A Common Framework for Complete and Incomplete Attitude Synchronization in Networks with Switching Topology

Pedro O. Pereira, Dimitris Boskos and Dimos V. Dimarogonas

***Abstract*—In this paper, we study attitude synchronization for elements in the unit sphere in $\mathbb{R}^3$ and for elements in the $3D$ rotation group, for a network with switching topology. The agents' angular velocities are assumed to be the control inputs, and a switching control law for each agent is devised that guarantees synchronization, provided that all elements are initially contained in a region, which we identify later in the paper. The control law is decentralized and it does not require a common orientation frame among all agents. We refer to synchronization of unit vectors in $\mathbb{R}^3$ as incomplete synchronization, and of $3D$ rotation matrices as complete synchronization. Our main contribution lies in showing that these two problems can be analyzed under a common framework, where all agents' dynamics are transformed into unit vectors dynamics on a sphere of appropriate dimension.**

I. Introduction

Decentralized control in a multi-agent environment has been a topic of active research, with applications in large scale robotic systems. Attitude synchronization in satellite formations is one of the relevant applications [1], [2], where the control goal is to guarantee that a network of fully actuated rigid bodies acquires a common attitude. Coordination of underwater vehicles in ocean exploration missions [3], and of unmanned aerial vehicles in aerial exploration missions, may also be casted as attitude synchronization problems.

In the literature, attitude synchronization strategies for elements in the special orthogonal group are found in [4]–[17], which focus on *complete* attitude synchronization; and in [15], [18]–[27], which focus on *incomplete* attitude synchronization. In this paper, we focus on both *complete* and *incomplete* attitude synchronization. We refer to *incomplete* attitude synchronization when the agents are unit vectors in $\mathbb{R}^3$ (synchronization in $\mathbb{S}^2$); and we refer to *complete* attitude synchronization, when the agents are $3D$ rotation matrices (synchronization in $\mathbb{SO}(3)$). Incomplete synchronization represents a relevant practical problem, when the goal among multiple agents is to share a common direction: for example, in flocking, moving along a common direction is a requirement; and, in a network of satellites, whose antennas are to point in a common direction, incomplete synchronization may be more pertinent than complete. Finally, incomplete synchronization can also describe chemical and biological oscillators [15].

In [5]–[8], [10]–[12], state dependent control laws for torques are presented which guarantee synchronization for elements in $\mathbb{SO}(3)$, while in [21], [22], [28] state dependent control laws for torques are presented which guarantee synchronization for elements in $\mathbb{S}^2$. In these works, all agents are dynamic and their angular velocities are part of the state of the system, rather than control inputs. In this paper, however, we consider kinematic agents and we design control laws for the agents' angular velocities, which are not exclusively state dependent, but are also time dependent, with the time dependency encapsulating the switching network topology. Regarding synchronization in $\mathbb{SO}(3)$, different parameterizations have been used in the literature, such as, unit quaternions, the modified Rodrigues parameters or rotation matrices [7], [29]–[31]. In this paper, we model the agents and present the control laws as functions of rotation matrices, while performing the analysis with unit quaternions. Incomplete synchronization is not necessarily accomplished under complete synchronization, and it is not considered in most works regarding complete synchronization [13]–[17], even though it shares a commonality with complete synchronization, which we show in this paper. Moreover, complete synchronization requires full angular velocity control, while incomplete synchronization can be accomplished with restricted angular velocity control, which further motivates the study of incomplete synchronization.

In [13], [15]–[17], [32], [33], consensus on non-linear spaces is analyzed with the help of a common weak non-smooth Lyapunov function, i.e., a Lyapunov function which is non-increasing along solutions, either for fixed network graphs or relying on a dwell time assumption between consecutive switches of network graphs. In our proposed framework, we relax the assumption of a dwell time by providing conditions for synchronization under average dwell time, and, in order to handle the non-smoothness of the proposed Lyapunov function, we present an invariance-like result. In brief, we propose control laws for angular velocities of unit vectors in $\mathbb{R}^3$ and $3D$ rotation matrices that guarantee synchronization for a network of agents with a switching topology. The control laws devised for unit vectors and rotation matrices achieve different goals, and differ in two aspects worth emphasizing. First, controlling rotation matrices requires more measurements when compared with controlling unit vectors; secondly, while controlling rotation matrices requires full actuation, i.e., all body components of the angular velocity need to be controllable, controlling unit vectors does not. Our main contribution compared to the aforementioned literature lies in analyzing both problems under a common framework, in order to allow for a unified stability analysis using the same common weak Lyapunov function. Particularly both problems are transformed into synchronization problems in $\mathbb{S}^m$ for an appropriate $m \in \mathbb{N}$, and where a dwell time between consecutive switches is not required. A preliminary version of this work is found in [34]. With respect to this preliminary version, this paper provides proofs which were omitted therein due to space constraints (especially the proof of Theorem 1, which is fundamental for the final result); it provides further details concerning the synchronization in $\mathbb{SO}(3)$; it provides the output maps for each agent, and the control laws as functions of the outputs; and the exposition of the results in Section V is simplified by exploiting an invariance property.

The authors are with the School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden. Email addresses: {ppereira, boskos, dimos}@kth.se. This work was supported by the EU H2020 Research and Innovation Programme under GA No.644128 (AEROWORKS), the Swedish Research Council (VR), the Swedish Foundation for Strategic Research (SSF) and the KAW Foundation.

II. Notation

Given $n \in \mathbb{N}$, we denote the inner product of $x, y \in \mathbb{R}^n$ as $\langle x, y\rangle$ and the identity matrix in $\mathbb{R}^{n\times n}$ as I_n. We denote $\mathcal{S} : \mathbb{R}^3 \ni a \mapsto \mathcal{S}(a) \in \{\text{antisymmetric matrices in } \mathbb{R}^{3\times 3}\}$, with $\mathcal{S}(a)$ as the matrix satisfying $\mathcal{S}(a)\, b = a \times b$ for any $a, b \in \mathbb{R}^3$; and with $\mathcal{S}^{-1}$ denoting its inverse. We denote by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$ the unit sphere in $\mathbb{R}^{n+1}$. The map $\Pi : \mathbb{S}^n \ni x \mapsto \Pi(x) \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined as the map satisfying $\Pi(x)\, y := y - \langle y, x\rangle x$ for any $y \in \mathbb{R}^{n+1}$, and represents the orthogonal projection of y to the subspace orthogonal to $x \in \mathbb{S}^n$. $e_1, \cdots, e_n \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote the canonical basis vectors in $\mathbb{R}^n$. Given $r > 0$, we denote $\mathcal{B}_n(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ and $\bar{\mathcal{B}}_n(r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ as the open and closed balls of radius r and centered around 0_n, respectively. Given a function $f : A \to B$, for some normed spaces A and B, $df(a) : A \to B$ denotes its derivative at some $a \in A$ (if f is differentiable); and given any $N \in \mathbb{N}$, we denote $f^N : A^N \ni a := (a_1, \cdots, a_N) \mapsto f^N(a) := (f(a_1), \cdots, f(a_N)) \in B^N$. Finally, given a manifold M, $T_m M$ denotes the tangent space to M at $m \in M$ and $TM := \amalg_{m\in M} T_q M$ its tangent bundle.

III. Preliminaries

Throughout the paper, we consider a network of $N \in \mathbb{N}$ agents indexed by $\mathcal{N} := \{1, \cdots, N\}$, with $n \in \mathbb{N}$ the dimension of each agent's state space. We associate a subset of all undirected network graphs to the set $\mathcal{P} := \{1, 2, \cdots, P\} \subset \mathbb{N}$, where $P \leq 2^{N(N-1)}$, and consider an average dwell time switching signal $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ [35]. We then consider the agents' trajectory $t \mapsto x(t) := (x_1(t), \cdots, x_N(t))$ initiated from $x(0) \in (\mathbb{R}^n)^N$, which satisfies

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \text{ for almost all } t \in \mathbb{R}_{\geq 0}, \tag{1}$$

where, for any $p \in \mathcal{P}$,

$$f_p := (f_{1,p}, \cdots, f_{N,p}) : (\mathbb{R}^n)^N \to (\mathbb{R}^n)^N, \tag{2}$$

and where the vector fields $\{f_p\}_{p\in\mathcal{P}}$ in (2) are assumed to be locally Lipschitz maps. Next, we present further conditions on the same vector fields which guarantee that $t \mapsto x(t)$ converges to the consensus set $\mathcal{C} := \{(x_1, \cdots, x_N) \in (\mathbb{R}^n)^N : x_1 = \cdots = x_N\}$. In later sections, when studying synchronization in $\mathbb{S}^2$ and $\mathbb{SO}(3)$, we will transform the agents' dynamics into the form (1) and leverage the following theorem.

Theorem 1. *Consider the vector fields $\{f_p\}_{p\in\mathcal{P}}$ in* (2)*, denote $x := (x_1, \cdots, x_N) \in (\mathbb{R}^n)^N$, $\mathcal{H}(x) := \arg\max_{j\in\mathcal{N}}(\|x_j\|) \subseteq \mathcal{N}$, and assume that for certain $r > 0$ and all $x \in \mathcal{B}_n(r)^N$ the following hold:*

1) when $x \notin \mathcal{C}$,

a) $\max_{p\in\mathcal{P}} \langle x_i, f_{i,p}(x)\rangle \leq 0$, for all $i \in \mathcal{H}(x)$,

b) $\forall p \in \mathcal{P} \exists i \in \mathcal{H}(x) : \langle x_i, f_{i,p}(x)\rangle < 0$,

2) when $x \in \mathcal{C}$, $\langle x_i, f_{i,p}(x)\rangle = 0$ for all $i \in \mathcal{N}$ and $p \in \mathcal{P}$.

