A Common Framework for Complete and Incomplete Attitude Synchronization in Networks with Switching Topology

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Abstract—In this paper, we study attitude synchronization for elements in the unit sphere in $\mathbb{R}^2$ and for elements in the 3D rotation group, for a network with switching topology. The agents’ angular velocities are assumed to be the control inputs, and a switching control law for each agent is devised that guarantees synchronization, provided that all elements are initially contained in a region, which we identify later in the paper. The control law is decentralized and it does not require a common orientation frame among all agents. We refer to synchronization of unit vectors in $\mathbb{R}^2$ as incomplete synchronization, and of 3D rotation matrices as complete synchronization. Our main contribution lies in showing that these two problems can be analyzed under a common framework, where all agents’ dynamics are transformed into unit vectors dynamics on a sphere of appropriate dimension.

I. INTRODUCTION

Decentralized control in a multi-agent environment has been a topic of active research, with applications in large scale robotic systems. Attitude synchronization in satellite formations is one of the relevant applications [1], [2], where the control goal is to guarantee that a network of fully actuated rigid bodies acquires a common attitude. Coordination of underwater vehicles in ocean exploration missions [3], and of unmanned aerial vehicles in aerial exploration missions, may also be casted as attitude synchronization problems.

In the literature, attitude synchronization strategies for elements in the special orthogonal group are found in [4]–[17], which focus on complete attitude synchronization; and in [15], [18]–[27], which focus on incomplete attitude synchronization. In this paper, we focus on both complete and incomplete attitude synchronization. We refer to incomplete attitude synchronization when the agents are unit vectors in $\mathbb{R}^2$ (synchronization in $S^1$); and we refer to complete attitude synchronization, when the agents are 3D rotation matrices (synchronization in $SO(3)$). Incomplete synchronization represents a relevant practical problem, when the agents are 3D rotation matrices that guarantee synchronization for a network of agents with a switching topology. The control laws devised for unit vectors and rotation matrices achieve different goals, and differ in two aspects worth emphasizing. First, controlling rotation matrices requires more measurements when compared with controlling unit vectors; secondly, while controlling rotation matrices requires full actuation, i.e., all body components of the angular velocity need to be controllable, controlling unit vectors does not. Our main contribution compared to the aforementioned literature lies in analyzing both problems under a common framework, in order to allow for a unified stability analysis using the same common weak Lyapunov function. Particularly both problems are transformed into synchronization problems in $S^m$ for an appropriate $m \in \mathbb{N}$, and where a dwell time between consecutive switches is not required. A preliminary version of this work is found in [34]. With respect to this preliminary version, this paper provides proofs which were omitted therein due to space constraints (especially the proof of Theorem 1, which is fundamental for the final result); it provides further details concerning the synchronization in $SO(3)$; it provides the output maps for each agent, and the control laws as functions of the outputs; and the exposition of the results in Section V is simplified by exploiting an invariance property.
II. Notation

Given $n \in \mathbb{N}$, we denote the inner product of $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle$ and the identity matrix in $\mathbb{R}^{n \times n}$ as $I_n$. We denote $\mathcal{S} : \mathbb{R}^3 \rightarrow \mathcal{S}(a) \in \{\text{antisymmetric matrices in } \mathbb{R}^{3 \times 3}\}$, with $\mathcal{S}(a)$ as the matrix satisfying $\mathcal{S}(a) b = a \times b$ for any $a, b \in \mathbb{R}^3$; and with $\mathcal{S}^{-1}$ denoting its inverse. We denote by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = 1\}$ the unit sphere in $\mathbb{R}^{n+1}$. The map $\Pi : \mathbb{S}^n \ni x \mapsto \Pi(x) \in (\mathbb{R}^{n+1})^{\times (n+1)}$ is defined as the map satisfying $\Pi(x) y := y - \langle y, x \rangle x$ for any $y \in \mathbb{R}^{n+1}$, and represents the orthogonal projection of $y$ to the subspace orthogonal to $x \in \mathbb{S}^n$. $e_1, \ldots, e_n \in \mathbb{S}^n \subset \mathbb{R}^n$ denote the canonical basis vectors in $\mathbb{R}^n$. Given $r > 0$, we denote $\mathcal{B}_r(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ and $\tilde{\mathcal{B}}_r(r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ as the open and closed balls of radius $r$ and centered around $0$, respectively. Given a function $f : A \rightarrow B$, for some normed spaces $A$ and $B$, $df(a) : A \rightarrow B$ denotes its derivative at some $a \in A$ (if $f$ is differentiable); and given any $N \in \mathbb{N}$, we denote $f^N : A^N \ni a := (a_1, \ldots, a_N) \mapsto f^N(a) := (f(a_1), \ldots, f(a_N)) \in B^N$. Finally, given a manifold $M$, $T_mM$ denotes the tangent space to $M$ at $m \in M$ and $TM := \sqcup_{m \in M} T_mM$ its tangent bundle.

III. Preliminaries

Throughout the paper, we consider a network of $N \in \mathbb{N}$ agents indexed by $\mathcal{N} := \{1, \ldots, N\}$, with $n \in \mathbb{N}$ the dimension of each agent’s state space. We associate to each of these undirected network graphs to the set $\mathcal{P} := \{1, 2, \ldots, P\} \subset \mathbb{N}$, where $P \leq 2^{N(N-1)}$, and consider an average dwell time switching signal $\sigma : \mathbb{R}^+ \rightarrow \mathcal{P}$ [35]. We then consider the agents’ trajectory $t \mapsto x(t) := (x_1(t), \ldots, x_N(t))$ initiated from $x(0) \in (\mathbb{R}^n)^N$, which satisfies

$$\dot{x}(t) = f_{\sigma(t)}(x(t)),$$

for almost all $t \in \mathbb{R}^+$.

Proof. For the proof, we invoke [36, Corollary 4.7], which is applicable to persistent dwell time signals. We also note that although this corollary requires Lyapunov functions to be continuously differentiable, it can be extended to continuous Lyapunov functions by replacing [36, Theorem 1] with [37, Corollary 4.4 b] (and making use of the Clarke generalized derivative). Finally, [36] considers multiple Lyapunov functions, one for each switching signal mode, whilst in this proof we restrict ourselves to a common Lyapunov function. For brevity, in what follows, we denote $x \mapsto f(x) := \langle \mathbb{R}^n \rangle \ni x \ni x_1, \ldots, x_N \mapsto f(x)$.

