Aerial slung-load position tracking under unknown wind forces

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***Abstract*—We propose a dynamic controller for position tracking of a point-mass load attached to an omni-directional aerial vehicle by means of a cable. Both the load and the aerial vehicle are subject to unknown wind forces. We model the dynamics of the slung-load system and put it into canonical form, i.e., a form which is independent of the system's physical parameters. Following a backstepping strategy, we design a dynamic control law for the canonical system that contains four estimators, since each of the two wind disturbances has two separate effects: an effect on the linear acceleration and another on the angular acceleration. Loosely speaking, the difference between the wind forces is an input-additive disturbance, while the wind force on the load is not, which makes removing the wind force on the load non-trivial. We identify conditions on the desired position trajectory and on the wind on the load which guarantee that a well-defined equilibrium trajectory exists. The designed controller guarantees simultaneously that *(i)* the latter trajectory is asymptotically tracked and *(ii)* the cable remains taut, provided that the system is initialized in a suitable set. Simulations illustrate our results.**

I. Introduction

Vertical take off and landing rotorcrafts with hover capabilities, hereafter UAVs, are vehicles whose popularity stems from their ability to be used in small spaces, their reduced mechanical complexity, and inexpensive components. Slung-load transportation consists of a UAV physically coupled to a point-mass load. The general aim is to make the load position track a given desired trajectory. Slung-load transportation has been considered in prior work, but not while formally accounting for the presence of unknown wind forces acting on the load and on the aerial vehicle, which is an inevitable reality if such a transportation is to be accomplished outdoors.

Different solutions to slung-load transportation can be found in the literature [1]–[15]. Different modeling approaches have been pursued, such as Euler-Lagrange formulations, Hamiltonian formulations, or Kane's method [1]–[5]. The slung-load system, as a mechanical system, is known to be under-actuated. Most works rely on local parametrizations of the configuration space, while others provide a coordinate-free modeling as well as a coordinate-free control law [6]–[9]. Some works have focused on the simpler problem of position stabilization [3], [4], [7], [8], while others have examined simplified two-dimensional settings [4], [16]. Vision has been used to estimate the load position with respect to the UAV [1], [10], [15], and a force sensor on the rope has been used to compensate and/or estimate the tension on the cable [1], [14]. Dynamic controllers, considering model uncertainties and/or input disturbances, are found in [2], [8], [9], with some relying on a discrete-time model, and a flexible cable has also been considered [7], [8]. Trajectory planning has been considered by exploring (hybrid) differential flatness [7], [16], or by minimizing the allowed swing motion [2], [3]. Implementations with partial-state or visual feedback are developed at the cost of some assumptions, such as the absence of wind, the restriction to planar motions, or by making use of simplified models [17], [18]. Slung-load transportation with multiple UAVs has also been considered [1], [11]–[13]. In the coordinate-free works cited above, the desired attitude for the cable is computed, which involves the normalization of a designed three-dimensional vector. This normalization is only valid if the latter vector is non-zero: in this paper, we guarantee this normalization is valid for any state, which includes the internal states of the dynamic controller. Our solution to achieve this involves using a bounded linear acceleration control law and a bounded disturbance estimator. Also, the latter works assume, without guarantees, that the cable remains taut – this constraint must be satisfied, otherwise the load behaves as a free-falling unactuated point-mass: instead, in this paper, the designed control law guarantees that the latter physical constraint is satisfied along a closed-loop trajectory, provided that the state is initialized in a set we identify.

Position tracking for the slung-load system shares similarities with position tracking for a standard UAV [19], [20]. Here, we use differential flatness [21] to compute the desired state and input trajectory and to describe feasible position trajectories. In designing a controller, we then follow a backstepping procedure, similar to that found in [22]–[26], but we do not feedback linearize the system by dynamic augmentation of the thrust (in our case, tension), as done in [24]–[26]. In position tracking, it is known that an *a priori* bounded linear acceleration control law is necessary [9], [23], [24], a problem we tackle; we also improve on these works by providing a smooth projector operator. Moreover, when controlling a UAV, one must guarantee that the thrust remains positive (either because the UAV rotors can only spin in one direction; or because dynamic augmentation of the thrust so requires), and [26] provides a region of initial states for which such a constraint is satisfied. In a similar fashion, in a slung-load system, one must guarantee that the cable remains taut, which is the case if the tension on the cable remains positive. Finally, we assume that the UAV is omni-directional, i.e., it can generate thrust force in any direction [27]–[29].

II. Strategy and Contributions

We summarize our problem solving strategy and main contributions. In Section III, we present the model of an aerial slung-load system in the presence of winds acting on the UAV