Then, for each initial condition $x(0) \in \mathcal{B}_n(r)^N$ with $r_0 := \max_{i\in\mathcal{N}} \|x_i(0)\| < r$, the set $\bar{\mathcal{B}}_n(r_0)^N$ is positively invariant for the solution $t \mapsto x(t)$ of (1)*. Moreover, if we define $V(x) := \max_{i\in\mathcal{N}} \frac{1}{2}\|x_i\|^2$, there exists a constant $V^\infty \in [0, V(x(0))]$ such that $\lim_{t\to\infty} V(x(t)) = V^\infty$; and, finally, the trajectory $t \mapsto x(t)$ converges asymptotically to $V^{-1}(V^\infty) \cap \mathcal{C} \subset \bar{\mathcal{B}}_n(r_0)^N \cap \mathcal{C}$.*

Proof. For the proof, we invoke [36, Corollary 4.7], which is applicable to persistent dwell time signals. We also note that although this corollary requires Lyapunov functions to be continuously differentiable, it can be extended to continuous Lyapunov functions by replacing [36, Theorem 1] with [37, Corollary 4.4 b)] (and making use of the Clarke generalized derivative). Finally, [36] considers multiple Lyapunov functions, one for each switching signal mode, whilst in this proof we restrict ourselves to a common Lyapunov function. For brevity, in what follows, we denote $x \mapsto f(x) :\Leftrightarrow (\mathbb{R}^n)^N \ni x := (x_1, \cdots, x_N) \mapsto f(x)$.

Consider then any $p \in \mathcal{P}$ and the vector field (2). Also, for brevity, denote $x \mapsto v_{i,p}(x) := \langle x_i, f_{i,p}(x)\rangle \in \mathbb{R}$, for all $i \in \mathcal{N}$, which are all continuous functions. Consider then the continuous function

$$x \mapsto V(x) := \max_{i\in\mathcal{N}} \frac{1}{2}\|x_i\|^2 \in \mathbb{R}_0^+, \tag{3}$$

whose generalized gradient (in the sense of Clarke) is given by (denote $\mathrm{co}(S)$ as the convex hull of a finite point set $S \subset \mathbb{R}^m$, for any $m \in \mathbb{N}$)

$$x \mapsto \partial V(x) = \mathrm{co}(\{e_i \otimes x_i\}_{i\in\mathcal{H}(x)}) \subseteq \mathbb{R}^{nN}, \tag{4}$$

and where we emphasize that $\cup_{v\leq\frac{1}{2}r^2} V^{-1}(v) = (\bar{\mathcal{B}}_n(r))^N$ for any $r \in \mathbb{R}_+$ (see Notation for definition of $\bar{\mathcal{B}}_n$). The generalized directional derivative of V along (2), for a mode $p \in \mathcal{P}$, is then given by

$$x \mapsto V_p^\circ(x) := \max_{dV\in\partial V(x)} \langle dV, f_p(x)\rangle \overset{(4)}{=} \max_{i\in\mathcal{H}(x)} v_{i,p}(x). \tag{5}$$

Recall that the Theorem's condition *1a)* reads as $v_{i,p}(x)|_{\forall i\in\mathcal{H}(x)} \leq 0$ and condition *1b)* reads as $\exists i \in \mathcal{H}(x)$ such that $v_{i,p}(x) < 0$, for any $p \in \mathcal{P}$. As such, the generalized derivative in (5) can be expressed equivalently as

$$x \mapsto V_p^\circ(x) = \begin{cases} v_{i,p}(x)|_{\text{any } i\in\mathcal{H}(x)} < 0 & x \in A_p \\ \max_{i\in\mathcal{H}(x)} v_{i,p}(x) \leq 0 & x \in B_p \\ 0 & x \in \mathcal{C} \end{cases}, \tag{6}$$

where

$$\begin{aligned} A_p &:= \{x \in \mathbb{R}^{nN} : \forall k, l \in \mathcal{H}(x), v_{k,p}(x) = v_{l,p}(x) < 0\}, \\ B_p &:= \{x \in \mathbb{R}^{nN} : \exists k, l \in \mathcal{H}(x), v_{k,p}(x) \neq v_{l,p}(x)\}. \end{aligned} \tag{7}$$

The function V in (3) is lower bounded and its generalized derivative along (1) is non-positive. This implies that $\lim_{t\to\infty} V(x(t)) =: V^\infty \in [0, V(x(0))]$; and that $\bar{\mathcal{B}}_n(r_0)^N$, with $r_0 := \max_{i\in\mathcal{N}} \|x_i(0)\|$, is positively invariant (since $V(x) \leq V(x(0)) \Leftrightarrow x \in \cup_{r\leq V(x(0))} V^{-1}(r) = \bar{\mathcal{B}}_n(r_0)^N$).

Moreover, it follows from (6) that $(V_p^\circ)^{-1}(0) \subseteq \mathcal{C} \cup B_p$, and that (see Proposition 2) $\overline{(V_p^\circ)^{-1}(0)} \subseteq \mathcal{C} \cup B_p$. We now wish to compute the largest invariant subset (in the sense of [36, Corollary 4.4]) of $V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)}$. For that purpose, consider a solution

$$\mathbb{R} \supset I \ni t \mapsto x(t) \in V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)}. \tag{8}$$

Composing (3) with (8) yields a constant function $I \ni t \mapsto V(x(t)) = V^\infty$, whose derivative is well defined, namely

$$I \ni t \mapsto \dot{V}(x(t)) = v_{i,p}(x(t))|_{\forall i\in\mathcal{H}(x(t))} = 0. \tag{9}$$

In fact, (9) implies that $v_{k,p}(x) = v_{l,p}(x)$ for all $k, l \in \mathcal{H}(x)$, which is not satisfied for any $x \in B_p$ as defined in (7). This implies that x in (8) does not belong to B_p. It then follows that the largest invariant subset of $V^{-1}(V^\infty) \cap \overline{(V_p^\circ)^{-1}(0)} \subseteq V^{-1}(V^\infty) \cap (\mathcal{C} \cup B_p)$ is, in fact, a subset of $V^{-1}(V^\infty) \cap \mathcal{C}$, which is independent of the mode $p \in \mathcal{P}$.

In brief, we used a Lyapunov function common to all modes, and verified that the largest invariant set for each mode is independent of the mode. Based on the previous observations, we can invoke [36, Corollary 4.7], from which it follows that a solution of (1) converges to $V^{-1}(V^\infty) \cap \mathcal{C} \subset \bar{\mathcal{B}}_n(r_0)^N \cap \mathcal{C}$. □

Proposition 2. *Let $p \in \mathcal{P}$, and consider the map V_p° in (6) and the set B_p in (7). The closure of the set where V_p° vanishes lies in $\mathcal{C} \cup B_p$, i.e., $\overline{(V_p^\circ)^{-1}(0)} \subseteq \mathcal{C} \cup B_p$.*

Proof. Consider a convergent sequence in $(V_p^\circ)^{-1}(0)$, namely

$$\{x^m \in (V_p^\circ)^{-1}(0)\}_{m\in\mathbb{N}}, \qquad (10)$$
$$x^\infty := \lim_{m\to\infty} x^m \in A_p \cup B_p \cup \mathcal{C}.$$

Let us prove that $\overline{(V_p^\circ)^{-1}(0)} \subseteq \mathcal{C} \cup B_p$ by assuming that $x^\infty \in A_p$, which will lead to a contraction. For that purpose, denote $B_\epsilon(x^\infty) := \{x \in \mathbb{R}^{nN} : \|x - x^\infty\| < \epsilon\}$ as an open ball of size $\epsilon > 0$ around x^∞, and denote $\mathcal{H}(x) := \arg\max_{j\in\mathcal{N}}(\|x_j\|)$.

a) Since the sequence (10) belongs to $(V_p^\circ)^{-1}(0) \subseteq \mathcal{C} \cup B_p$, it follows that for any $m \in \mathbb{N}$, there exist $i \in \mathcal{H}(x^m)$ s.t. $v_{i,p}(x^m) = 0$. *b)* Since the sequence (10) is convergent, it follows that for any $\epsilon > 0$ there exits $M \in \mathbb{N}$ s.t., for all $m \geq M$, $x^m \in B_\epsilon(x^\infty)$. *c)* By definition $\mathcal{H}(x) := \arg\max_{j\in\mathcal{N}}(\|x_j\|)$, therefore it follows that for any $x \in \mathbb{R}^{nN}$ there exists $\epsilon_1 > 0$ s.t., for all $y \in B_{\epsilon_1}(x)$, $\mathcal{H}(y) \subseteq \mathcal{H}(x)$. *d)* Since $v_{1,p}, \ldots, v_{N,p}$ (with $v_{i,p}(x) := \langle x_i, f_{i,p}(x)\rangle$) are continuous, it follows that for any $x \in A_p$ (see (7)) there exists $\epsilon_2 \in (0, \epsilon_1)$ s.t., for all $y \in B_{\epsilon_1}(x)$, $v_{i,p}(y) < 0 \forall i \in \mathcal{H}(y) \overset{c)}{\subseteq} \mathcal{H}(x)$. *e)* Combining *b)* with *d)*, it follows that for $\epsilon = \epsilon_2$ there exists $M \in \mathbb{N}$ s.t., for all $m \geq M$, $x^m \in B_{\epsilon_2}(x^\infty) \overset{d)}{\Rightarrow} v_{i,p}(x^m) < 0 \forall i \in \mathcal{H}(x^m) \subseteq \mathcal{H}(x^\infty)$. However, *e)* contradicts *a)*, which implies that $x^\infty \notin A_p$. □

IV. Synchronization

In the next subsections, we study synchronization of agents in $\mathbb{S}^2$ and $\mathbb{SO}(3)$. More specifically, we first present feedback control laws for the angular velocities of the agents, with which we determine the closed loop dynamics. Afterwards, by means of appropriate transformations, those dynamics are rewritten in a common form that allows us to study synchronization in $\mathbb{S}^2$ and $\mathbb{SO}(3)$ under a common framework. Additionally, in [38], we also show that consensus in $\mathbb{R}^n$ can be casted as a synchronization problem in a subset of $\mathbb{S}^n$, for any $n \in \mathbb{N}$. It is noted that in all the above cases the agents evolve in appropriate subsets of $\mathbb{S}^n$. These are specified through the following definition.