Consider then any $p \in \mathcal{P}$ and the vector field (2). Also, for brevity, denote $x \mapsto v_{i,p}(x) := \langle x_i, f_{i,p}(x) \rangle \in \mathbb{R}$, for all $i \in \mathcal{N}$, which are all continuous functions. Consider then the continuous function

$$x \mapsto V(x) := \max_{i \in \mathcal{N}} \frac{1}{2}\|x_i\|_2^2 \in \mathbb{R}^+,$$

whose generalized gradient (in the sense of Clarke) is given by (denote $\text{co}(S)$ as the convex hull of a finite point set $S \subset \mathbb{R}^n$, for any $m \in \mathbb{N}$)

$$x \mapsto \partial V(x) = \text{co}\{e_i \otimes x_i, i \in H(x)\} \subset \mathbb{R}^n,$$

and where we emphasize that $\cup_{p \in \mathcal{P}} V^{-1}(v) = (\mathcal{B}_r(r))^\mathcal{N}$ for any $r \in \mathbb{R}$ (see Notation for definition of $\mathcal{B}_r$). The generalized directional derivative of $V$ along (2), for a mode $p \in \mathcal{P}$, is then given by

$$x \mapsto V_p^\mathcal{N} := \max_{dV \in \partial V(x)} \langle dV, f_p(x) \rangle \equiv \max_{i \in H(x)} v_{i,p}(x).$$

Recall that the Theorem’s condition 1a) reads as $v_{i,p}(x)|_{i \in H(x)} \leq 0$ and condition 1b) reads as $\exists i \in H(x)$ such that $v_{i,p}(x) < 0$, for any $p \in \mathcal{P}$. As such, the generalized derivative in (5) can be expressed equivalently as

$$x \mapsto V_p^\mathcal{N} = \begin{cases} v_{i,p}(x)|_{i \in H(x)} < 0 & x \ni A_p \ni \max_{i \in H(x)} v_{i,p}(x) \leq 0, \\ x \ni B_p, \end{cases}$$

where

$$A_p := \{x \ni \mathbb{R}^n : \forall k, l \ni \mathcal{H}(x), v_{k,p}(x) = v_{l,p}(x) < 0\},$$

$$B_p := \{x \ni \mathbb{R}^n : \exists k, l \ni \mathcal{H}(x), v_{k,p}(x) \neq v_{l,p}(x)\}.$$

The function $V$ in (3) is lower bounded and its generalized derivative along (1) is non-positive. This implies that $\lim_{t \rightarrow \infty} V(x(t)) := \sup_{t \ni \mathbb{R}} V(x(t))$; and that $\mathcal{B}_r(r)^\mathcal{N}$, with $r_0 := \max_{i \in \mathcal{N}} \|x_i(0)\|$ is positively invariant (since $V(x) \leq V(x(0)) \equiv x \ni \cup_{p \in \mathcal{P}} V^{-1}(r) = (\mathcal{B}_r(r))^\mathcal{N}$). Moreover, it follows from (6) that $(V_p^\mathcal{N})^{-1}(0) \subset \mathcal{C} \cup B_p$, and that (see Proposition 2) $(V_p^\mathcal{N})^{-1}(0) \subset \mathcal{C} \cup B_p$. We now wish to compute the largest invariant subset (in the sense of [36, Corollary 4.4]) of $V^{-1}(V(x(0)) \cap (V_p^\mathcal{N})^{-1}(0)$. For that purpose, consider a solution

$$x \ni \mathcal{I} \ni \mathcal{I} \ni x \rightarrow x(t) \ni V^{-1}(V(x(t)) \ni V_p^\mathcal{N})^{-1}(0).$$

Composing (3) with (8) yields a constant function $I \ni x \rightarrow V(x(t)) \ni \mathbb{R}$, whose derivative is well defined, namely

$$x \ni \mathcal{I} \ni x \ni x \ni v_{i,p}(x(t)|_{i \in H(x(t))} = 0.$$
which is not satisfied for any \( x \in B_p \) as defined in (7). This implies that \( x \) in (8) does not belong to \( B_p \). It then follows that the largest invariant subset of \( V^{-1}(V^\infty) \cap (\overline{V_p})^{-1}(0) \subseteq V^{-1}(V^\infty) \cap \mathcal{C} \cup B_p \) is, in fact, a subset of \( V^{-1}(V^\infty) \cap \mathcal{C} \), which is independent of the mode \( p \in \mathcal{P} \).

In brief, we used a Lyapunov function common to all modes, and verified that the largest invariant set for each mode is independent of the mode. Based on the previous observations, we can invoke [36, Corollary 4.7], from which it follows that a solution of (1) converges to \( V^{-1}(V^\infty) \cap \mathcal{C} \subseteq \overline{B}_x(r_x) \cap \mathcal{C} \). \( \square \)

**Proposition 2.** Let \( p \in \mathcal{P} \) and consider the map \( V_p^\alpha \) in (6) and the set \( B_p \) in (7). The closure of the set where \( V_p^\alpha \) vanishes lies in \( \mathcal{C} \cup B_p \), i.e., \( (\overline{V_p^\alpha})^{-1}(0) \subseteq \mathcal{C} \cup B_p \).

