Hierarchical control for uncertain discrete-time nonlinear systems under signal temporal logic specifications

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***Abstract*— This paper studies the hierarchical control of uncertain discrete-time nonlinear systems under input constraints. First, the notion of robust approximate simulation relation is defined. We show that by properly designing a control interface, the robust approximate simulation relation can be constructed from a low-complexity, deterministic (abstract) system to the original system. Then, we apply the hierarchical control approach to the robust control synthesis under signal temporal logic specifications. The results show that this approach reduces the computational complexity of the control synthesis, and is in some cases applicable to a larger set of initial states. The effectiveness of the proposed approach is verified by a simulation example.**

I. Introduction

Hierarchical control has been widely used in the context of control systems [1] and robotics [2]. The fact that real-world systems, e.g., intelligent vehicle systems [3], are often nonlinear, high-dimensional, and subject to uncertainties, makes them difficult to control in order to achieve complex tasks. This leads naturally to the hierarchical control framework, in which a simpler system (referred to as abstract system), is introduced for the purpose of simplifying planning and control.

Early techniques of hierarchical control were based on the notion of simulation relations [1], [4]. This relation requires that the original system (referred to as concrete system) and its abstraction have exactly the same trajectories. It was later pointed out that this requirement may be too strong [5]. To this end, approximate (bi)simulation relations were introduced [6], [7], which allow the trajectories of the concrete and abstract systems match only approximately. Such a relaxation has made hierarchical control applicable to more general class of systems. In [8]–[12], approximate (bi)simulation relations have been constructed for transition systems [8], continuous-time nonlinear systems [9], large-scale interconnected systems [10], [11], as well as stochastic hybrid systems [12]. Moreover, to further account for uncertainties (which is crucial for physical systems operating in the real-world, e.g., robots), the notion of robust approximate simulation relation has been proposed recently [13], [14]. In [13], continuous-time linear systems subject to additive disturbances were studied. In [14], uncertain continuous-time nonlinear systems were considered. However, to the best of our knowledge, conditions that enforce robust approximate simulation relations have not been developed for general uncertain discrete-time nonlinear systems.

In the area of robot motion planning, increasing attention has been paid to the control synthesis under high level specifications, such as linear temporal logic (LTL) and signal temporal logic (STL) specifications. LTL focuses on the Boolean satisfaction of properties over given signals [15]. STL further allows the specification of quantitative spatial and temporal properties on the system [16], which is beneficial for cyber-physical systems [17]. In [18]–[20], hierarchical control approaches have been successfully applied to the controller synthesis under LTL specifications. For STL control synthesis, these approaches have not been investigated so far. Existing methods that deal with STL control synthesis include optimization-based [21], [22], control barrier function [23], [24], and learning-based [25], [26] methods. In our recent work [27], the notion of tube-based temporal logic tree (tTLT) is proposed for the robust control synthesis under STL specifications. Nevertheless, most of the existing methods are either suffering from the computational complexity issue or not efficient in dealing with uncertainties.

Motivated by the above considerations, this work concerns the hierarchical control of uncertain discrete-time nonlinear systems and the application to robust control synthesis under STL specifications. The main contributions are as follows. i) For uncertain discrete-time nonlinear systems, we define the notion of robust approximate simulation relation. It is shown that with a properly designed control interface, such a relation can be constructed from a (possibly) low-dimensional, deterministic abstract system to the concrete system. ii) The application of the hierarchical control to robust control synthesis under STL specifications is investigated. It is shown that this approach can reduce the computational complexity of the control synthesis, and is in some cases applicable to a larger set of initial states.

II. Preliminaries

Notation. Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$. Denote $\mathbb{R}^n$ as the n dimensional real vector space, $\mathbb{R}^{n\times m}$ as the $n \times m$ real matrix space. Throughout this paper, vectors are denoted in italics, $x \in \mathbb{R}^n$, and boldface $\boldsymbol{x}$ is used for discrete-time signals. Let $\|x\|$ and $\|A\|$ be the Euclidean norm of vector x and matrix A. The operators $\cup$ and $\cap$ represent set union and intersection, respectively. In addition, we use $\wedge$ and $\vee$ to denote the logical operators

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AND and OR, respectively. The set difference $A \setminus B$ is defined by $A \setminus B := \{x : x \in A \wedge x \notin B\}$.

A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

A. System dynamics

We consider an uncertain discrete-time nonlinear system Σ of the form

$$\Sigma : \begin{cases} x_{k+1} = f(x_k, u_k, w_k), \\ y_{k+1} = h(x_{k+1}), \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}, y_k \in \mathbb{R}^{n_y}, u_k \in U \subseteq \mathbb{R}^{n_u}, w_k \in W \subseteq \mathbb{R}^l, k \in \mathbb{N}$ are the state, output, input, and disturbance at time k, respectively. We assume that the functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^l \to \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ are continuous maps. The input and disturbance of (1) are constrained to compact sets U and W, respectively. Assume that (1) is obtained by sampling of a continuous-time system. Let $\tau : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be the corresponding sampling function, which satisfies $\tau(0) = 0$ and $\tau(k) < \tau(k+1), \forall k \in \mathbb{N}$. For simplicity, we define $\tau_k := \tau(k)$. Then, one has that $x_k = x(\tau_k), y_k = y(\tau_k), u_k = u(\tau_k)$, and $w_k = w(\tau_k)$. For a given continuous-time interval $[a, b], a \leq b$, define the set $\Omega(a, b) := \{k \in \mathbb{N} : a \leq \tau_k \leq b\}$ as the corresponding discrete-time counterpart.