Definition 1. *Given $n \in \mathbb{N}$, $\alpha \in (0, \pi]$ and $\bar{\nu} \in \mathbb{S}^n$, the open α–ball $\mathcal{C}(\alpha, \bar{\nu})$ is defined as $\mathcal{C}(\alpha, \bar{\nu}) := \{\nu \in \mathbb{S}^n : \langle \bar{\nu}, \nu\rangle > \cos(\alpha) \Leftrightarrow \arccos(\langle \bar{\nu}, \nu\rangle) < \alpha\}$, representing the set of unit vectors that form an angle less than α with $\bar{\nu}$ (see Fig. 2). Similarly, we define the closed α–ball $\bar{\mathcal{C}}(\alpha, \bar{\nu}) := \{\nu \in \mathbb{S}^n : \langle \bar{\nu}, \nu\rangle \geq \cos(\alpha) \Leftrightarrow \arccos(\langle \bar{\nu}, \nu\rangle) \leq \alpha\}$.*

Given a group of unit vectors $\nu = (\nu_1, \cdots, \nu_N)$, we say that ν belongs to an open (closed) α–ball, where $\alpha \in (0, \pi]$, if $\nu \in \mathcal{C}(\alpha, \bar{\nu})^N (\bar{\mathcal{C}}(\alpha, \bar{\nu})^N)$ for a certain $\bar{\nu} \in \mathbb{S}^n$. We also say that ν is synchronized if $\nu_1 = \cdots = \nu_N$. Before presenting the common form of the agents' dynamics, recall that their network is modeled as a time varying digraph $\mathcal{G}(\sigma(t)) = \{\mathcal{N}, \mathcal{E}(\sigma(t))\}$, where $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ represents the switching signal, and with $\mathcal{G}(p)$ and $\mathcal{E}(p)$ as the graph and edge set corresponding to mode $p \in \mathcal{P}$. We also denote $\mathcal{N}_i(p) \equiv \{j_1, \cdots, j_{|\mathcal{N}_i(p)|}\} \subset \mathcal{N}$ as the neighbor set of agent $i \in \mathcal{N}$ corresponding to mode $p \in \mathcal{P}$. Next, consider a group $\nu = (\nu_1, \cdots, \nu_N)$ of unit vectors in $\mathbb{S}^n$, which evolves according to the dynamics

$$\dot{\nu}(t) = \tilde{f}_{\sigma(t)}(\nu(t)) = \begin{bmatrix} \tilde{f}_{1,\sigma(t)}(\nu(t)) \\ \vdots \\ \tilde{f}_{N,\sigma(t)}(\nu(t)) \end{bmatrix}, \nu(0) \in (\mathbb{S}^n)^N, \quad (11)$$

where $\tilde{f}_{i,\sigma(t)} : (\mathbb{S}^n)^N \to T\mathbb{S}^n$ is defined as

$$\tilde{f}_{i,\sigma(t)}(\nu) := \sum_{j\in\mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\nu_i, \nu_j)\Pi(\nu_i)\nu_j, \qquad (12)$$

for all $i \in \mathcal{N}$; i.e., $\dot{\nu}_i(t) = \tilde{f}_{i,\sigma(t)}(\nu(t))$. Recall that $\Pi(\nu_i)\nu_j := \nu_j - \langle \nu_j, \nu_i\rangle\nu_j$ (see Notation) and thus $\tilde{f}_{i,\sigma(t)}$ vanishes when $\nu_1 = \cdots = \nu_N$. Each function $\tilde{w}_{ij} : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ in (12) assigns the weight $\tilde{w}_{ij}(\nu_i, \nu_j)$ to the deviation between agent i and its neighbor j. We require that those functions are continuous and non-negative on $\mathcal{C}(\alpha, \bar{\nu})^2$ for a certain $\alpha \in (0, \pi]$ and $\bar{\nu} \in \mathbb{S}^n$. In addition, we require that

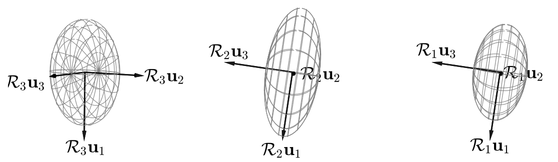
$$\tilde{w}_{ij}(\nu_i, \nu_j) > 0 \ \forall(\nu_i, \nu_j) \in \mathcal{C}(\alpha, \bar{\nu})^2 \text{ with } \langle \nu_i, \nu_j\rangle \neq 1. \quad (13)$$

The dynamics (11)–(12) are time dependent due to each agent's time varying neighbor set in (12). Furthermore, the system (11)–(12) is the standard form all problems are transformed into, with the unit vectors evolving in $\mathbb{S}^2$ and $\mathbb{S}^3$, for incomplete and complete synchronization, respectively. In particular, we will show that the proposed control laws guarantee asymptotic synchronization of the unit vectors, provided that they are initially contained in an open $\alpha^\star$–ball, with $\alpha^\star = \frac{\pi}{2}$ for synchronization in $\mathbb{S}^2$, and $\alpha^\star = \frac{\pi}{4}$ for synchronization in $\mathbb{SO}(3)$.

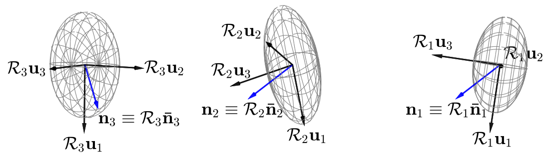
A. Complete synchronization in $\mathbb{SO}(3)$ casted as synchronization in $\mathbb{S}^3$

Consider a group of N agents $\mathcal{R}_1, \ldots, \mathcal{R}_N \in \mathbb{SO}(3) := \{\mathcal{R} \in \mathbb{R}^{3\times 3} : \mathcal{R}^T\mathcal{R} = \mathcal{R}\mathcal{R}^T = I_3, \det(\mathcal{R}) = 1\}$, where, for every $i \in \mathcal{N}$, $\mathcal{R}_i$ represents the orientation frame of agent i, with respect to an unknown inertial orientation frame. We say that the agents are synchronized if they all share the same *complete* orientation, i.e., if $\mathcal{R}_1 = \cdots = \mathcal{R}_N$, as illustrated in Fig. 1a. The term *complete* synchronization is used in juxtaposition with *incomplete* synchronization as described in the next subsection. In incomplete synchronization, rather than synchronizing all three bodies axes, the agents synchronize only one body direction. Furthermore, as explained in the next section, complete synchronization does not guarantee incomplete synchronization (and vice-versa).

For each $i \in \mathcal{N}$, $\omega_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ denotes the body-framed angular velocity of agent i, which is considered as its input. Each rotation matrix $\mathcal{R}_i$ evolves according to the dynamics



(a) Two on the right completely synchronized; two on the left not completely synchronized.



(b) Two on the right incompletely synchronized, i.e., $\mathcal{R}_1\bar{n}_1 = \mathcal{R}_2\bar{n}_2$; two on the left not incompletely synchronized, i.e., $\mathcal{R}_2\bar{n}_2 \neq \mathcal{R}_3\bar{n}_3$ (where $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 3^{-\frac{1}{2}}(1,1,1)$).

Fig. 1: In complete synchronization, N agents synchronize their rotation matrices. In incomplete synchronization, N agents align the unit vectors $n_i := \mathcal{R}_i\bar{n}_i$, where $\bar{n}_i$ is fixed in rigid body $i \in \mathcal{N}$ (u_1,u_2 and u_3 stand for the canonical basis vectors in $\mathbb{R}^3$).

$\dot{\mathcal{R}}_i(t) = f_{\mathcal{R}}(\mathcal{R}_i(t), \omega_i(t))$ with $f_{\mathcal{R}} : \mathbb{SO}(3) \times \mathbb{R}^3 \to T\mathbb{SO}(3)$ given as

$$f_{\mathcal{R}}(\mathcal{R}, \omega) := \mathcal{R}\mathcal{S}(\omega). \tag{14}$$

If, at a time instant $t \in \mathbb{R}_{\geq 0}$, agent i can measure the relative attitude between itself and another agent k, then $k \in \mathcal{N}_i(\sigma(t)) \equiv \{j_1, \cdots, j_{|\mathcal{N}_i(\sigma(t))|}\}$. This motivates the definition of the measurement function $h_i(t,\cdot) : \mathbb{SO}(3)^N \to \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|}$ for each time instant $t \in \mathbb{R}_{\geq 0}$, given as

$$h_i(t, \mathcal{R}) := (h_{ij_1}(\mathcal{R}), \cdots, h_{ij_{|\mathcal{N}_i(\sigma(t))|}}(\mathcal{R})), \tag{15}$$

$$\text{where } h_{ij}(\mathcal{R}) := \mathcal{R}_i^T\mathcal{R}_j, \tag{16}$$

for each $\mathcal{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_N)$ and $j \in \mathcal{N}_i(\sigma(t))$. Thus, at each $t \in \mathbb{R}_{\geq 0}$, agent i measures the rotation matrices corresponding to its neighbors' orientation with respect to its own orientation. We emphasize that the measurement in (16) does not require an agent to be aware of its own rotation matrix neither of its neighbor's rotation matrix (which are both specified with respect to an unknown inertial frame). Instead, an agent can measure its neighbors' three axes with respect to its own orientation fame.