**Proof.** Consider a convergent sequence in \( (\overline{V_p^\alpha})^{-1}(0) \), namely
\[
\{x^m \in (\overline{V_p^\alpha})^{-1}(0)\}_{m \in \mathbb{N}},
\]
\[
x^\infty := \lim_{m \to \infty} x^m \in \mathcal{A}_p \cup B_p \cup \mathcal{C}.
\]
Let us prove that \( (\overline{V_p^\alpha})^{-1}(0) \subseteq \mathcal{C} \cup B_p \) by assuming that \( x^\infty \in \mathcal{A}_p \), which will lead to a contradiction. For that purpose, denote \( B_\epsilon(x^\infty) := \{x \in \mathbb{R}^{n \times n} : \|x - x^\infty\| < \epsilon\} \) as an open ball of size \( \epsilon > 0 \) around \( x^\infty \), and denote \( \mathcal{H}(x) := \arg \max_{i \in \mathcal{N}_i}(\|x_i\|) \).

a) Since the sequence (10) belongs to \( (\overline{V_p^\alpha})^{-1}(0) \subseteq \mathcal{C} \cup B_p \), it follows that for any \( m \in \mathbb{N} \), there exists \( h \in \mathcal{H}(x^m) \) s.t. \( v_i(x^m) = 0 \). b) Since the sequence (10) is convergent, it follows that for any \( \epsilon > 0 \) there exists \( M \in \mathbb{N} \) s.t., for all \( m \geq M \), \( x^m \in B_\epsilon(x^\infty) \). c) By definition \( \mathcal{H}(x) := \arg \max_{i \in \mathcal{N}_i}(\|x_i\|) \), therefore it follows that for any \( x \in \mathcal{R}^n \) there exists \( \epsilon_i > 0 \), s.t., for all \( y \in B_\epsilon(x) \), \( \mathcal{H}(y) \leq \mathcal{H}(x) \).

Since \( v_i, p(x) := (x, f_i, p(x)) \) are continuous, it follows that for any \( x \in B_\epsilon \) (see (7)) there exists \( \epsilon_i \in (0, \epsilon) \), s.t., for all \( y \in B_\epsilon(x) \), \( v_i(y) < 0 \). Combining a) and b), it follows that for \( \epsilon = \epsilon_i \), there exists \( M \in \mathbb{N} \) s.t., for all \( m \geq M \), \( x^m \in B_\epsilon(x^\infty) \) \( \Rightarrow v_i(x^m) = 0 \). Combining c), it contradicts a), which implies that \( x^\infty \notin \mathcal{A}_p \). \( \square \)

**IV. SYNCHRONIZATION**

In the next subsections, we study synchronization of agents in \( S^2 \) and \( \mathbb{S}O(3) \). More specifically, we first present feedback control laws for the angular velocities of the agents, with which we determine the closed loop dynamics. Afterwards, by means of appropriate transformations, those dynamics are rewritten in a common form that allows us to study synchronization in \( S^2 \) and \( \mathbb{S}O(3) \) under a common framework. Additionally, in [38], we also show that consensus in \( \mathbb{R}^n \) can be casted as a synchronization problem in a subset of \( S^2 \), for any \( n \in \mathbb{N} \). It is noted that in all the above cases the agents evolve in appropriate subsets of \( S^2 \). These are specified through the following definition.

**Definition 1.** Given \( n \in \mathbb{N}, \alpha \in (0, \pi] \) and \( \bar{\nu} \in S^2 \), the open \( \alpha \)-ball \( \mathcal{C}(\alpha, \bar{\nu}) \) is defined as \( \mathcal{C}(\alpha, \bar{\nu}) := \{\nu \in S^2 : \langle \bar{\nu}, \nu \rangle > \cos(\alpha) \Leftrightarrow \arccos(\langle \bar{\nu}, \nu \rangle) < \alpha\} \), representing the set of unit vectors that form an angle less than \( \alpha \) with \( \bar{\nu} \) (see Fig. 2). Similarly, we define the closed \( \alpha \)-ball \( \overline{\mathcal{C}}(\alpha, \bar{\nu}) := \{\nu \in S^2 : \langle \bar{\nu}, \nu \rangle \geq \cos(\alpha) \Leftrightarrow \arccos(\langle \bar{\nu}, \nu \rangle) \leq \alpha\} \).

Given a group of unit vectors \( \nu = (\nu_1, \ldots, \nu_n) \), we say that \( \nu \) belongs to an open (closed) \( \alpha \)-ball, when \( \alpha \in (0, \pi] \). if \( \nu \in \mathcal{C}(\alpha, \bar{\nu}) \) \( \Leftrightarrow \overline{\mathcal{C}}(\alpha, \bar{\nu}) \) for a certain \( \bar{\nu} \in S^2 \). We also say that \( \nu \) is synchronized if \( \nu_i = \cdots = \nu_{\nu} \). Before presenting the common form of the agents’ dynamics, recall that their network is modeled as a time varying digraph \( G(\sigma(t)) = (\mathcal{N}, \mathcal{E}(\sigma(t))) \), where \( \sigma : \mathbb{R}_{\geq 0} \to \mathcal{P} \) represents the switching signal, and with \( G(p) \) and \( \mathcal{E}(p) \) as the graph and edge set corresponding to mode \( p \in \mathcal{P} \). We also denote \( \mathcal{N}(p) := \{j, \ldots, j_{\mathcal{N}(p)}\} \subseteq \mathcal{N} \) as the neighbor set of agent \( i \in \mathcal{N} \) corresponding to mode \( p \in \mathcal{P} \). Next, consider a group \( \nu = (\nu_1, \ldots, \nu_n) \) of unit vectors in \( S^2 \), which evolves according to the dynamics
\[
\dot{\nu}(t) = \bar{f}_{\sigma, \nu}(\nu(t)) = \begin{bmatrix} \bar{f}_1(\nu_1) & \cdots & \bar{f}_n(\nu_n) \end{bmatrix}, \nu(0) \in (S^2)^n, \]
\[
\text{where } \bar{f}_{\sigma, \nu}(\nu(t)) := \sum_{j \in \mathcal{N}(\nu(t))} \bar{\nu}_j(\nu_j, \nu_j) \Pi(\nu_j, \nu_j),
\]
for all \( i \in \mathcal{N} \); i.e., \( \dot{\nu}(t) = \bar{f}_{\sigma, \nu}(\nu(t)) \).

The dynamics (11)–(12) are time dependent due to each agent’s time varying neighbor set in (12). Furthermore, the system (11)–(12) is the standard form all problems are transformed into, with the unit vectors evolving in \( S^2 \) and \( \mathbb{S}^1 \), for incomplete and complete synchronization, respectively. In particular, we will show that the proposed control laws guarantee asymptotic synchronization of the unit vectors, provided that they are initially contained in an open \( \alpha \)-ball, with \( \alpha = \frac{\pi}{2} \) for synchronization in \( S^2 \), and \( \alpha^* = \frac{\pi}{4} \) for synchronization in \( \mathbb{S}O(3) \).