Denote by $\mathcal{U}_{\geq k} := \{u_k u_{k+1} \ldots : u_l \in U, \forall l \geq k\}$ and $\mathcal{W}_{\geq k} := \{w_k w_{k+1} \ldots : w_l \in W, \forall l \geq k\}$. The solution of (1) is defined as a discrete-time signal $\boldsymbol{x} := x_0 x_1 \ldots$. We call $\boldsymbol{x}$ a *trajectory* of (1) if there exists a control signal $\boldsymbol{u} \in \mathcal{U}_{\geq 0}$ and a disturbance signal $\boldsymbol{w} \in \mathcal{W}_{\geq 0}$ satisfying (1). We call $\boldsymbol{y} := y_0 y_1 \ldots$ an *output trajectory* of (1) if $y_k = h(x_k), \forall k \in \mathbb{N}$. We use $\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}(k)$ to denote the trajectory point reached at time k under the control signal $\boldsymbol{u}$ and the disturbance signal $\boldsymbol{w}$ from initial state x_0. In this work, bounded disturbances are considered. Therefore, we have the following assumption.

Assumption 2.1: There exists a constant $\bar{w} > 0$ such that $\|w\| \leq \bar{w}, \forall w \in W$.

B. Signal temporal logic

STL [16] is a predicate logic consisting of predicates μ, which are obtained after evaluation of a predicate function $g_\mu : \mathbb{R}^n \to \mathbb{R}$ as $\mu := \top$ if $g_\mu(x) \geq 0$ and $\mu := \bot$, otherwise.

In [22], it was shown that each STL formula has an equivalent STL formula in positive normal form (PNF), *i.e.,* negations only occur adjacent to predicates. The syntax of the PNF STL is given by

$$\varphi ::= \top \mid \mu \mid \neg\mu \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathsf{U}_{\mathrm{I}} \varphi_2 \mid \mathsf{G}_{\mathrm{I}} \varphi. \tag{2}$$

where $\varphi, \varphi_1, \varphi_2$ are STL formulas and I is a closed or half-closed interval of the form $[a, b]$ or $[a, b)$ with $a, b \in \mathbb{R}_{\geq 0} \cup \infty$ and $a \leq b$.

Definition 2.1 (STL semantics): [21] The validity of an STL formula φ with respect to a discrete-time signal $\boldsymbol{x}$ at time k, is defined inductively as follows:

$$\begin{aligned} (\boldsymbol{x}, k) \vDash \mu \quad &\Leftrightarrow \quad g_\mu(\boldsymbol{x}_k) \geq 0, \\ (\boldsymbol{x}, k) \vDash \neg\varphi \quad &\Leftrightarrow \quad \neg((\boldsymbol{x}, k) \vDash \varphi), \\ (\boldsymbol{x}, k) \vDash \varphi_1 \wedge \varphi_2 \quad &\Leftrightarrow \quad (\boldsymbol{x}, k) \vDash \varphi_1 \wedge (\boldsymbol{x}, k) \vDash \varphi_2, \\ (\boldsymbol{x}, k) \vDash \varphi_1 \mathsf{U}_{[a,b]} \varphi_2 \quad &\Leftrightarrow \quad \exists k' \in \Omega(\tau_k + a, \tau_k + b) \text{ s.t.} \\ &\qquad (\boldsymbol{x}, k') \vDash \varphi_2 \wedge \\ &\qquad \forall k'' \in \Omega(\tau_k, \tau_{k'}), (\boldsymbol{x}, k'') \vDash \varphi_1. \end{aligned}$$

In Definition 2.1, the satisfaction relation $(\boldsymbol{x}, k) \vDash \varphi$ denotes that the discrete signal $\boldsymbol{x}$ satisfies φ from time step k. If $(\boldsymbol{x}, 0) \vDash \varphi$, this will be denoted by $\boldsymbol{x} \vDash \varphi$.