Problem 1. *For each agent $i \in \mathcal{N}$ and time instant $t \geq 0$, given the measurement function* (15)*, design time-varying decentralized feedback laws $\omega_i^h(t,\cdot) : \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|} \to \mathbb{R}^3$, such that asymptotic synchronization of $\mathcal{R} := (\mathcal{R}_1, \cdots, \mathcal{R}_N) : \mathbb{R}_{\geq 0} \to \mathbb{SO}(3)^N$ is attained, where, for every $i \in \mathcal{N}$,*

$$\dot{\mathcal{R}}_i(t) = f_{\mathcal{R}}(\mathcal{R}_i(t), \omega_i^h(t, h_i(t, \mathcal{R}(t)))). \tag{17}$$

Problem 1 may be restated as finding a control law for each agent that depends exclusively on the measurement function as defined in (15), and which encodes the partial state information available to each agent at a given time instant.

Definition 2. *The angular distance between two rotation matrices (Riemannian distance on $\mathbb{SO}(3)$) is given by $\theta : \mathbb{SO}(3) \times \mathbb{SO}(3) \to [0, \pi]$, with $\theta(\mathcal{R}_1, \mathcal{R}_2) := \arccos\left(\frac{tr(\mathcal{R}_1^T\mathcal{R}_2)-1}{2}\right)$.*

For each agent $i \in \mathcal{N}$ and each time instant $t \in \mathbb{R}_{\geq 0}$, consider the control law $\omega_i^h(t,\cdot) : \mathbb{SO}(3)^{|\mathcal{N}_i(\sigma(t))|} \to \mathbb{R}^3$ defined as

$$\omega_i^h(t, h_i) := \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(I_3, h_{ij}))\mathcal{S}^{-1}\left(\frac{h_{ij} - h_{ij}^T}{2}\right), \tag{18}$$

where $h_i := (h_{ij_1}, \cdots, h_{ij_{|\mathcal{N}_i(\sigma(t))|}})$, and $w_{ij} : [0, \pi] \to \mathbb{R}_{\geq 0}$ is continuous and satisfies $w_{ij}(\theta) > 0$ for all $\theta \in (0, \pi]$. Notice that w_{ij} corresponds to a weight on the feedback law (18) that agent i assigns to the displacement between itself and its neighbor j. Thus, given the feedback law (18), one can write the closed-loop dynamics in (17).

Recalling that we wish to analyze all problems under a common framework, we will cast complete synchronization in $\mathbb{SO}(3)$ in the form (11)–(12), through a quaternion-based coordinate transformation. This change of variables serves only the purpose of analysis, while the implemented control law is still that in (18). Details on unit quaternions are found in [39], [40], and the identities shown next are also verified in [41]. We next fix a rotation matrix $\bar{\mathcal{R}} \in \mathbb{SO}(3)$ and denote

$$\tilde{\mathbb{SO}}(3) := \left\{\mathcal{R} \in \mathbb{SO}(3) : \theta(\bar{\mathcal{R}}, \mathcal{R}) < \pi\right\}, \tag{19a}$$

$$\mathcal{C}\left(\frac{\pi}{2}, e_1\right) := \{q \in \mathbb{S}^3 : \langle q, e_1\rangle > 0\}. \tag{19b}$$

Consider then the map $\phi : \tilde{\mathbb{SO}}(3) \to \mathcal{C}\left(\frac{\pi}{2}, e_1\right)$, together with its inverse, which are defined as

$$\phi(\mathcal{R}) := \frac{1}{2}\left(\sqrt{1 + \text{tr}(\bar{\mathcal{R}}^T\mathcal{R})}, \frac{\mathcal{S}^{-1}\left(\bar{\mathcal{R}}^T\mathcal{R} - \mathcal{R}^T\bar{\mathcal{R}}\right)}{\sqrt{1 + \text{tr}(\bar{\mathcal{R}}^T\mathcal{R})}}\right) \tag{20a}$$

$$\phi^{-1}(q) := \bar{\mathcal{R}}\left(I_3 + 2\bar{q}\mathcal{S}(\hat{q}) + 2\mathcal{S}(\hat{q})\mathcal{S}(\hat{q})\right), \tag{20b}$$

where $q = (\bar{q}, \hat{q})$ and $\hat{q} \in \mathbb{R}^3$. Note that since $\theta(I_3, \mathcal{R}) < \pi \Leftrightarrow 1 + \text{tr}(\mathcal{R}) > 0$, (20a) is well defined. Intuitively, ϕ provides a coordinate transformation between rotation matrices in $\tilde{\mathbb{SO}}(3)$ and unit vectors in $\mathcal{C}\left(\frac{\pi}{2}, e_1\right)$, i.e., unit quaternions with a positive first component.

The idea followed next is to *(i)*, given a rotation matrix $\mathcal{R} \in \tilde{\mathbb{SO}}(3)$, consider the quaternion $q = \phi(\mathcal{R}) \in \mathcal{C}\left(\frac{\pi}{2}, e_1\right)$; and *(ii)*, given the closed loop dynamics of the rotation matrix in (17), to compute the closed loop dynamics of the quaternion, which will be in the common form (12). We first define $Q : \mathbb{S}^3 \to \mathbb{R}^{4\times 3}$ as the map satisfying $Q(q)v := \langle q, e_1\rangle\tilde{v} - \langle q, \tilde{v}\rangle e_1 + \text{diag}(0, \mathcal{S}(\hat{q}))\tilde{v}$, for any $v \in \mathbb{R}^3$, where $q = (\bar{q}, \hat{q})$ and $\tilde{v} = (0, v)$. It can be also checked that $\langle q, Q(q)v\rangle = 0$, and that $Q(q)Q(q)^T = \Pi(q)$. Given the kinematics $f_{\mathcal{R}}$ in (14) and the coordinate change ϕ in (20), one may verify that

$$\dot{q} = d\phi(\mathcal{R})\mathcal{R}\mathcal{S}(\omega)|_{\mathcal{R}=\phi^{-1}(q)} = \frac{1}{2}Q(q)\omega, \tag{21}$$

where $d\phi(\mathcal{R}) : \mathbb{R}^{3\times 3} \to \mathbb{R}^4$ is the derivative of ϕ at $\mathcal{R} \in \tilde{\mathbb{SO}}(3)$; and that, for $\mathcal{R}_1 = \phi^{-1}(q_1)$ and $\mathcal{R}_2 = \phi^{-1}(q_2)$,

$$\mathcal{S}^{-1}\left(\mathcal{R}_1^T\mathcal{R}_2 - \mathcal{R}_2^T\mathcal{R}_1\right) = 2\langle q_1, q_2\rangle Q(q_1)^T q_2, \tag{22a}$$

$$\cos(\theta(I_3, \mathcal{R}_1^T\mathcal{R}_2)) = 2(\langle q_1, q_2\rangle)^2 - 1. \tag{22b}$$

With (21) and (22a)–(22b) in mind, one can then compute the closed-loop dynamics of the unit quaternions based on the closed-loop dynamics of the rotation matrices in (17); i.e., given $q := (q_1, \cdots, q_N) \in \mathcal{C}\left(\frac{\pi}{2}, e_1\right)^N$, it follows that

$$\dot{q}_i = d\phi(\mathcal{R}_i)f_{\mathcal{R}}(\mathcal{R}_i, \omega_i^h(t, h_i(t, \mathcal{R})))|_{\mathcal{R}_i=\phi^{-1}(q_i), i\in\mathcal{N}}$$

$$\overset{(21),(22)}{=} \sum_{j\in\mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(q_i,q_j)Q(q_i)Q(q_i)^T q_j$$
$$\overset{(12)}{=:} \tilde{f}_{i,\sigma(t)}(q). \quad (23)$$

where

$$\tilde{w}_{ij}(q_i,q_j) := \langle q_i,q_j\rangle w_{ij}(\arccos(2(\langle q_1,q_2\rangle)^2-1)). \quad (24)$$

Now, recall that (24) needs to satisfy (13), which forces us to restrict the sets in (19a)–(19b). The next proposition, which can be proved by exploiting the triangular inequality, sheds some light into the domain restriction we must impose.

Proposition 3. *Let $(q_1,\ldots,q_N)\in\mathcal{C}\left(\frac{\pi}{4},e_1\right)\times\ldots\times\mathcal{C}\left(\frac{\pi}{4},e_1\right)$. Then $\langle q_i,q_j\rangle>0$ for all $i,j\in\{1,\ldots,N\}$, and therefore the weight $\tilde{w}_{ij}(q_i,q_j)$, as defined in* (24)*, is positive when $q_i\neq q_j$, and non-negative otherwise.*

With the Proposition 3 in mind, if one considers the sets $\tilde{\mathbb{SO}}(3) := \left\{\mathcal{R}\in\mathbb{SO}(3):\theta(\bar{\mathcal{R}},\mathcal{R})<\pi/2\right\}$ and $\mathcal{C}\left(\frac{\pi}{4},e_1\right) := \left\{q\in\mathbb{S}^3:\langle q,e_1\rangle>\cos(\frac{\pi}{4})\right\}$ (instead of those in (19)), then ϕ and ϕ^{-1}, as defined in (20a) and (20b), still constitute diffeomorphisms, and the closed-loop dynamics of the unit quaternions in (23) are then restricted to $\mathcal{C}\left(\frac{\pi}{4},e_1\right)^N$ (instead of $\mathcal{C}\left(\frac{\pi}{2},e_1\right)^N$) for which the weights in (24) are non-negative (i.e., they satisfy (13)). In Section V, we show that any set $\bar{\mathcal{C}}\left(\alpha,e_1\right)^N$, with $\alpha<\frac{\pi}{4}$ is positively invariant under (23); and therefore (by means of the inverse map ϕ^{-1}) that $\left\{\mathcal{R}\in\mathbb{SO}(3):\theta(\bar{\mathcal{R}},\mathcal{R})\leq 2\alpha\right\}$ is positively invariant under (17) (the factor 2 relates with the fact that $\mathbb{S}^3$ provides a double covering of $\mathbb{SO}(3)$ [40]). That is, if all rotation matrices start α *close* to some rotation matrix $\bar{\mathcal{R}}$, they remain forever α *close* to it.

