\left\|\hat{u}_i(t_{\sigma_i}^i - h)\right\| \leq \hat{C}e^{\eta Rh} e^{-\eta(t_{\sigma_i}^i - h)} + c_2 e^{-\alpha(t_{\sigma_i}^i - h)}. \tag{60}$$

Substituting (60) into (57), one can get

$$\begin{aligned}\|x_i(t_{\sigma_i}^i) - \hat{y}_i(t_{\sigma_i}^{i-})\| \leq & c_1 e^{(\rho_1+\alpha)h} e^{-\alpha t_{\sigma_i}^i} \\ & + h\left(\hat{C}e^{\eta Rh} + c_2 + \frac{c_4\sqrt{n}\Delta}{2}\right) e^{-\eta(t_{\sigma_i}^i - h)}.\end{aligned} \tag{61}$$

In addition, $\|\hat{y}_i(t_{\sigma_i}^{i-}) - \varsigma_i(t_{\sigma_i}^i)\| \leq e^{\rho_1(t_{\sigma_i}^i - t_{\sigma_i-1}^i)}\|x_i(t_{\sigma_i-1}^i) - \hat{x}_i(t_{\sigma_i-1}^i)\|$ and $\|x_i(t_{\sigma_i-1}^i) - \hat{x}_i(t_{\sigma_i-1}^i)\| \leq g_1(t_{\sigma_i-1}^i)\sqrt{n}\Delta/2$. Thus,

$$\|y_i(t_{\sigma_i}^{i-}) - \varsigma_i(t_{\sigma_i}^i)\| \leq e^{\rho_1 Rh} c_3 e^{-\eta(t_{\sigma_i-1}^i)}\sqrt{n}\Delta/2. \tag{62}$$

Combining (59), (61) and (62), one has $\|\chi_i^x(\sigma_i)\| \leq (c_1 e^{(\rho_1+\alpha)h} e^{-\alpha t_{\sigma_i}^i} + hCe^{-\eta(t_{\sigma_i}^i - h)})/c_3 e^{-\eta t_{\sigma_i}^i} + (e^{\rho_1 Rh} c_3 e^{-\eta(t_{\sigma_i-1}^i)}\sqrt{n}\Delta/2)/c_3 e^{-\eta t_{\sigma_i}^i} \leq \Delta M_1^i$.

Next, the control quantizer is analyzed. For all $k \geq 0$, one has

$$\|\chi_i^u(k)\| = \left\|\frac{\hat{u}_i(T_k^i)}{g_2(T_k^i)}\right\| \leq \left\|\frac{\hat{u}(T_k^i)}{g_2(T_k^i)}\right\| \leq \frac{Ce^{-\eta T_k^i}}{g_2(T_k^i)} \leq \Delta M_2,$$

where M_2 is given in (40). ■

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