Definition 2.2 (STL robust semantics): [21] The space robustness of an STL formula φ is defined inductively as follows:

$$\begin{aligned} &\rho^{\mu}(\boldsymbol{x}, k) := g_\mu(\boldsymbol{x}_k), \\ &\rho^{\neg\varphi}(\boldsymbol{x}, k) := -\rho^{\varphi}(\boldsymbol{x}, k), \\ &\rho^{\varphi_1 \wedge \varphi_2}(\boldsymbol{x}, k) := \min\{\rho^{\varphi_1}(\boldsymbol{x}, k), \rho^{\varphi_2}(\boldsymbol{x}, k)\}, \\ &\rho^{\varphi_1 \mathsf{U}_{[a,b]} \varphi_2}(\boldsymbol{x}, k) := \max_{k' \in \Omega(\tau_k + a, \tau_k + b)} \min \big\{\rho^{\varphi_2}(\boldsymbol{x}, k'), \\ &\qquad\qquad \min_{k'' \in \Omega(\tau_k, \tau_{k'})} \rho^{\varphi_1}(\boldsymbol{x}, k'')\big\}. \end{aligned}$$

One can conclude that $\boldsymbol{x} \vDash \varphi$ if and only if $\rho^{\varphi}(\boldsymbol{x}, 0) \geq 0$.

C. Tube-based temporal logic tree

In our previous work [27], a notion of tTLT is defined for STL formulas. In addition, an online control synthesis algorithm is designed for the robust control synthesis under STL specifications. We recap here the main elements of [27] that will be used later.

Definition 2.3: A tTLT is a tree for which

- each node is either a *tube* node, *i.e.,* a node that maps from the nonnegative time axis $\mathbb{R}_{\geq 0}$ to a subset of $\mathbb{R}^n$, or an *operator* node, *i.e.,* a node that belongs to $\{\wedge, \vee, \mathsf{U}_{\mathrm{I}}, \mathsf{F}_{\mathrm{I}}, \mathsf{G}_{\mathrm{I}}\}$;
- the root node and the leaf nodes are *tube* nodes;
- if a *tube* node is not a leaf node, its unique child is an *operator* node;
- the children of any *operator* node are *tube* nodes.

Lemma 2.1: [27] Given the uncertain system Σ in (1) and an STL formula φ in PNF, a tTLT, denoted by $\mathcal{T}_\varphi$, can be constructed from Σ and φ.

III. Robust approximate simulation relation

From now on, we will refer to the uncertain system Σ defined in (1) as the concrete system. Control of Σ will be synthesized hierarchically via an abstract system Σ' and with a control interface u_v. The abstract system is defined by:

$$\Sigma' : \begin{cases} z_{k+1} = g(z_k, v_k), \\ q_{k+1} = \kappa(z_{k+1}) \end{cases} \tag{3}$$

where $z_k \in \mathbb{R}^{n_z}, q_k \in \mathbb{R}^{n_y}$, and $v_k \in U' \subseteq \mathbb{R}^{n_v}$. Note that we abstract the uncertain system Σ by a deterministic system Σ' and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_x}$ is the abstracted dynamics.

The systems Σ and Σ' have the same output space (i.e., $\mathbb{R}^{n_y}$), but may have different state and input spaces. In addition, the input set U' is a design parameter that will be specified later. The trajectory and output trajectory of (3) are denoted by discrete-time signals $\boldsymbol{z} := z_0 z_1 \ldots$ and $\boldsymbol{q} := q_0 q_1 \ldots$, respectively. We use $\boldsymbol{z}_{z_0}^{\boldsymbol{v}}(k)$ to denote the trajectory point reached at time k under the control signal $\boldsymbol{v}$ from initial state z_0.

The control interface $u_v : \mathbb{R}^{n_v} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_u}$ is given by

$$u_k = u_v(v_k, x_k, z_k). \tag{4}$$

Let $\varepsilon > 0$ be a given precision that we want to maintain between Σ and Σ'. Define

$$X_0 := \{(x_0, z_0) : \|h(x_0) - \kappa(z_0)\| \leq \varepsilon\}.$$

In the following, we assume without loss of generality that $\forall x_0 \in \mathbb{R}^{n_x}, \exists z_0 \in \mathbb{R}^{n_z}$ such that $(x_0, z_0) \in X_0$. Denote by $\mathcal{U}_{[0,k]} := \{u_0 \ldots u_k : u_l \in U, \forall l = 0, \ldots, k\}$, $\mathcal{W}_{[0,k]} := \{w_0 \ldots w_k : w_l \in W, \forall l = 0, \ldots, k\}$ and $\mathcal{U}'_{[0,k]} := \{v_0 \ldots v_k : v_l \in U', \forall l = 0, \ldots, k\}$. Then, we have the following definition.

Definition 3.1: The control interface $u_v : \mathbb{R}^{n_v} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_u}$ is called *admissible* if there exists a set $U' \neq \emptyset$ such that

- $u_0 = u_v(v_0, x_0, z_0) \in U, \forall (x_0, z_0) \in X_0, \forall v_0 \in U'$,
- $u_k = u_v(v_k, \boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}(k), \boldsymbol{z}_{z_0}^{\boldsymbol{v}}(k)) \in U, \forall (x_0, z_0) \in X_0, \forall \boldsymbol{u} \in \mathcal{U}_{[0,k-1]}, \forall \boldsymbol{v} \in \mathcal{U}'_{[0,k-1]}, \forall \boldsymbol{w} \in \mathcal{W}_{[0,k-1]}, \forall k \geq 1$.