Remark 4. *Consider the set $\tilde{\mathbb{SO}}(3)$ in* (19) *and the set $\mathcal{C}\left(\pi,e_1\right) := \left\{q\in\mathbb{S}^3:\langle q,e_1\rangle>\cos(\pi)\right\}$. Suppose that we had chosen the map $\phi:\tilde{\mathbb{SO}}(3)\to\mathcal{C}\left(\pi,e_1\right)$ defined as $\phi(\mathcal{R}) := \left(\frac{tr(\bar{\mathcal{R}}^T\mathcal{R})-1}{2},\mathcal{S}^{-1}\left(\frac{\bar{\mathcal{R}}^T\mathcal{R}-\mathcal{R}^T\bar{\mathcal{R}}}{2}\right)\right)$ (instead of the map ϕ in* (20a)*). For this choice, the closed-loop dynamics under the new coordinates does not come in the form* (12)*, and thus the common framework cannot be leveraged for this alternative map ϕ. This illustrates the importance of the choice of the change of coordinates if the common framework is to be leveraged for other problems.*

B. Incomplete synchronization in $\mathbb{SO}(3)$ casted as synchronization in $\mathbb{S}^2$

In this section, we consider again a group of N agents operating in $\mathbb{SO}(3)$, but with a different synchronization objective. As in Section IV-A, for each $i\in N$, $\mathcal{R}_i$ represents the orientation frame of agent i. Additionally, for each agent i there is a constant body direction $\bar{n}_i\in\mathbb{S}^2$, known by the agent and its neighbors, which is required to synchronize with all the other agents' body directions. The goal of incomplete attitude synchronization in $\mathbb{SO}(3)$ is that all agents share the same orientation along the chosen body directions; i.e., given $(\mathcal{R}_1,\cdots,\mathcal{R}_N)\in\mathbb{SO}(3)^N$ and $(\bar{n}_1,\cdots,\bar{n}_N)\in(\mathbb{S}^2)^N$, *incomplete* synchronization is accomplished when $\mathcal{R}_1\bar{n}_1=\cdots=\mathcal{R}_N\bar{n}_N$, as illustrated in Fig. 1b. We note that the requirement for incomplete synchronization is independent from that of complete synchronization: i.e., complete synchronization does not imply incomplete synchronization (consider the case where $\bar{n}_1\neq\bar{n}_2$), and vice-versa (see Figure 1b, where two agents are incompletely synchronized but not completely synchronized).

As in Section IV-A, ω_i denotes the body-framed angular velocity of agent $i\in\mathcal{N}$, which is taken as the input. Again, each rotation matrix $\mathcal{R}_i$ evolves according to $\dot{\mathcal{R}}_i(t)=f_{\mathcal{R}}(\mathcal{R}_i(t),\omega_i(t))$ with $f_{\mathcal{R}}$ as defined in (14). If, additionally, we consider some constant $\bar{n}_i\in\mathbb{S}^2$, then $n_i:=\mathcal{R}_i\bar{n}_i\in\mathbb{S}^2$, evolves according to $\dot{n}_i(t)=f_n(\mathcal{R}_i(t),\omega_i(t),\bar{n}_i)$ where $f_n:\mathbb{SO}(3)\times\mathbb{R}^3\times\mathbb{S}^2\to\mathbb{S}^2$ is defined as

$$f_n(\mathcal{R},\omega,\bar{n}) := f_R(\mathcal{R},\omega)\bar{n}=\mathcal{R}\mathcal{S}\left(\omega\right)\bar{n}=\mathcal{S}\left(\mathcal{R}\bar{n}\right)^T\mathcal{R}\omega \quad (25)$$

If, at time $t\in\mathbb{R}_{\geq 0}$, agent $i\in\mathcal{N}$ is aware of the relative attitude of agent's j unit vector to be synchronized, then $j\in\mathcal{N}_i(\sigma(t))$. We therefore define for each $t\in\mathbb{R}_{\geq 0}$ the measurement function $h_i(t,\cdot):\mathbb{SO}(3)^N\to(\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|}$ as

$$h_i(t,\mathcal{R}) := (h_{ij_1}(\mathcal{R}),\cdots,h_{ij_{|\mathcal{N}_i(\sigma(t))|}}(\mathcal{R})), \quad (26)$$

where $h_{ij}(\mathcal{R}) := \mathcal{R}_i^T\mathcal{R}_j\bar{n}_j$, $\mathcal{R}=(\mathcal{R}_1,\cdots,\mathcal{R}_N)$, for each $j\in\mathcal{N}_i(\sigma(t))$. Thus, at each time instant $t\in\mathbb{R}_{\geq 0}$, agent $i\in\mathcal{N}$ measures the $|\mathcal{N}_i(\sigma(t))|$ unit vectors corresponding to the projection of a neighbor's unit vector onto agent's i orientation frame (note that the agent does not need to be aware of $\mathcal{R}_i$ and $\mathcal{R}_j$, but rather of the product $\mathcal{R}_i^T\mathcal{R}_j\bar{n}_j$).

Problem 2. *For each agent $i\in\mathcal{N}$ and time instant $t\geq 0$, given the measurement function* (26)*, design time-varying decentralized feedback laws $\omega_i^h(t,\cdot):(\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|}\to\mathbb{R}^3$, such that asymptotic synchronization of $(\mathcal{R}_1\bar{n}_1,\cdots,\mathcal{R}_N\bar{n}_N):\mathbb{R}_{\geq 0}\to(\mathbb{S}^2)^N$ is accomplished, where, for every $i\in\mathcal{N}$,*

$$\dot{\mathcal{R}}_i(t)=f_{\mathcal{R}}(\mathcal{R}_i(t),\omega_i^h(t,h_i(t,\mathcal{R}(t)))). \quad (27)$$

Problem 2 may be restated as finding a control law for each agent that depends exclusively on the measurement function as defined in (26), and which encodes the partial state information available to each agent at a given time instant.

Definition 3. *The angular displacement between two unit vectors (Riemannian distance on $\mathbb{S}^2$) is given by $\theta:\mathbb{S}^2\times\mathbb{S}^2\to[0,\pi]$, with $\theta(n_1,n_2):=\arccos(\langle n_2,n_2\rangle)$.*

For each $t\in\mathbb{R}_{\geq 0}$ and agent $i\in\mathcal{N}$, consider the control law $\omega_i^h(t,\cdot):(\mathbb{S}^2)^{|\mathcal{N}_i(\sigma(t))|}\to\mathbb{R}^3$ defined as

$$\omega_i^h(t,h_i) := \sum_{j\in\mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\bar{n}_i,h_{ij}))\mathcal{S}\left(\bar{n}_i\right)h_{ij}, \quad (28)$$

where $h_i:=(h_{ij_1},\cdots,h_{ij_{|\mathcal{N}_i(\sigma(t))|}})$, and $w_{ij}:[0,\pi]\to\mathbb{R}_{\geq 0}$ is a continuous function satisfying $w_{ij}(\theta)>0$ for all $\theta\in(0,\pi]$; and corresponding to a weight on the feedback law (28) that agent i assigns to the angular displacement between itself and its neighbor j. We note that (28) is orthogonal to $\bar{n}_i$, which implies that full angular velocity control is not necessary, i.e., we only need to control the angular velocity along the two directions orthogonal to $\bar{n}_i$ ($\omega_i^h(\cdot,\cdot)\in T_{\bar{n}_i}\mathbb{S}^2\subset\mathbb{R}^3$). At this point, given the feedback law (28), one can write the closed-loop dynamics in (27).