**A. Complete synchronization in \( \mathbb{S}O(3) \) casted as synchronization in \( S^2 \)**

Consider a group of \( N \) agents \( \mathcal{R}_1, \ldots, \mathcal{R}_N \in \mathbb{S}O(3) := \{R \in \mathbb{R}^{3 \times 3} : R^T R = R R^T = I_3, \det(R) = 1\} \), where, for every \( i \in \mathcal{N}, \mathcal{R}_i \) represents the orientation frame of agent \( i \), with respect to an unknown inertial orientation frame. We say that the agents are synchronized if they all share the same complete orientation, i.e., if \( \mathcal{R}_1 = \cdots = \mathcal{R}_N \), as illustrated in Fig. 1a. The term complete synchronization is used in juxtaposition with incomplete synchronization as described in the next subsection. In incomplete synchronization, rather than synchronizing all three bodies axes, the agents synchronize only one body direction. Furthermore, as explained in the next section, complete synchronization does not guarantee incomplete synchronization (and vice-versa).

For each \( i \in \mathcal{N}, \omega_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \) denotes the body-framed angular velocity of agent \( i \), which is considered as its input. Each rotation matrix \( \mathcal{R}_i \) evolves according to the dynamics
(a) Two on the right completely synchronized; two on the left not completely synchronized.

(b) Two on the right ... i.e.,
given $q := (q_1, \ldots, q_N) \in \mathbb{C} \left(\frac{\pi}{2}, e_1\right)^N$, it follows that
$$\dot{q}_i = d\phi(R_i)fR(R_i, \omega_h^i(t, h_i(t, R)))|_{R_i=\phi^{-1}(q_i), i\in\mathcal{N}}$$

For each agent $i \in \mathcal{N}$ and time instant $t \in \mathbb{R}_{\geq 0}$, consider the control law $\omega^h_\ell(t, h_i) : \mathbb{SO}(3)^{\mathcal{N}(\sigma(t))} \to \mathbb{R}^3$ defined as
$$\omega^h_\ell(t, h_i) := \sum_{j \in \mathcal{N}(\sigma(t))} w_{ij} (\theta(I_j, h_j)) S^{-1} \left( h_{ij} - h_i^T \right), \quad (18)$$
where $h_i := (h_{ij_1}, \ldots, h_{ij_N}^{\mathcal{N}(\sigma(t))})$ and $w_{ij} : [0, \pi] \to \mathbb{R}_{\geq 0}$ is continuous and satisfies $w_{ij}(\theta) > 0$ for all $\theta \in [0, \pi]$. Notice that $w_{ij}$ corresponds to a weight on the feedback law (18) that agent $i$ assigns to the displacement between itself and its neighbor $j$. Thus, given the feedback law (18), one can write the closed-loop dynamics in (17).

Recalling that we wish to analyze all problems under a common framework, we will cast complete synchronization in $\mathbb{SO}(3)$ in the form (11)–(12), through a quaternion-based coordinate transformation. This change of variables serves only the purpose of analysis, while the implemented control law is still that in (18). Details on unit quaternions are found in [39], [40], and the identities shown next are also verified in [41]. We next fix a rotation matrix $R \in \mathbb{SO}(3)$ and denote
$$\mathbb{SO}(3) := \{ R \in \mathbb{SO}(3) : \theta(R, R) < \pi \},$$

$$C \left( \frac{\pi}{2}, e_1 \right) := \left\{ q \in \mathbb{S}^3 : \langle q, e_i \rangle > 0 \right\}, \quad (19a)$$

Consider then the map $\phi : \mathbb{SO}(3) \to C \left( \frac{\pi}{2}, e_1 \right)$, together with its inverse, which are defined as
$$\phi(R) := \frac{1}{2} \left( \sqrt{1 + \text{tr}(R^T R)} - \frac{S^{-1} \left( R^T R - R^T \bar{R} \right)}{\sqrt{1 + \text{tr}(R^T R)}} \right), \quad (20a)$$
$$\phi^{-1}(q) := \left( I_3 + 2qS(\tilde{q}) + 2S(\tilde{q})S(\tilde{q}) \right)^{-1}, \quad (20b)$$

where $q = (\tilde{q}, \tilde{q})$ and $\tilde{q} \in \mathbb{R}^3$. Note that since $\theta(I_3, R) < \pi \leftrightarrow 1 + \text{tr}(R^T R) > 0$, (20a) is well defined. Intuitively, $\phi$ provides a coordinate transformation between rotation matrices in $\mathbb{SO}(3)$ and unit vectors in $C \left( \frac{\pi}{2}, e_1 \right)$, i.e., unit quaternions with a positive first component.

The idea followed next is to (i), given a rotation matrix $R \in \mathbb{SO}(3)$, consider the quaternion $q = \phi(R) \in C \left( \frac{\pi}{2}, e_1 \right)$; and (ii), given the closed loop dynamics of the rotation matrix in (17), to compute the closed loop dynamics of the quaternion, which will be in the common form (12). We first define $Q : \mathbb{S}^3 \to \mathbb{R}^{3 \times 3}$ as the map satisfying $Q(q)v := \langle q, e_i \rangle v - \langle q, \tilde{v} \rangle e_i + \text{diag}(0, S(q))\tilde{v}$, for any $\nu, \tilde{\nu} \in \mathbb{R}^3$, where $q = (\tilde{q}, \tilde{q})$ and $\tilde{v} = (0, \nu)$. It can be also checked that $Q(q, q)v = 0$, and that $Q(q)Q(q)^T = I(q)$. Given the kinematics $f_R$ in (14) and the coordinate change $\phi$ in (20), one may verify that
$$\dot{q} = d\phi(R)f_R S(\omega)|_{\omega=\phi^{-1}(\ell)} = \frac{1}{2}Q(q)\omega, \quad (21)$$
where $d\phi(R) : \mathbb{R}^{3 \times 3} \to \mathbb{R}^4$ is the derivative of $\phi$ at $R \in \mathbb{SO}(3)$; and that, for $R_1 = \phi^{-1}(q_1)$ and $R_2 = \phi^{-1}(q_2)$,
$$S^{-1}(R_1^T R_2 - R_2^T R_1) = 2(q_1, q_2)Q(q_1)q_2, \quad (22a)$$
$$\cos(\theta(I_3, R_1^T R_2)) = 2(q_1, q_2)^2 - 1. \quad (22b)$$