One can see from Definition 3.1 that if a control interface u_v is admissible, then the synthesized control signal $\boldsymbol{u}$ is admissible, *i.e.,* $\boldsymbol{u} \in \mathcal{U}_{\geq 0}$.

The robust approximate simulation relation is defined in terms of a Lyapunov-like simulation function as follows.

Definition 3.2: Given the concrete system Σ in (1) and the abstract system Σ' in (3), a function $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ is called a *robust simulation function* for the system pair (Σ, Σ') and u_v is the associated control interface if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}, \sigma$ and a constant $\gamma > 0$ such that:

i) $\forall x, x' \in \mathbb{R}^{n_x}$,

$$\underline{\alpha}(\|h(x) - \kappa(x')\|) \leq V(x, x') \leq \bar{\alpha}(\|h(x) - \kappa(x')\|), \tag{5}$$

ii) $\forall x, x' \in \mathbb{R}^{n_x}, \forall v \in U'$,

$$\begin{aligned} &V\big(f(x, u_v(v, x, x'), w), g(x', v)\big) - V(x, x') \\ &\quad \leq -\gamma V(x, x') + \sigma(\|w\|), \forall w : \|w\| \leq \bar{w}. \end{aligned} \tag{6}$$

Remark 3.1: In [13], the notion of robust simulation function is defined for unconstrained continuous-time nonlinear systems, whereas we here define the robust simulation function for uncertain discrete-time nonlinear systems under input constraint. Moreover, in Definition 3 of [13], it is required that $V(x, x') > \gamma_1(\|v\|) + \gamma_2(\|w\|), \forall v, d$, where γ_1, γ_2 are two class $\mathcal{K}_\infty$ functions. This means that the design of the robust simulation function is dependent on the disturbance w and input v. This requirement is relaxed in Definition 3.2 in this work.

Definition 3.3: Given the concrete system Σ in (1) and the abstract system Σ' in (3), we say that Σ *robustly approximately simulates* Σ' *with parameters* $(\varepsilon, \bar{w})$, denoted by $\Sigma' \preceq_{\mathcal{S}}^{(\varepsilon, \bar{w})} \Sigma$, if:

i) $\forall x_0 \in \mathbb{R}^{n_x}, \exists z_0 \in \mathbb{R}^{n_z}$ such that $\|h(x_0) - \kappa(z_0)\| \leq \varepsilon$,

ii) $\forall x, x'$ s.t. $\|h(x) - \kappa(x')\| \leq \varepsilon$, $\forall v \in U', \exists u \in U$ such that

$$\|h(f(x, u, w)) - \kappa(g(x', v))\| \leq \varepsilon, \forall w : \|w\| \leq \bar{w}.$$

Then, we have the following result.

Theorem 3.1: Given the concrete system Σ in (1) and the abstract system Σ' in (3), suppose that Assumption 2.1 holds and there exists a robust simulation function V for (Σ, Σ') with u_v being the associated control interface. Assume furthermore that

i) $\bar{w} < \sigma^{-1}(\underline{\alpha}(\varepsilon))$,

ii) u_v is admissible, and

iii) the constant γ satisfies

$$1 - \frac{\underline{\alpha}(\varepsilon) - \sigma(\bar{w})}{\bar{\alpha}(\varepsilon)} \leq \gamma \leq 1, \tag{7}$$

where $\underline{\alpha}, \bar{\alpha}, \sigma, \gamma$ are defined in Definition 3.2, $\bar{w}$ is defined in Assumption 2.1, and ε is the desired precision. Then, $\Sigma' \preceq_{\mathcal{S}}^{(\varepsilon, \bar{w})} \Sigma$.

Proof: Given x, x' such that $\|h(x) - \kappa(x')\| \leq \varepsilon$ and $v \in U'$, define $\hat{x} := f(x, u_v(v, x, x'), w)$ and $\hat{x}' := g(x', v)$. Since the control interface u_v is admissible, one has that $u_v(v, x, x') \in U$. To prove item ii) of Definition 3.3, it is sufficient to prove that $\|h(\hat{x}) - \kappa(\hat{x}')\| \leq \varepsilon, \forall w : \|w\| \leq \bar{w}$.

Since V is a robust simulation function, then (5) and (6) hold, which gives $V(\hat{x}, \hat{x}') \leq (1 - \gamma)V(x, x) + \sigma(\bar{w})$. In addition, γ satisfies (7), then one has that

$$\begin{aligned} V(\hat{x}, \hat{x}') &\leq (1 - \gamma)\bar{\alpha}(\|h(x) - \kappa(x')\|) + \sigma(\bar{w}) \\ &\leq \frac{\underline{\alpha}(\varepsilon) - \sigma(\bar{w})}{\bar{\alpha}(\varepsilon)} \bar{\alpha}(\varepsilon) + \sigma(\bar{w}) \\ &\leq \underline{\alpha}(\varepsilon) \end{aligned}$$

if $\bar{w} < \sigma^{-1}(\underline{\alpha}(\varepsilon))$. Therefore,

$$\|h(\hat{x}) - \kappa(\hat{x}')\| \leq \underline{\alpha}^{-1}(V(\hat{x}, \hat{x}')) \leq \underline{\alpha}^{-1}(\underline{\alpha}(\varepsilon)) \leq \varepsilon$$

for all $w : \|w\| \leq \bar{w}$. Item ii) of Definition 3.3 holds. In addition, one has that item i) of Definition 3.3 holds by assumption. Therefore, $\Sigma' \preceq_{\mathcal{S}}^{(\varepsilon, \bar{w})} \Sigma$. ■