Recalling that we wish to analyze the problems under a common framework, we need to transform the dynamics in (27) to the form (11)–(12). In fact, it turns out that the kinematics (25), when composed with the proposed control law in (28) and the output function in (26), yield (12). Indeed, given $\mathcal{R} := (\mathcal{R}_1, \cdots, \mathcal{R}_N) \in (\mathbb{SO}(3))^N$ and denoting

$$n := (n_1, \cdots, n_N) = (\mathcal{R}_1 \bar{n}_1, \cdots, \mathcal{R}_N \bar{n}_N) \in (\mathbb{S}^2)^N, \quad (29)$$

we get that $\dot{n}_i = \dot{\mathcal{R}}_i \bar{n}_i = f_n(\mathcal{R}_i, \omega_i^h(t, h_i(t, \mathcal{R}(t))), \bar{n}_i)$. Indeed, it follows from (25), (28), and (26) that

$$\begin{aligned} \dot{n}_i &= f_n(\mathcal{R}_i, \omega_i^h(t, h_i(t, \mathcal{R}(t))), \bar{n}_i) \\ &= \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(n_i, n_j) \mathcal{S}(\mathcal{R}_i \bar{n}_i)^T \mathcal{S}(\mathcal{R}_i \bar{n}_i) \mathcal{R}_j \bar{n}_j \\ &\overset{(29)}{=:} \tilde{f}_{i,\sigma(t)}(n), \end{aligned} \quad (30)$$

where

$$\tilde{w}_{ij}(n_i, n_j) := w_{ij}(\theta(n_i, n_j)), \quad (31)$$

which is the desired form (12). Indeed, (30) follows from the facts that $\mathcal{S}(\cdot)^T \mathcal{S}(\cdot) = \Pi(\cdot)$ and that $\mathcal{R}\mathcal{S}(\cdot) = \mathcal{S}(\mathcal{R}\cdot)\mathcal{R}$ for any $\mathcal{R} \in \mathbb{SO}(3)$. Furthermore, we have that $\tilde{w}_{ij}(n_i, n_j) = w_{ij}(\theta(\bar{n}_i, \mathcal{R}_i^T \mathcal{R}_j \bar{n}_j))$ and thus the weights satisfy (13) for any $\alpha \in [0, \pi]$ and $\bar{\nu} \in \mathbb{S}^2$ (note that $\theta(n_1, n_2) = \theta(\mathcal{R}^T n_1, \mathcal{R}^T n_2)$ for any $\mathcal{R} \in \mathbb{SO}(3)$ and any $n_1, n_2 \in \mathbb{S}^2$). We have thus casted this problem in the form (11)–(12) with $\nu \equiv n \in (\mathbb{S}^2)^N$.

V. Analysis

In this section, we analyze the solutions of (11)–(12), and show that given a wide set of initial conditions, asymptotic synchronization is guaranteed. Specifically, asymptotic synchronization is guaranteed if all unit vectors are initially contained in an open $\alpha^\star$–ball, i.e., if $\nu(0) \in \mathcal{C}(\alpha^\star, \bar{\nu})^N$ for certain $\bar{\nu} \in \mathbb{S}^n$, where

$$\begin{cases} \alpha^\star = \frac{\pi}{2} & \text{for incomplete synchronization,} \\ \alpha^\star = \frac{\pi}{4} & \text{for complete synchronization.} \end{cases} \quad (32)$$

Next, we introduce a coordinate transformation that allows us to cast the dynamics (12) into a form that satisfies the conditions of Theorem 1. Also, and without loss of generality, we assume that $\nu(0) \in \mathcal{C}(\alpha^\star, e_1)^N$. Indeed, for the general case where $\nu(0) \in \mathcal{C}(\alpha^\star, \bar{\nu})^N$ for some $\bar{\nu} \in \mathbb{S}^n$, select a rotation matrix $\bar{\mathcal{R}} \in \mathbb{SO}(n+1)$, with $\bar{\mathcal{R}}\bar{\nu} = e_1$ and consider the transformation $\varphi(\nu) := (\bar{\mathcal{R}}^T \nu_1, \ldots, \bar{\mathcal{R}}^T \nu_N)$ between $\mathcal{C}(\alpha^\star, \bar{\nu})^N$ and $\mathcal{C}(\alpha^\star, e_1)^N$. Then, due to the fact that $\tilde{w}_{ij}(\cdot, \cdot) = \tilde{w}_{ij}(\bar{\mathcal{R}}^T \cdot, \bar{\mathcal{R}}^T \cdot)$ for the selected weights in (24) and (31), it follows from (11) that $d\varphi(\nu)\tilde{f}_{\sigma(t)}(\nu) = \tilde{f}_{\sigma(t)}(\varphi(\nu))$, i.e., the dynamics are the same in the transformed coordinates.

Next, given $\alpha \in \left(0, \frac{\pi}{2}\right]$, let $r = \sin(\alpha)$, and consider the matrix $Q := \begin{bmatrix} e_2 & \cdots & e_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}$ and the diffeomorphism $h : \mathcal{C}(\alpha, e_1) \to \mathcal{B}_n(r)$ (see Notation for the definition of $\mathcal{B}_n$), defined as

$$h(\nu_i) := Q^T \nu_i, \quad (33)$$

namely, the projection of the ball $\mathcal{C}(\alpha, e_1)$ to the subspace of $\mathbb{R}^{n+1}$ orthogonal to e_1 (see Fig. 2). Its inverse $h^{-1} : \mathcal{B}_n(r) \to \mathcal{C}(\alpha, e_1)$ is given by $h^{-1}(x_i) = \sqrt{1 - \|x_i\|^2} e_1 + Q x_i$. For the subsequent analysis we will also use the following propositions, whose proofs are elementary and can be found in [38].

Proposition 5. *Let $\nu_1, \nu_2 \in \mathcal{C}(\frac{\pi}{2}, e_1)$. Then, $\langle e_1, \nu_1 \rangle = \sqrt{1 - \|h(\nu_1)\|^2} > 0$ and the following implications hold: $\|h(\nu_1)\| > \|h(\nu_2)\| \Leftrightarrow 0 < \langle e_1, \nu_1 \rangle < \langle e_1, \nu_2 \rangle$, and $\|h(\nu_1)\| \geq \|h(\nu_2)\| \Leftrightarrow 0 < \langle e_1, \nu_i \rangle \leq \langle e_1, \nu_j \rangle$.*

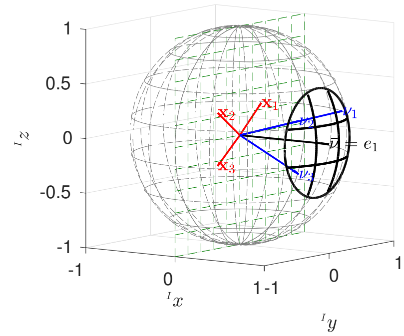


Fig. 2: Illustration of the ball $\bar{\mathcal{C}}(30°, e_1)$, with $\nu_1, \nu_2, \nu_3 \in \bar{\mathcal{C}}(30°, e_1)$. And illustration of the diffeomorphism $h(\cdot)$ in (33): $x_i = h(\nu_i) = Q^T \nu_i$, and green lines are those spanned by the columns of Q (which form a plane).

Proposition 6. *Let ν_1, $\nu_2 \in \mathbb{S}^n$ such that $0 < \langle e_1, \nu_1 \rangle \leq \langle e_1, \nu_2 \rangle$. Then* (a) *$\langle e_1, \Pi(\nu_1) \nu_2 \rangle = 0$ iff $\nu_1 = \nu_2$, and* (b) *$\langle e_1, \Pi(\nu_1) \nu_2 \rangle > 0$ iff $\nu_2 \neq \nu_1$.*

Consider now the solution $\nu : \mathbb{R}_{\geq 0} \to (\mathbb{S}^n)^N$ of (11)–(12) with $\nu(0) \in \mathcal{C}(\alpha, e_1)^N$ with $\alpha \in [0, \alpha^\star]$, which, as shown later in Theorem 8, remains in $\mathcal{C}(\alpha, e_1)^N$ for all $t \geq 0$. Then, by defining $x := h^N \circ \nu$ (see Notation for h^N), with $h(\cdot)$ as given in (33), it follows that $\dot{x}(t) = f_{\sigma(t)}(x(t))$, where

$$f_p(x) = dh^N(\nu) \tilde{f}_p(\nu)|_{\nu = (h^{-1})^N(x)}, \quad (34)$$

for all $p \in \mathcal{P}$. In addition, since $x_i = h \circ \nu_i$, we have that $\dot{x}_i(t) = f_{i,\sigma(t)}(x(t))$, where

$$\begin{aligned} f_{i,p}(x) &= dh(\nu_i) \tilde{f}_{i,p}(\nu)|_{\nu = (h^{-1})^N(x)} \\ &\overset{(12),(33)}{=} Q^T \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j|_{\nu = (h^{-1})^N(x)}. \end{aligned} \quad (35)$$

for each $p \in \mathcal{P}$. Furthermore, by taking into account that $QQ^T = \Pi(e_1)$, and that $\langle \nu_1, \Pi(\nu_1)\nu_2 \rangle = 0$ for any $\nu_1, \nu_2 \in \mathbb{S}^n$, it follows from (35) that

$$\begin{aligned} \langle x_i, f_{i,p}(x) \rangle &\overset{(35)}{=} \langle Q^T \nu_i, Q^T \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \Pi(\nu_i) \nu_j \rangle \\ &= -\langle e_1, \nu_i \rangle \sum\nolimits_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\nu_i, \nu_j) \langle e_1, \Pi(\nu_i) \nu_j \rangle, \end{aligned} \quad (36)$$

for any $\nu \in \mathcal{C}(\alpha, e_1)^N$ and $p \in \mathcal{P}$, and where $x = h^N(\nu) \in \mathcal{B}_n(\sin(\alpha))^N$. Next, by exploiting (36) and Propositions 5 and 6, we show that the conditions of Theorem 1 are satisfied for the vector fields in (34)–(35) (and recall that $\alpha^\star = \frac{\pi}{2}$ for synchronization in $\mathbb{S}^2$, and $\alpha^\star = \frac{\pi}{4}$ for synchronization in $\mathbb{SO}(3)$ as indicated in (32)).

Proposition 7. *Consider the vector fields in* (34)–(35) *and assume that $\mathcal{P}$ encodes strongly connected network digraphs. Then, the conditions of Theorem 1 are satisfied for $r = \sin(\alpha^\star)$, with $\alpha^\star$ as in* (32).