With (21) and (22a)–(22b) in mind, one can then compute the closed-loop dynamics of the unit quaternions based on the closed-loop dynamics of the rotation matrices in (17); i.e., given $q := (q_1, \ldots, q_N) \in C \left( \frac{\pi}{2}, e_1 \right)^N$, it follows that
$$\dot{q}_i = d\phi(R_i)f_R(S(\omega^h(t, h_i, R)))|_{\omega=\phi^{-1}(\ell), i\in\mathcal{N}},$$
\[ \omega(t,\cdot) := \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(q_i,q_j)Q(q_i)Q(q_i)^T q_j \]

where

\[ \tilde{w}_{ij}(q_i,q_j) := \langle q_i,q_j \rangle w_{ij}(\arccos(2\langle q_i,q_j \rangle^2 - 1)). \]

Now, recall that (24) needs to satisfy (13), which forces us to restrict the sets in (19a)–(19b). The next proposition, which can be proved by exploiting the triangular inequality, sheds some light into the domain restriction we must impose.

**Proposition 3.** Let \( \{q_1, \ldots, q_N\} \in C(\mathbb{R}, e_1) \times \cdots \times C(\mathbb{R}, e_1) \). Then \( \langle q_i, q_j \rangle > 0 \) for all \( i, j \in \{1, \ldots, N\} \), and therefore the weight \( \tilde{w}_{ij}(q_i,q_j) \), as defined in (24), is positive when \( q_i \neq q_j \), and non-negative otherwise.

With the Proposition 3 in mind, if one considers the sets \( \mathcal{S}(3) := \{ R \in \mathcal{S}(3) \colon \theta(R,R) < \pi/2 \} \) and \( C(\mathbb{R}, e_1) := \{ q \in \mathbb{S}^3 \colon \langle q, e_1 \rangle > \cos(\pi/2) \} \) (instead of those in (19)), then \( \phi \) and \( \phi^{-1} \), as defined in (20a) and (20b), still constitute diffeomorphisms, and the closed-loop dynamics of the unit quaternions in (23) are then restricted to \( C(\mathbb{R}, e_1)^N \) (instead of \( C(\mathbb{R}, e_1)^N \)) for which the weights in (24) are non-negative (i.e., they satisfy (13)). In Section V, we show that any set \( \mathcal{C}(\alpha, e_i) \), with \( \alpha < \pi/2 \) is positively invariant under (23); and therefore (by means of the inverse map \( \phi^{-1} \)) that \( \{ R \in \mathcal{S}(3) \colon \theta(R,R) < \pi/2 \} \) is positively invariant under (17) (the factor 2 relates with the fact that \( \mathbb{S}^3 \) provides a double covering of \( \mathcal{S}(3) \) [40]). That is, if all rotation matrices start \( \alpha \) close to some rotation matrix \( R \), they remain forever \( \alpha \) close to it.

**Remark 4.** Consider the set \( \mathcal{S}(3) \) in (19) and the set \( \mathcal{C}(\pi, e_1) := \{ q \in \mathbb{S}^3 \colon \langle q, e_1 \rangle > \cos(\pi) \} \). Suppose that we had chosen the map \( \phi : \mathcal{S}(3) \rightarrow \mathcal{C}(\pi, e_1) \) defined as \( \phi(R) := \left( \frac{\rho(R^TR - I)}{2}, S^{-1} \left( \frac{R^TR - I}{2} R \right) \right) \) (instead of the map \( \phi \) in (20a)). For this choice, the closed-loop dynamics under the new coordinates does not come in the form (12), and thus the common framework cannot be leveraged for this alternative map \( \phi \). This illustrates the importance of the choice of the change of coordinates if the common framework is to be leveraged for other problems.

**B. Incomplete synchronization in \( \mathcal{S}(3) \) casted as synchronization in \( \mathbb{S}^3 \)**

In this section, we consider again a group of \( N \) agents operating in \( \mathcal{S}(3) \), but with a different synchronization objective. As in Section IV-A, for each \( i \in N \), \( R_i \) represents the orientation frame of agent \( i \). Additionally, for each agent \( i \) there is a constant body direction \( \bar{n}_i \in \mathbb{S}^2 \), known by the agent and its neighbors, which is required to synchronize with all the other agents’ body directions. The goal of incomplete attitude synchronization in \( \mathcal{S}(3) \) is that all agents share the same orientation along the chosen body directions; i.e., given \( \{ R_1, \cdots, R_N \} \in \mathcal{S}(3)^N \) and \( \{ \bar{n}_1, \cdots, \bar{n}_N \} \in (\mathbb{S}^2)^N \), incomplete synchronization is accomplished when \( \mathcal{R}_1, \bar{n}_1 = \cdots = \mathcal{R}_N, \bar{n}_N \), as illustrated in Fig. 1b. We note that the requirement for incomplete synchronization is independent from that of complete synchronization; i.e., complete synchronization does not imply incomplete synchronization (consider the case where \( \bar{n}_1 \neq \bar{n}_2 \), and vice-versa (see Figure 1b, where two agents are incompletely synchronized but not completely synchronized).