Remark 3.2: From Theorem 3.1, one can see that a properly designed control interface, *i.e.,* a u_v that is admissible and guarantees the existence of a robust simulation function V and the satisfaction of condition (7), is crucial for the existence of the robust approximate simulation relation from Σ' to Σ.

IV. Application to STL control synthesis

In this section, we show the application of the proposed hierarchical control approach to the robust control synthesis under STL specifications.

Assume a PNF STL formula φ over a set of predicates $\{\mu_1,\ldots,\mu_m\}$. Each predicate $\mu_i, i=1,\ldots,m$ is defined over the output signal $\boldsymbol{y}$. In addition, define

$$\mu_i^{\varepsilon} := \begin{cases} \top, & \text{if} \quad g_{\mu_i}(y_k) \geq \varepsilon, \\ \bot, & \text{if} \quad g_{\mu_i}(y_k) < \varepsilon. \end{cases}$$

Then, one can further define the STL formula φ^{ε}, where φ^{ε} is obtained by replacing each predicate μ_i with μ_i^{ε}. For example, given $\varphi = \mathsf{F}_{[a_1,b_1]}\mathsf{G}_{[a_2,b_2]}\mu_1 \wedge \mu_2 \mathsf{U}_{[a_3,b_3]}\mu_3$, then $\varphi^{\varepsilon} = \mathsf{F}_{[a_1,b_1]}\mathsf{G}_{[a_2,b_2]}\mu_1^{\varepsilon} \wedge \mu_2^{\varepsilon} \mathsf{U}_{[a_3,b_3]}\mu_3^{\varepsilon}$.

Theorem 4.1: Given the concrete system Σ in (1), the abstract system Σ' in (3), and the STL formula φ, suppose that Assumption 2.1 holds and there exists a robust simulation function V for (Σ,Σ') with u_v being the associated control interface. If furthermore, one has that u_v satisfies items ii)-iii) of Theorem 3.1, then,

$$\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}}) \models \varphi^{\varepsilon} \Rightarrow h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}) \models \varphi, \forall (x_0,z_0) \in X_0, \forall \boldsymbol{w} \in \mathcal{W}_{\geq 0},$$

where $u_k = u_v(v_k,x_k,z_k), \forall k \in \mathbb{N}$, $\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}})$ and $h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}})$ are the output trajectories of (1) and (3), respectively.

Proof: From the definition of φ^{ε} and Definition 2.2, one has that $(\boldsymbol{y},0) \models \varphi^{\varepsilon} \Rightarrow \rho^{\varphi}(\boldsymbol{y},0) \geq \varepsilon$.

Since there exists a robust simulation function V for (Σ,Σ') and u_v satisfies items ii)-iii) of Theorem 3.1, one can get from Theorem 3.1 that $\Sigma' \preceq_{\mathcal{S}}^{(\varepsilon,\bar{w})} \Sigma$. According to Definition 3.3, it further implies that $\forall (x_0,z_0) \in X_0$,

$$\|\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}}(k)) - h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}(k))\| \leq \varepsilon, \forall k \in \mathbb{N}, \forall \boldsymbol{w} \in \mathcal{W}_{\geq 0}.$$

Therefore, $\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}}) \models \varphi^{\varepsilon} \Rightarrow \rho^{\varphi}(\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}},0)) \geq \varepsilon \Rightarrow \rho^{\varphi}(h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}},0)) \geq 0, \forall (x_0,z_0) \in X_0, \forall \boldsymbol{w} \in \mathcal{W}_{\geq 0} \Rightarrow h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}) \models \varphi, \forall (x_0,z_0) \in X_0, \forall \boldsymbol{w} \in \mathcal{W}_{\geq 0}$. ■

Remark 4.1: Theorem 4.1 allows us to transform the robust control synthesis problem for the uncertain system Σ to the control synthesis problem for the deterministic system Σ'. The latter one can be solved by many existing approaches. For instance, a mixed integer program formulation in [22] when φ is bounded can be used, whereas an online control synthesis algorithm is proposed in [27].