Proof. For the proof, we exploit (36) and the fact that h^N is a bijection between $\mathcal{C}(\alpha^\star, e_1)^N$ and $\mathcal{B}_n(\sin(\alpha^\star))^N$. In order to verify condition *1)* of Theorem 1, let $x = (x_1, \cdots, x_N) \in \mathcal{B}_n(r)^N$ and set $\nu = (\nu_1, \cdots, \nu_N) = (h^{-1})^N(x) \in \mathcal{C}(\alpha^\star, e_1)^N$. Thus, it follows that $\langle e_1, \nu_i \rangle > \cos(\alpha^\star) \geq 0$ for all $i \in \mathcal{N}$. In addition, from (13) and continuity of the weight functions

$\tilde{w}_{ij}(\cdot,\cdot)$, it follows that $\tilde{w}_{ij}(\nu_i,\nu_j) \geq 0$ for any $\nu_i, \nu_j \in \mathcal{C}(\alpha^\star, e_1)$. The latter two properties imply by virtue of (36) that the sign of each term $\langle x_i, f_{i,p}(x)\rangle$ is the opposite of $\langle e_1, \Pi(\nu_i)\nu_j\rangle$. Next, recall that $\mathcal{H}(x) := \arg\max_{i\in\mathcal{N}} \|x_i\|$ and pick $p \in \mathcal{P}$, $i \in \mathcal{H}(x)$ and $j \in \mathcal{N}$. From Proposition 5, it follows that $\langle e_1, \nu_i\rangle \leq \langle e_1, \nu_j\rangle$; combining the latter with Proposition 6, it follows that $\langle e_1, \Pi(\nu_i)\nu_j\rangle \geq 0$. Thus, we conclude that $\langle x_i, f_{i,p}(x)\rangle \leq 0$ and hence, condition *1)a)* is satisfied. The verification of the other conditions of Theorem 1 is found in [34]. $\square$

Finally, we use the result of Proposition 7 to prove that the solution of the system in (11) will reach consensus.

Theorem 8. *Consider the solution* $\nu : \mathbb{R}_{\geq 0} \to (\mathbb{S}^n)^N$ *of* (11) *with* $\nu(0) \in \mathcal{C}(\alpha^\star, e_1)^N$, *with* $\alpha^\star$ *as in* (32). *Then, for a network digraph strongly connected at all time instants,* i) $\nu(t) \in \bar{\mathcal{C}}(\alpha, e_1)^N$ *for all* $t \geq 0$, *where* $\alpha = \arccos(\max_{i\in\mathcal{N}}\langle e_1, \nu_i(0)\rangle) \in [0, \alpha^\star)$*;* ii) ν *synchronizes asymptotically and* $\lim_{t\to\infty}\langle e_1, \nu_i(t)\rangle$ *exists for all* $i \in \mathcal{N}$*; and* iii) *all unit vectors converge to a constant unit vector.*

Proof. Consider a solution ν of (11), and $x = h^N \circ \nu$. Since $\alpha^\star$ is either $\frac{\pi}{2}$ or $\frac{\pi}{4}$, then $\nu(0)$ is within the domain of h^N and moreover $\bar{\mathcal{B}}_n(r_0)^N \subset \mathcal{B}_n(\sin(\alpha^\star))^N$ where $r_0 \overset{(33)}{=} \max_{i\in\mathcal{N}} \|Q^T\nu_i(0)\| \overset{\text{Prop. 5}}{=} \sqrt{1 - \max_{i\in\mathcal{N}}(\langle e_1, \nu_i(0)\rangle)^2} < \sin(\alpha^\star)$ (where the latter inequality follows from the fact that $\nu(0) \in \mathcal{C}(\alpha^\star, e_1)^N$). From Proposition 7, the dynamics (34) satisfy Theorem's 1 conditions and therefore the set $\bar{\mathcal{B}}_n(r_0)^N$ is positively invariant for trajectories of $\dot{x}(t) = f_{\sigma(t)}(x(t))$. This, in turn, implies that the closed ball $\bar{\mathcal{C}}(\alpha, e_1)^N = (h^{-1})^N(\bar{\mathcal{B}}_n(r_0)^N)$, with α as given in the statement of the theorem, is positively invariant for the trajectories of $\dot{\nu}(t) = \tilde{f}_{\sigma(t)}(\nu(t))$, which establishes the validity of part *i)*.

Let us now focus on part *ii)* of the Theorem. From Proposition 7, the dynamics (34) satisfy Theorem's 1 conditions. It follows from Theorem 1 that $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ for all $i, j \in \mathcal{N}$, which implies that $\lim_{t\to\infty}\nu_i(t) - \nu_j(t) = 0$, for all $i, j \in \mathcal{N}$ (see Proposition 5). Moreover, it follows that the Lyapunov function in Theorem 1 converges to a constant, i.e., $\lim_{t\to\infty} V(x(t)) := \lim_{t\to\infty}\max_{i\in\mathcal{N}} \frac{1}{2}\|x_i(t)\|^2 = \lim_{t\to\infty}\frac{1}{2}\|x_1(t)\|^2 = V^\infty$, for some constant $0 \leq V^\infty \leq V(0) < \frac{1}{2}$. From Proposition 5, it follows that $\lim_{t\to\infty}\langle e_1, \nu_i(t)\rangle = \lim_{t\to\infty}\sqrt{1 - \|x_i(t)\|^2} = \sqrt{1 - 2V^\infty}$.

We now prove part *iii)* of the Theorem. Since $\nu(0) \in \mathcal{C}(\alpha^\star, e_1)^N$, there exist $n+1$ linearly independent unit vectors $\{\bar{\nu}_1, \cdots, \bar{\nu}_{n+1}\}$ such that $\nu(0) \in \mathcal{C}(\alpha^\star, \bar{\nu}_k)^N$ for all $k \in \{1, \cdots, n+1\}$ (see [38, Proposition 11]). From part *ii)* of this Theorem, it follows that, for each $k \in \{1, \cdots, n+1\}$, there exists a constant $V_k^\infty < \frac{1}{2}$ such that $\lim_{t\to\infty}\langle \bar{\nu}_k^T, \nu_1(t)\rangle = \sqrt{1 - 2V_k^\infty}$. Thus, it follows that $\lim_{t\to\infty} A\nu_1(t) = b \Leftrightarrow \lim_{t\to\infty}\nu_1(t) = A^{-1}b$, where $A^T = [\bar{\nu}_1 \cdots \bar{\nu}_{n+1}]$ is nonsingular, since $\bar{\nu}_1, \cdots, \bar{\nu}_{n+1}$ are linearly independent, and $b = (\sqrt{1 - 2V_1^\infty}, \cdots, \sqrt{1 - 2V_{n+1}^\infty})$. Since synchronization is asymptotically reached, $\lim_{t\to\infty}\nu_i(t) = A^{-1}b$ for all $i \in \mathcal{N}$, i.e., all unit vectors converge to the constant unit vector $A^{-1}b$. $\square$

Remark 9. *Regarding incomplete synchronization, the domain of attraction (in the form of* $\mathcal{C}(\cdot,\cdot)^N$*) is not larger than* $\mathcal{C}\left(\frac{\pi}{2}, e_1\right)^N$*. Regarding complete synchronization, the domain of attraction (in the form of* $\mathcal{C}(\cdot,\cdot)^N$*) is not larger than* $\mathcal{C}\left(\frac{\pi}{4}, e_1\right)^N$*. In both cases, one can find equilibria lying in the boundary of those sets, which implies that the domains of attraction cannot be extended under the proposed control laws (under relative measurements).*

VI. Simulations

In this section, we present simulations that illustrate some of the derived results. All simulations are provided for a network of six agents, i.e., $\mathcal{N} = \{1, \cdots, 6\}$, with time-varying neighboring sets. In particular, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, and $\mathcal{N}_6$, alternate between $\{2\}$ and $\{2,4\}$, $\{3\}$ and $\{3,6\}$, $\{4\}$ and $\{4,5\}$, $\{5\}$ and $\{5,1\}$, $\{6\}$ and $\{6,3\}$, and $\{1\}$ and $\{1,2\}$, respectively. For these time-varying neighboring sets, the network digraph is strongly connected at all time instants. The switching time instants for the neighbor set of each agent $i \in \mathcal{N}$ are those from the sequences $\mathcal{T}^i = \{\frac{1}{i} + ki\}_{k\in\mathbb{N}}$, which are shown in the time axes in Fig 3. Regarding the weight functions, for agents whose $i \in \mathcal{N}$ is even, $w_{ij}(\theta) = j$ for both simulations; and for agents whose $i \in \mathcal{N}$ is odd, $w_{ij}(\theta) = j(2 - \cos(\theta))$ (simulations with different weight functions are found in [38]).

In Fig. 3a, six unit vectors are randomly initialized in an open $\frac{\pi}{2}$-ball around $(1,0,0) \in \mathbb{S}^2$. In the same figure, the trajectories of the unit vectors on the unit sphere are shown, and a visual inspection indicates convergence to a synchronized network. In Fig. 3c, the angular distance, i.e., θ as in Definition 3, between some agents is presented, and it indicates convergence to a synchronized network.

In Fig. 3b, six rotation matrices were randomly initialized such that $\theta(I_3, \mathcal{R}_i) \leq \frac{\pi}{2}$ for all $i \in \mathcal{N}$. In the same figure, the trajectories of the rotation matrices are shown on a sphere of π radius[1], and a visual inspection indicates convergence to a synchronized network. In Fig. 3d, the angular distance, i.e., θ as in Definition 2, between some agents is presented, and it indicates convergence to a synchronized network.

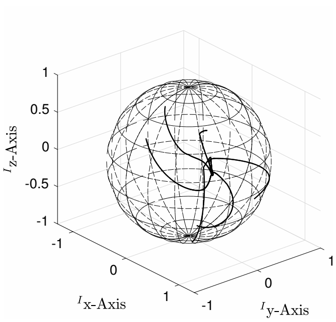
VII. Conclusions

In this paper, we studied complete and incomplete attitude synchronization for a group of agents under switching strongly connected network digraphs. We proposed switching output feedback control laws for each agent's angular velocity, which are decentralized and do not require a common orientation frame among agents. Our main contribution lied in transforming those two problems into a common framework, where all agents dynamics are transformed into unit vectors' dynamics on a sphere of appropriate dimension. Convergence to a synchronized network was guaranteed for a wide range of initial conditions. Directions for future work include extending all results to agents controlled at the torque level, rather than the angular velocity level.