As in Section IV-A, \( \omega_i \) denotes the body-framed angular velocity of agent \( i \in \mathcal{N} \), which is taken as the input. Again, each rotation matrix \( R_i \) evolves according to \( \mathcal{R}_i(t) = f_R(\mathcal{R}_i(t), \omega_i(t), q_i(t)) \) with \( \omega_i \) as defined in (14). If, additionally, we consider some constant \( \bar{n}_i \in \mathbb{S}^2 \), then \( n_i := R_i \bar{n}_i \in \mathbb{S}^2 \), evolves according to \( \dot{n}_i(t) = f_n(\mathcal{R}_i(t), \omega_i(t), \bar{n}_i) \) where \( f_n : \mathcal{S}(3)^3 \times \mathbb{S}^2 \rightarrow \mathbb{S}^2 \) is defined as

\[ f_n(\mathcal{R}, \omega, \bar{n}) := f_n(\mathcal{R}, \omega) \bar{n} = \mathcal{R}_S(\omega) \bar{n} = \mathcal{R} \bar{n}, \]  

where \( \mathcal{R}_S(\omega) := \mathcal{R}_T \mathcal{R}_n, \mathcal{R}_n = (\mathcal{R}_1, \cdots, \mathcal{R}_N) \), for each \( i \in \mathcal{N}(\sigma(t)) \). Thus, at each time instant \( t \in \mathbb{R}^+ \), agent \( i \in N \) measures the \( |\mathcal{N}(\sigma(t))| \) unit vectors corresponding to the projection of a neighbor’s unit vector onto agent’s \( i \) orientation frame (note that the agent does not need to be aware of \( R_i \) and \( \bar{n}_i \), but rather of the product \( R_i^T \mathcal{R}_n \)).

**Problem 2.** For each agent \( i \in \mathcal{N} \) and time instant \( t \geq 0 \), given the measurement function (26), design time-varying decentralized feedback laws \( \omega_i(t, \cdot) : (\mathbb{S}^2)^N \rightarrow (\mathbb{S}^2)^N \rightarrow \mathbb{R}^3 \), such that asymptotic synchronization of \( (\mathcal{R}_i, n_1, \cdots, \mathcal{R}_n^N, \bar{n}_N) : \mathbb{R}^+ \rightarrow (\mathbb{S}^2)^N \) is accomplished, where, for every \( i \in \mathcal{N} \),

\[ \dot{\mathcal{R}}_i(t) = f_R(\mathcal{R}_i(t), \omega_i^R(t, h_i(t, R_i(t)))), \]  

where \( \mathcal{R}_i(t) = (\mathcal{R}_1(t), \cdots, \mathcal{R}_N(t)) \), and \( h_i(t, R_i(t)) \rightarrow \mathbb{R}^3 \)

Each for \( t \in \mathbb{R}^+ \) and agent \( i \in \mathcal{N} \), consider the control law \( \omega_i^R(t, \cdot) : (\mathbb{S}^2)^N \rightarrow \mathbb{S}^2 \) defined as

\[ \omega_i^R(t, h_i) := \sum_{j \in \mathcal{N}_i(\sigma(t))} w_{ij}(\theta(\bar{n}_i, h_i)) \mathcal{S}(\bar{n}_i) h_i, \]  

where \( h_i := (h_{i1}, \cdots, h_{iN}) \) and \( \mathcal{S}(\bar{n}_i) \) is a continuous function satisfying \( w_{ij}(\theta) > 0 \) for all \( \theta \in (0, \pi] \), and corresponding to a weight on the feedback law (28) that agent \( i \) assigns to the angular displacement between itself and its neighbor \( j \). We note that (28) is orthogonal to \( \bar{n}_i \), which implies that full angular velocity control is not necessary, i.e., we only need to control the angular velocity along the two directions orthogonal to \( \bar{n}_i \), \( \omega_i^R(t, \cdot) \in T_{\bar{n}_i} \mathbb{S}^2 \subset \mathbb{R}^3 \). At this point, given the feedback law (28), one can write the closed-loop dynamics in (27).

Recalling that we wish to analyze the problems under a common framework, we need to transform the dynamics in (27) to the form (11)–(12). In fact, it turns out that the kinematics (25), when composed with the proposed control law in (28) and the output function in (26), yield (12). Indeed,
given $R := (R_1, \ldots, R_N) \in (SO(3))^N$ and denoting
\[ n := (n_1, \ldots, n_N) = (R_1\bar{n}_1, \ldots, R_N\bar{n}_N) \in (S^2)^N, \]
we get that $\dot{n}_i = \mathcal{R}i\bar{n}_i = f_S(R_i, \omega_i(t, h(t, R(t))))$. Indeed, it follows from (25), (28), and (26) that
\[ \dot{n}_i = f_S(R_i, \omega_i(t, h(t, R(t)))), \]
\[ = \sum_{j \in \mathcal{N}(i)} \tilde{w}_{ij}(n_i, n_j)S(R_i\bar{n}_i)^T S(R_i, \bar{n}_i)R_i\bar{n}_j, \]
where
\[ \tilde{w}_{ij}(n_i, n_j) := w_{ij}(\theta(n_i, n_j)), \]
which is the desired form (12). Indeed, (30) follows from the facts that $S(\cdot)^T S(\cdot) = \Pi(\cdot)$ and that $\mathcal{R}S(\cdot) = S(\mathcal{R}(\cdot) R)$ for any $\mathcal{R} \in SO(3)$. Furthermore, we have that $\tilde{w}_{ij}(n_i, n_j) = w_{ij}(\theta(n_i, R^T \mathcal{R} n_j))$ and thus the weights satisfy (13) for any $\alpha \in [0, \pi]$ and $\bar{v} \in S^2$ (note that $\theta(n_i, n_j) = \theta(R^T n_i, R^T n_j)$ for any $\mathcal{R} \in SO(3)$ and any $n_i, n_j \in S^2$). We have thus cast this problem in the form (11)–(12) with $\nu \equiv n \in (S^2)^N$.