In the following, we outline the procedure of the hierarchical control, where Algorithm *onlineControlSyntheis* (Algorithm 5, [27]) is adopted for the control synthesis of Σ'. We note that other approaches, such as the mixed integer program in [22], can also be adopted. Firstly, an initialization process (Algorithm 1) is required, where a tTLT $\mathcal{T}_{\varphi^{\varepsilon}}$ is constructed from Σ' and φ^{ε} using Algorithm *tTLT-Construction* (Algorithm 1, [27]). Then, one round (compute (v_k,z_{k+1}) and (u_k,x_{k+1}) given (x_k,z_k,k)) of the online control synthesis is outlined in Algorithm 2. Using Algorithm *onlineControlSyntheis*, a feasible control input set $\mathbb{U}(z_k,k)$ can be obtained at each k given $(\mathcal{T}_{\varphi^{\varepsilon}},z_k,k)$ (line 1). The control input v_k can be chosen as any element of $\mathbb{U}(z_k,k)$ (line 2), and then one can get z_{k+1} (line 3). The control input u_k is obtained via the admissible control interface u_v (line 4), and then we implement u_k and measure x_{k+1} (line 5).

Now, let us recap the following definitions from [27].

Algorithm 1 *Initialization*

Input: Σ' and φ.
Return: $\mathcal{T}_{\varphi^{\varepsilon}}$.
obtain φ^{ε} from φ,
$\mathcal{T}_{\varphi^{\varepsilon}} \leftarrow$ *tTLTConstruction*$(\Sigma',\varphi^{\varepsilon})$.

Algorithm 2 *hierarchicalControlSynthesis*

Input: $\mathcal{T}_{\varphi^{\varepsilon}},\Sigma,\Sigma'$ and (x_k,z_k,k).
Return: (v_k,z_{k+1}) and (u_k,x_{k+1}).
1: $\mathbb{U}(z_k,k) \leftarrow$ *onlineControlSynthesis*$(\mathcal{T}_{\varphi^{\varepsilon}},z_k,k)$,
2: choose $v_k \in \mathbb{U}(z_k)$,
3: $z_{k+1} \leftarrow g(z_k,v_k)$,
4: $u_k \leftarrow u_v(v_k,x_k,z_k)$,
5: implement u_k and obtain x_{k+1}.

Definition 4.1 (Satisfiability): Let the system Σ' in (3) and the STL formula φ. We say that φ is satisfiable from the initial state z_0 if there exists a control signal $\boldsymbol{v} \in \mathcal{U}'_{\geq 0}$ such that $\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}}) \models \varphi$.

Definition 4.2 (Robust satisfiability): Let the uncertain system Σ in (1) and the STL formula φ. We say that φ is robust satisfiable from the initial state x_0 if there exists a control signal $\boldsymbol{u} \in \mathcal{U}_{\geq 0}$ such that $h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}) \models \varphi, \forall \boldsymbol{w} \in \mathcal{W}_{\geq 0}$.

Let $\mathbb{S}_{\varphi^{\varepsilon}}^{\Sigma'} := \{z_0 \in \mathbb{R}^{n_z} | \varphi^{\varepsilon}$ is satisfiable for Σ' from $z_0\}$ be the set of initial states of Σ' from which φ^{ε} is satisfiable. Denote by $\boldsymbol{v} = v_0v_1\ldots$ and $\boldsymbol{u} = u_0u_1\ldots$ the control signals for Σ' and Σ, respectively.

Theorem 4.2: Given the concrete system Σ in (1), the abstract system Σ' in (3), and the STL formula φ. Assume that the conditions in Theorem 4.1 hold and $(x_0,z_0) \in X_0$. If $z_0 \in \mathbb{S}_{\varphi^{\varepsilon}}^{\Sigma'}$ and $\kappa(\boldsymbol{z}_{z_0}^{\boldsymbol{v}}) \models \varphi^{\varepsilon}$, then by implementing the control interface u_v, i.e., $u_k = u_v(v_k,x_k,z_k), \forall k$, one can guarantee that,

$$\forall \boldsymbol{w} \in \mathcal{W}_{\geq 0}, h(\boldsymbol{x}_{x_0}^{\boldsymbol{u},\boldsymbol{w}}) \models \varphi.$$

Let

$$\mathbb{S}_{\varphi}^{\Sigma} := \{x_0 \in \mathbb{R}^{n_x} | \varphi \text{ is robust satisfiable for } \Sigma \text{ from } x_0\}$$

be the set of initial states of Σ from which φ is robust satisfiable. In the following, we use a simple example to show that in some cases, one can have $\{x_0 \in \mathbb{R}^{n_x} : (x_0,z_0) \in X_0, z_0 \in \mathbb{S}_{\varphi^{\varepsilon}}^{\Sigma'}\} \supseteq \mathbb{S}_{\varphi}^{\Sigma}$, *i.e.,* the hierarchical control approach applies to a larger set of initial states.