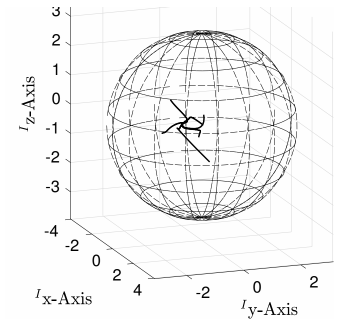
References

[1] J. R. Lawton and R. W. Beard, "Synchronized multiple spacecraft rotations," *Automatica*, vol. 38, no. 8, pp. 1359–1364, 2002.

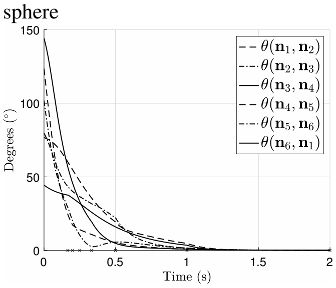
[1]For each rotation matrix $\mathcal{R}_i$, we plot $\theta_i n_i$ where $\theta_i = \theta(I_3, \mathcal{R}_i) \in [0, \pi]$ and $n_i = \frac{1}{2\sin(\theta_i)}\mathcal{S}^{-1}(\mathcal{R}_i - \mathcal{R}_i^T) \in \mathbb{S}^2$.



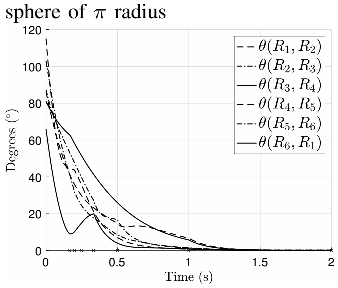
(a) Trajectories of unit vectors in unit sphere



(b) Trajectories of rotation matrices in sphere of π radius



(c) Angular distance between some pairs of unit vectors



(d) Angular distance between some pairs of rotation matrices

Fig. 3: Simulations for complete and incomplete synchronization.

[2] A. Abdessameud and A. Tayebi, “Attitude synchronization of a group of spacecraft without velocity measurements,” *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2642–2648, 2009.

[3] N. Leonard, D. Paley, F. Lekien, R. Sepulchre, D. Fratantoni, and R. Davis, “Collective motion, sensor networks, and ocean sampling,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 48–74, Jan 2007.

[4] T. Hatanaka, Y. Igarashi, M. Fujita, and M. Spong, “Passivity-based pose synchronization in three dimensions,” *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 360–375, 2012.

[5] W. Ren, “Distributed attitude consensus among multiple networked spacecraft,” in *American Control Conference*. IEEE, June 2006, pp. 1760–1765.

[6] A. Sarlette, R. Sepulchre, and N. E. Leonard, “Autonomous rigid body attitude synchronization,” *Automatica*, vol. 45, no. 2, pp. 572–577, 2009.

[7] D. V. Dimarogonas, P. Tsiotras, and K. Kyriakopoulos, “Leader–follower cooperative attitude control of multiple rigid bodies,” *Systems & Control Letters*, vol. 58, no. 6, pp. 429–435, 2009.

[8] H. Bai, M. Arcak, and J. T. Wen, “Rigid body attitude coordination without inertial frame information,” *Automatica*, vol. 44, no. 12, pp. 3170–3175, 2008.

[9] R. Tron, B. Afsari, and R. Vidal, “Intrinsic consensus on SO(3) with almost-global convergence.” in *Conference on Decision and Control*, 2012, pp. 2052–2058.

[10] S. Chung, S. Bandyopadhyay, I. Chang, and F. Hadaegh, “Phase synchronization control of complex networks of lagrangian systems on adaptive digraphs,” *Automatica*, vol. 49, no. 5, pp. 1148–1161, 2013.

[11] A. K. Bondhus, K. Y. Pettersen, and J. T. Gravdahl, “Leader/follower synchronization of satellite attitude without angular velocity measurements,” in *Conference on Decision and Control and European Control Conference*. IEEE, 2005, pp. 7270–7277.

[12] T. Krogstad and J. Gravdahl, “Coordinated attitude control of satellites in formation,” in *Group Coordination and Cooperative Control*. Springer, 2006, pp. 153–170.

[13] J. Thunberg, W. Song, E. Montijano, Y. Hong, and X. Hu, “Distributed attitude synchronization control of multi-agent systems with switching topologies,” *Automatica*, vol. 50, no. 3, pp. 832–840, 2014.

[14] Y. Igarashi, T. Hatanaka, M. Fujita, and M. W. Spong, “Passivity-based attitude synchronization in $\mathcal{SE}(3)$,” *IEEE Transactions on Control Systems Technology*, vol. 17, no. 5, pp. 1119–1134, 2009.

[15] F. Dörfler and F. Bullo, “Synchronization in complex networks of phase oscillators: A survey,” *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[16] R. Sepulchre, “Consensus on nonlinear spaces,” *Annual reviews in control*, vol. 35, no. 1, pp. 56–64, 2011.

[17] A. Sarlette and R. Sepulchre, “Consensus optimization on manifolds,” *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 56–76, 2009.

[18] K. Oh and H. Ahn, “Formation control and network localization via orientation alignment,” *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 540–545, 2014.

[19] R. Olfati-Saber, “Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks,” in *Conference on Decision and Control*. IEEE, 2006, pp. 5060–5066.

[20] N. Moshtagh and A. Jadbabaie, “Distributed geodesic control laws for flocking of nonholonomic agents,” *Transactions on Automatic Control*, vol. 52, no. 4, pp. 681–686, 2007.

[21] D. A. Paley, “Stabilization of collective motion on a sphere,” *Automatica*, vol. 45, no. 1, pp. 212–216, 2009.

[22] W. Li and M. W. Spong, “Unified cooperative control of multiple agents on a sphere for different spherical patterns,” *Transactions on Automatic Control*, vol. 59, no. 5, pp. 1283–1289, 2014.

[23] A. Sarlette, S. E. Tuna, V. Blondel, and R. Sepulchre, “Global synchronization on the circle,” in *Proceedings of the 17th IFAC world congress*, 2008, pp. 9045–9050.

[24] A. Sarlette and R. Sepulchre, “Synchronization on the circle,” *arXiv preprint arXiv:0901.2408*, 2009.

[25] L. Moreau, “Stability of continuous-time distributed consensus algorithms,” in *Conference on Decision and Control*, vol. 4. IEEE, 2004, pp. 3998–4003.

[26] ——, “Stability of multiagent systems with time-dependent communication links,” *Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[27] H. Zhang, C. Zhai, and Z. Chen, “A general alignment repulsion algorithm for flocking of multi-agent systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 430–435, 2011.

[28] P. O. Pereira and D. V. Dimarogonas, “Family of controllers for attitude synchronization on the sphere,” *Automatica*, vol. 75, pp. 271 – 281, 2017.

[29] J. Thunberg, J. Goncalves, and X. Hu, “Consensus and formation control on SE(3) for switching topologies,” *Automatica*, vol. 66, no. Supplement C, pp. 109 – 121, 2016.

[30] H. Du and S. Li, “Attitude synchronization for flexible spacecraft with communication delays,” *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3625–3630, Nov 2016.

[31] H. Du, S. Li, and C. Qian, “Finite-time attitude tracking control of spacecraft with application to attitude synchronization,” *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2711–2717, Nov 2011.

[32] Z. Lin, B. Francis, and M. Maggiore, “State agreement for continuous-time coupled nonlinear systems,” *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.

[33] J. Thunberg, X. Hu, and J. Gonçalves, “Local lyapunov functions for consensus in switching nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 12, pp. 6466–6472, Dec 2017.

[34] P. O. Pereira, D. Boskos, and D. V. Dimarogonas, “A common framework for attitude synchronization of unit vectors in networks with switching topology,” in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 3530–3536.

[35] J. L. Mancilla-Aguilar and R. A. García, “An extension of LaSalle's invariance principle for switched systems,” *Systems & Control Letters*, vol. 55, no. 5, pp. 376–384, 2006.

[36] R. Goebel, R. G. Sanfelice, and A. R. Teel, “Invariance principles for switching systems via hybrid systems techniques,” *Systems & Control Letters*, vol. 57, no. 12, pp. 980–986, 2008.

[37] R. G. Sanfelice, R. Goebel, and A. R. Teel, “Invariance principles for hybrid systems with connections to detectability and asymptotic stability,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, 2007.

[38] P. O. Pereira, D. Boskos, and D. V. Dimarogonas, “A common framework for attitude synchronization of unit vectors in networks with switching topology,” *arXiv*, 2017. [Online]. Available: https://arxiv.org/pdf/1509.08343.pdf

[39] D. Pence, *Spacecraft Attitude, Rotations and Quaternions*. Birkhäuser, 1984, vol. 5, no. 2.

[40] C. Mayhew, R. Sanfelice, J. Sheng, M. Arcak, and A. Teel, “Quaternion-based hybrid feedback for robust global attitude synchronization,” *Transactions on Automatic Control*, vol. 57, no. 8, pp. 2122–2127, Aug 2012.

[41] P. Pereira, D. Boskos, and D. V. Dimarogonas, “Auxiliary mathematica files,” in *https://github.com/KTH-SML/common-framework-for-complete-and-incomplete-attitude-synchronization.git*.