V. Analysis

In this section, we analyze the solutions of (11)–(12), and show that given a wide set of initial conditions, asymptotic synchronization is guaranteed. Specifically, asymptotic synchronization is guaranteed if all unit vectors are initially contained in an open $\alpha^*$-ball, i.e., if $\nu(0) \in C(\alpha^*, \tilde{v})$ for certain $\tilde{v} \in S^*$, where
\[ \left\{ \begin{array}{ll}
\alpha^* = \frac{\pi}{2} & \text{for incomplete synchronization,} \\
\alpha^* = \frac{\pi}{4} & \text{for complete synchronization.}
\end{array} \right. \]

Next, we introduce a coordinate transformation that allows us to cast the dynamics (12) into a form that satisfies the conditions of Theorem 1. Also, without loss of generality, we assume that $\nu(0) \in C(\alpha^*, e_1)$. Indeed, for the general case where $\nu(0) \in C(\alpha^*, \nu^*)$ for some $\tilde{v} \in S^*$, select a rotation matrix $\mathcal{R} \in SO(n + 1)$, with $\mathcal{R}\bar{v} = e_1$ and consider the transformation $\nu(\cdot) = (\mathcal{R}^T \nu_1, \ldots, \mathcal{R}^T \nu_n)$ between $C(\alpha^*, \tilde{v})$ and $C(\alpha^*, e_1)$. Then, due to the fact that $\tilde{w}_{ij}(\cdot, \cdot) = \tilde{w}_{ij}(\mathcal{R}^T \cdot, \mathcal{R}^T \cdot)$ for the selected weights in (24) and (31), it follows from (11) that $d\varphi(v)\tilde{f}_{\alpha^*}(\nu) = \tilde{f}_{\alpha^*}(\varphi(\nu))$, i.e., the dynamics are the same in the transformed coordinates.

Next, given $\alpha \in (0, \frac{\pi}{2})$, let $r = \sin(\alpha)$, and consider the matrix $Q := \begin{bmatrix} e_2 & \cdots & e_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}$ and the diffeomorphism $h : C(\alpha, e_1) \to B_\alpha(r)$ (see Notation for the definition of $B_\alpha$), defined as
\[ h(\nu) := Q^T \nu, \]
where $Q^T$ is the projection of the ball $C(\alpha, e_1)$ to the subspace of $\mathbb{R}^{n+1}$ orthogonal to $e_1$ (see Fig. 2). Its inverse $h^{-1} : B_\alpha(r) \to C(\alpha, e_1)$ is given by $h^{-1}(x) = \sqrt{1 - \|x\|^2}e_1 + Qx$. For the subsequent analysis we will also use the following propositions, whose proofs are elementary and can be found in [38].

Proposition 5. Let $\nu_1, \nu_2 \in C(\frac{\pi}{2}, e_1)$. Then, $(e_1, \nu_i) = \sqrt{1 - \|h(\nu_i)\|^2} > 0$ and the following implications hold:

Consider now the solution $\nu : \mathbb{R}_+ \to (S^2)^N$ of (11)–(12) with $\nu(0) \in C(\alpha, e_1)$ with $\nu \in [0, \alpha^*]$, which, as shown later in Theorem 8, remains in $C(\alpha, e_1)$. Then, by defining $x := h^{\alpha} \circ \nu$ (see Notation for $h^{\alpha}$), with $h(\cdot)$ as given in (33), it follows that $\dot{x}(t) = f_{\alpha^*}(x(t))$,
\[ f_{\alpha^*}(x) = dh(\nu)\tilde{f}_{\alpha^*}(\nu)|_{\nu = h^{-1}(x)}, \]
Then, in order to account for the vector fields in (34)–(35) (and recall that $\alpha^* = \frac{\pi}{2}$ for synchronization in $S^2$, and $\alpha^* = \frac{\pi}{4}$ for synchronization in $SO(3)$ as indicated in (32)).

Proposition 7. Consider the vector fields in (34)–(35) and assume that $\mathcal{P}$ encodes strongly connected network digraphs. Then, the conditions of Theorem 1 are satisfied for $r = \sin(\alpha^*)$, with $\alpha^*$ as in (32).
\( \tilde{w}_{ij}(\cdot, \cdot), \) it follows that \( \tilde{w}_{ij}(\nu_i, \nu_j) \geq 0 \) for any \( \nu_i, \nu_j \in \mathcal{C}(\alpha^*, e_i) \). The latter two properties imply by virtue of (36) that the sign of each term \( \langle x_i, f(x_i) \rangle \) is the opposite of \( \langle e_i, \Pi(e_i) \rangle \). Next, recall that \( \mathcal{H}(x) := \max_{e \in \mathbb{N}} \|x\| \) and pick \( p \in \mathcal{P}, i \in \mathcal{H}(x) \) and \( j \in \mathcal{N} \). From Proposition 5, it follows that \( \langle e_i, \Pi(e_i) \rangle \leq \langle e_i, \nu_j \rangle \); combining the latter with Proposition 6, it follows that \( e_i, \Pi(e_i) \nu_j \rangle \geq 0 \). Thus, we conclude that \( \langle x_i, f(x_i) \rangle \leq 0 \) and hence, condition (a) is satisfied. The verification of the other conditions of Theorem 1 is found in [34].

Finally, we use the result of Proposition 7 to prove that the solution of the system in (11) will reach consensus.

**Theorem 8.** Consider the solution \( \nu : \mathbb{R}_{\geq 0} \rightarrow (\mathbb{S}^n)^N \) of (11) with \( \nu(0) \in \mathcal{C}(\alpha^*, e_i)^N \), with \( \alpha^* \) as in (32). Then, for a network digraph strongly connected at all time instants, i) \( \nu(t) \in \mathcal{C}(\alpha, e_i)^N \) for all \( t \geq 0 \), where \( \alpha = \arccos(\max_{e \in \mathbb{N}} \langle e_i, \nu(0) \rangle) \) and \( \nu \) satisfies the conditions of Theorem 1 and the set \( B_6(\nu_i)^N \) is positively invariant for trajectories of \( \dot{\nu} = f_{\nu(i)}(\nu(t)) \). This, in turn, implies that the closed ball \( \bar{C}(\alpha, e_i)^N \) is positively invariant for the trajectories of \( \nu(t) = f_{\nu(i)}(\nu(t)) \), which establishes the validity of part i).