Example 4.1: Consider the following uncertain discrete-time linear system

$$\Sigma: \begin{cases} x_{k+1} = 3x_k + u_k + w_k, \\ y_{k+1} = x_{k+1}, \end{cases}$$

where $x_k, y_k \in \mathbb{R}^2, u_k \in U := \{u \in \mathbb{R}^2 : ||u|| \leq 5.2\}, w_k \in W := \{w \in \mathbb{R}^2 : ||w|| \leq 0.2\}, \forall k \in \mathbb{N}$. Without loss of generality, we assume that $x_k = x(\tau_k) = x(k)$, *i.e.,* $\tau_k = k, \forall k$. The task φ is given by $\varphi = \mathsf{G}_{[5,10]}\mu$, where $g_{\mu}(y_k) = 10 - \|y_k\|$. Then, one can compute that

$$\mathbb{S}_{\varphi}^{\Sigma} = \{x \in \mathbb{R}^2 : ||x|| \leq \frac{10}{3}\}.$$

Let the abstract system Σ' be given by

$$\Sigma' : z_{k+1} = 3z_k + v_k, q_{k+1} = z_{k+1},$$

where $v_k \in U', \forall k$. The control interface u_v is designed as

$$u_k = u_v(v_k, x_k, z_k) = v_k - 2.5(x_k - z_k).$$

One can verify that $V(x, x') = \|x - x'\|$ is a robust simulation function for (Σ, Σ') with $\underline{\alpha}(s) = \bar{\alpha}(s) = s, \gamma = 0.5, \delta(s) = s$. Choosing the desired precision $\varepsilon = 0.5$ and the input set $U' = \{u \in \mathbb{R}^2 : ||u|| \leq 3.95\}$, one can verify that items i)-iii) of Theorem 3.1 hold. Therefore, $\Sigma' \preceq_{\mathcal{S}}^{(\varepsilon,\bar{w})} \Sigma$. Then, one can further compute that

$$\mathbb{S}_{\varphi^\varepsilon}^{\Sigma'} = \{z \in \mathbb{R}^2 : ||z|| \leq \frac{269}{60}\},$$

and thus $\{x_0 \in \mathbb{R}^2 : (x_0, z_0) \in X_0, z_0 \in \mathbb{S}_{\varphi^\varepsilon}^{\Sigma'}\} \supset \mathbb{S}_{\varphi}^{\Sigma}$.

Remark 4.2: We note that for more general uncertain discrete-time nonlinear systems and STL formulas, similar results (as in Example 4.1) can be obtained. Therefore, we argue that a larger set of initial conditions is achievable in some cases with the proposed hierarchical control approach.

V. Simulation

A simulation example is provided in this section to validate the effectiveness of the theoretical results. Consider an uncertain discrete-time nonlinear system

$$\Sigma_1 : \begin{cases} x_{k+1} = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} u_k + p(x_k) + w_k, \\ y_{k+1} = x_{k+1}, \end{cases}$$

where the input set $U = [-3.5, 3.5] \times [-3.5, 3.5]$ and the disturbance set $W = [-0.2, 0.2] \times [-0.2, 0.2]$. The sampling interval is 0.5s, that is, $\tau(k) = 0.5k, \forall k \in \mathbb{N}$. The nonlinear function $p(q) = 0.1 \sin(q)$, where the sinusoidal function $\sin(\cdot)$ is defined element-wise.

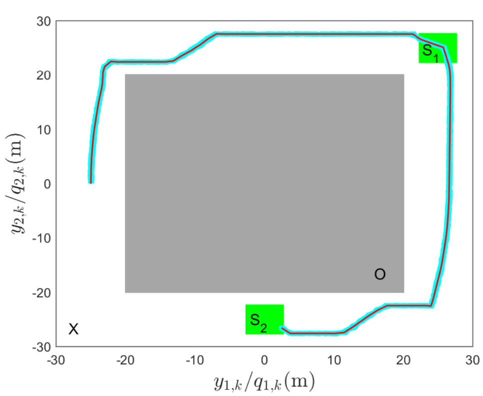

Fig. 1. Output trajectories $\boldsymbol{y}$ of the concrete system Σ_1 (light blue lines) for 100 realizations of disturbance signals and output trajectory $\boldsymbol{q}$ of the abstract system Σ_1' (red line).

The problem is to control Σ_1 to move in the bounded workspace X shown in Fig. 1, where the grey solid polygon O represents an obstacle and the green solid polygons S_1, S_2 represent two target regions. The task specification is expressed as an STL formula $\varphi = \mathsf{G}_{[0,\infty)}(X \wedge \neg O) \wedge \mathsf{F}_{[0,35]}\mathsf{G}_{[0,5]}S_1 \wedge \mathsf{F}_{[50,75]}S_2$.

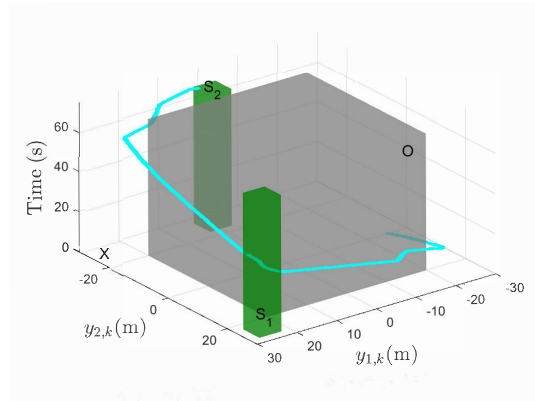

Fig. 2. The evolution of the 100 output trajectories $\boldsymbol{y}$ (light blue lines) with respect to time, where the output trajectories reach S_1 at either 29.5s or 30s, leave S_1 at either 36.5s or 37.5s, and reach S_2 at either 68s or 68.5s. Recall that the sampling interval is 0.5s.

The abstract system Σ_1' is given by

$$\Sigma_1' : \begin{cases} z_{k+1} = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} z_k + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} v_k, \\ q_{k+1} = z_{k+1} \end{cases}$$

with the input set U'. Let $\varepsilon = 0.6$ be the desired precision. The control interface u_v is designed as

$$u_v(v_k, x_k, z_k) = v_k - \begin{bmatrix} 2 & 0.01 \\ 0.01 & 2 \end{bmatrix} (x_k - z_k).$$

Then, by choosing $U' = [-2.2, 2.2] \times [-2.2, 2.2]$, one can guarantee that the control interface u_v is admissible and $\Sigma_1' \preceq_{\mathcal{S}}^{(\varepsilon,\bar{w})} \Sigma_1$.

Firstly, a tTLT $\mathcal{T}_{\varphi^\varepsilon}$ is constructed for Σ_1' using Algorithm 1. Then, the control signals $\boldsymbol{v}$ and $\boldsymbol{u}$ for Σ_1' and Σ_1 are obtained by implementing Algorithm 2 iteratively. The output trajectory $\boldsymbol{q}$ for Σ_1' is plotted in Fig. 1 (solid red line), where $q_{1,k}, q_{2,k}$ are the two components of q_k. Furthermore, in order to validate robustness, we run 100 realizations of the disturbance trajectories. The resulting output trajectories $\boldsymbol{y}$ for Σ_1 for these 100 realizations are shown (by the solid light blue lines) in Fig. 1, where $y_{1,k}, y_{2,k}$ are the two components of y_k. The evolution of the 100 output trajectories $\boldsymbol{y}$ with respect to time is depicted in Fig. 2. One can see that all the output trajectories $\boldsymbol{y}$ satisfy the STL formula φ. The evolution of the output error $\|y_k - q_k\|$ for the 100 realizations of disturbance signals is depicted in Fig. 3, and one can see that the desired precision $\varepsilon = 0.6$ is preserved at all times. In addition, the evolution of the input components $v_{1,k}, v_{2,k}$ and $u_{1,k}, u_{2,k}$ for the abstract system Σ_1' and the concrete system Σ_1 is plotted in Fig. 4, respectively. One can see that $u_k \in U, \forall k \in \mathbb{N}$ (i.e., the input constraint is satisfied at any time). We note that the use of the hierarchical control approach for uncertain discrete-time nonlinear systems under STL specifications is novel. In addition, it is shown in Example 4.1 that this approach can be applied to a larger set of initial conditions in some cases as compared to [27].

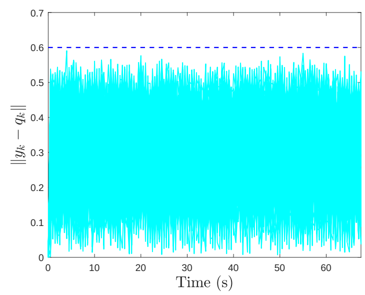

Fig. 3. The evolution of $\|y_k - q_k\|$ for 100 realizations of disturbance signals.

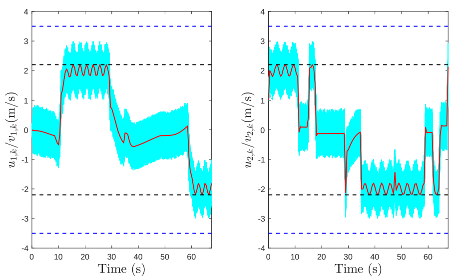

Fig. 4. The evolution of the input components $u_{1,k}, u_{2,k}$ (light blue lines) for 100 realizations of disturbance signals and $v_{1,k}, v_{2,k}$ (red lines), where the dash blue and black lines represent the bounds of the input sets U and U', respectively.

Finally, we report the computation time of this example, which is run in Matlab R2018b on a Dell laptop with Windows 10, Intel i7-6600U CPU2.80 GHz and 16.0 GB RAM. We perform reachability analysis for constructing the tTLT $\mathcal{T}_{\varphi^\epsilon}$ offline, which takes 2.4692 seconds. For online control synthesis, the average computation time at a single time step over 100 realizations is 0.322 seconds.

VI. Conclusion

A notion of robust approximate simulation relation was proposed for the hierarchical control of uncertain discrete-time nonlinear systems. First, it was shown that the robust approximate simulation relation can be constructed with a properly designed control interface. Then, the application of the hierarchical control to the robust control synthesis under STL specifications was investigated. Future work includes the extension of this approach to other control problems as well as experimental validation.

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