Let us now focus on part ii) of the Theorem. From Proposition 7, the dynamics (34) satisfy Theorem 1’s conditions. It follows from Theorem 1 that \( \lim_{t \to \infty} x_i(t) - x_j(t) = 0 \) for all \( i, j \in \mathcal{N} \), which implies that \( \lim_{t \to \infty} \nu_i(t) - \nu_j(t) = 0 \), for all \( i, j \in \mathcal{N} \) (see Proposition 5). Moreover, it follows that the Lyapunov function in Theorem 1 converges to a constant, i.e., \( \lim_{t \to \infty} \frac{1}{2} \|x_i(t)\|^2 = \lim_{t \to \infty} \frac{1}{2} \|x_i(t)\|^2 = V_\infty \), for some constant \( 0 \leq V_\infty \leq V(0) < \frac{1}{2} \). From Proposition 5, it follows that \( \lim_{t \to \infty} \langle e_i, \nu_i(t) \rangle = \lim_{t \to \infty} \sqrt{1 - \|x_i(t)\|^2} = \sqrt{1 - 2V_\infty} \).

We now prove part iii) of the Theorem. Since \( \nu(0) \in \mathcal{C}(\alpha^*, e_i)^N \), there exist \( n^1 \) linearly independent unit vectors \( \{\nu_{i1}, \cdots, \nu_{in^1}\} \) such that \( \nu(0) \in \mathcal{C}(\alpha^*, \nu_{i1})^N \) for all \( k \in \{1, \cdots, n + 1\} \) (see [38, Proposition 11]). From part ii) of this Theorem, it follows that, for each \( k \in \{1, \cdots, n + 1\} \), there exists a constant \( V_k^\infty < \frac{1}{2} \) such that \( \lim_{t \to \infty} \langle \nu_{ik}, \nu_{i1}(t) \rangle = \sqrt{1 - 2V_k^\infty} \). Thus, it follows that \( \lim_{t \to \infty} Ax_i(t) = b \iff \lim_{t \to \infty} \nu_i(t) = A^{-1}b \), where \( A^{-1} = [\nu_{i1} \cdots \nu_{in^1}] \) is nonsingular, since \( \nu_{i1}, \cdots, \nu_{in^1} \) are linearly independent, and \( b = (\sqrt{1 - 2V_\infty}, \cdots, \sqrt{1 - 2V_\infty}) \). Since synchronization is asymptotically reached, \( \lim_{t \to \infty} \nu_i(t) = A^{-1}b \) for all \( i \in \mathcal{N} \), i.e., all unit vectors converge to the constant unit vector \( A^{-1}b \).

**Remark 9.** Regarding incomplete synchronization, the domain of attraction (in the form of \( \mathcal{C}(\cdot, \cdot)^N \)) is not larger than \( \mathcal{C}(\bar{\alpha}, e_i)^N \). Regarding complete synchronization, the domain of attraction (in the form of \( \mathcal{C}(\cdot, \cdot)^N \)) is not larger than \( \mathcal{C}(\bar{\alpha}, e_i)^N \). In both cases, one can find equilibria lying in the boundary of those sets, which implies that the domains of attraction cannot be extended under the proposed control laws (under relative measurements).

**VI. SIMULATIONS**

In this section, we present simulations that illustrate some of the derived results. All simulations are provided for a network of six agents, i.e., \( \mathcal{N} = \{1, \cdots, 6\} \), with time-varying neighboring sets. In particular, \( \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6 \), and \( \mathcal{N}_L \), alternate between \( \{2\} \) and \( \{2, 4\} \), \( \{3\} \) and \( \{3, 6\} \), \( \{4\} \) and \( \{4, 5\} \), \( \{5\} \) and \( \{5, 1\} \), \( \{6\} \) and \( \{6, 3\} \), and \( \{1\} \) and \( \{1, 2\} \), respectively. For these time-varying neighboring sets, the network digraph is strongly connected at all time instants. The switching time instants for the neighbor set of each agent \( i \in \mathcal{N} \) are those from the sequences \( T_i = \{1, 2, \cdots, T_i \} \), which are shown in the time axes in Fig. 3. Regarding the weight functions, for agents whose \( i \in \mathcal{N} \) is even, \( w_i(\theta) = j \) for both simulations; and for agents whose \( i \in \mathcal{N} \) is odd, \( w_i(\theta) = j (2 - \cos(\theta)) \) (simulations with different weight functions are found in [38]).

In Fig. 3a, six unit vectors are randomly initialized in an open \( \mathbb{S}^2 \) ball around \( (1, 0, 0) \). In the same figure, the trajectories of the unit vectors on the unit sphere are shown, and a visual inspection indicates convergence to a synchronized network. In Fig. 3c, the angular distance, i.e., \( \theta \) as in Definition 3, between some agents is presented, and it indicates convergence to a synchronized network.

In Fig. 3b, six rotation matrices were randomly initialized such that \( \theta(I_3, R_i) \leq \frac{\pi}{2} \) for all \( i \in \mathcal{N} \). In the same figure, the trajectories of the rotation matrices are shown on a sphere of \( \pi \) radius, and a visual inspection indicates convergence to a synchronized network. In Fig. 3d, the angular distance, i.e., \( \theta \) as in Definition 2, between some agents is presented, and it indicates convergence to a synchronized network.

**VII. CONCLUSIONS**

In this paper, we studied complete and incomplete attitude synchronization for a group of agents under switching strongly connected network digraphs. We proposed switching output feedback control laws for each agent’s angular velocity, which are decentralized and do not require a common orientation frame among agents. Our main contribution lied in transforming those two problems into a common framework, where all agents dynamics are transformed into unit vectors dynamics on a sphere of appropriate dimension. Convergence to a synchronized network was guaranteed for a wide range of initial conditions. Directions for future work include extending all results to agents controlled at the torque level, rather than the angular velocity level.

**REFERENCES**


\[1\] For each rotation matrix \( R_i \), we plot \( \theta_i n_i \), where \( \theta_i = \theta(I_3, R_i) \in [0, \pi] \) and \( n_i = \frac{1}{2 \sin(\theta_i)} R_i^{-1} (R_i - R_i^T) \in \mathbb{S}^2 \).
(a) Trajectories of unit vectors in unit sphere of $\pi$ radius

(c) Angular distance between some pairs of unit vectors

(b) Trajectories of rotation matrices in sphere of radius $\pi$

(d) Angular distance between some pairs of rotation matrices

Fig. 3: Simulations for complete and incomplete synchronization.

References