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and on the load, which are unknown by the controller[1]. In Section IV-A, we show that the system is differentially flat with respect to the load's position and, given some desired load's position trajectory, we compute the (two) desired system's trajectories, and the (two) desired input trajectories (one solution is physically feasible – cable is under tension – while the other is unfeasible – cable is under compression). In the same section, we introduce the notion of feasible trajectories as those where the load is not buoyant in the air at any time instant. In Section IV-B, we provide a coordinate transformation that puts the system's vector field in a canonical form: this form is agnostic to the system's physical parameters and displays a characteristic cascaded structure, which we exploit in the controller design. We find out that there are two types of disturbances, where one is input additive (associated with the difference between the winds on the UAV and on the load), while the other is not (associated with the wind on the load). Dealing with the latter motivates us to introduce a smooth projection that guarantees that a disturbance estimator remains in a pre-specified domain and whose derivatives, of any order, can be computed. In Section V, we provide our main result, which establishes sufficient conditions for inferring stability and attractivity of equilibria sets, a result which is particularly useful for non-contractible state spaces and where some form of disturbance removal needs to be considered. In Section VI, we provide a strategy for designing a smooth update law, accompanied with an appropriate Lyapunov function, and which does not grow unbounded in an unbounded domain (i.e., if the load starts *far away* from its desired position, the estimators do not immediately saturate). In Section VII, we follow a six-step backstepping procedure which exploits the cascaded structure of the transformed problem. In the initial steps, the disturbances are assumed known, and these steps are immediately followed by steps where estimators for those disturbances are designed. In Section VII-A, we provide a bounded analytic control law for a double integrator, with a companion analytic Lyapunov function whose derivative is negative definite. Also, at each step, we present the equilibria sets and we formally characterize their stability and attractivity properties, which follow from an application of the results presented in Section V. Finally, at those steps where an estimator is designed, we follow the procedure presented in Section VI. Ultimately, given some feasible trajectory, we verify that the final dynamic controller guarantees that both the system's position and input trajectories converge to their desired trajectories, even if the estimators do not converge to the corresponding unknown disturbances. When the disturbances are only partially estimated, we can show attractivity while, when the disturbances are correctly estimated, we can also show stability. We also provide an excitation criterion which, if satisfied, guarantees that the estimator of the non-input-additive disturbance converges to the unknown disturbance. Finally, we show that the proposed controller guarantees that the cable remains taut along a closed-loop trajectory, given that its initial condition lies in a set, which we characterize.

[1]We consider a fully-actuated UAV, given that we wish to focus on unknown wind forces. Full actuation is available for an omni-directional UAV or it may be assumed for a UAV with a fast attitude inner loop.

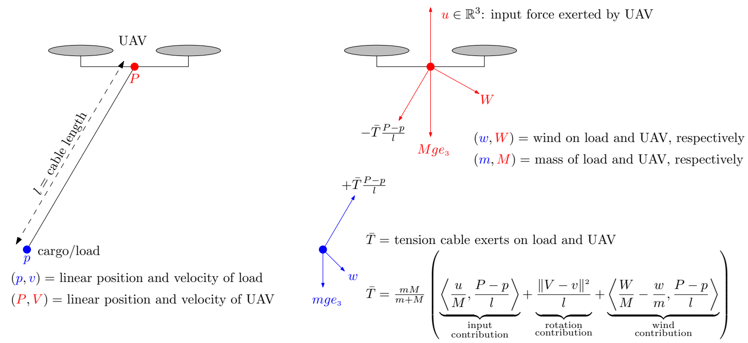


Fig. 1: Modeling of the aerial slung-load system subject to wind forces. Left: system of two point-masses physically coupled by a cable. Right: distribution of forces on each point-mass (the tension formula is discussed after (5)).

III. Problem Description

We consider[2] a UAV and a point-mass load physically coupled to each other by a massless cable, which behaves as a rigid link, cf. Fig. 1. For simplicity, we assume the UAV is fully-actuated, a simplification that is necessary due to space constraints (in our extended report [30], we deal with the under-actuated case). The system state is

$$z \in \mathbb{R}^{12} :\Leftrightarrow (p, P, v, V) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{1}$$

where p and P stand for the linear positions of the load and the UAV, while v and V stand for their linear velocities. The cable connecting the load and the UAV imposes two constraints described by $f(z) = 0_2$, where $f : \mathbb{R}^{12} \to \mathbb{R}^2$ is given by

$$f(z) := (l^{-2}\|P - p\|^2 - 1, 2l^{-2}\langle P - p, V - v\rangle). \tag{2}$$

The first component describes a geometric constraint, requiring the positions of the load and of the UAV to be apart by the length of the cable. The second component describes a holonomic constraint and it follows from differentiating the geometric constraint composed with the kinematics of the system. The constraint map f in (2) allows us to define the state space as well as its tangent space, namely,

$$\mathbb{Z} := \{z \in \mathbb{R}^{12} : f(z) = 0_2\}, \tag{3a}$$

$$T_z\mathbb{Z} := \{\delta z \in \mathbb{R}^{12} : df(z)\delta z = 0_2\}, \text{ for } z \in \mathbb{Z}. \tag{3b}$$

The linear accelerations of the load and the UAV may be found by considering the net force applied on each point-mass.

[2]Here we introduce basic notation used throughout the paper. Let $\mathcal{S} : \mathbb{R}^3 \ni x \mapsto \mathcal{S}(x) \in \mathbb{R}^{3\times 3}$ be the map that yields a skew-symmetric matrix satisfying $\mathcal{S}(x)y = x \times y$, for any $y \in \mathbb{R}^3$. $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\| = 1\}$ denotes the set of unit vectors in $\mathbb{R}^3$. The map $\Pi : \mathbb{S}^2 \to \mathbb{R}^{3\times 3}$, defined as $\Pi(x)y := y - \langle y, x\rangle x$ for any $y \in \mathbb{R}^3$ and $x \in \mathbb{S}^2$, yields a matrix that represents the projection of y onto the subspace orthogonal to x. For $n \in \mathbb{N}$, we denote by $e_1^n, \cdots, e_n^n \in \mathbb{R}^n$ the canonical basis vectors in $\mathbb{R}^n$. Let $n \in \mathbb{N}$ and $r > 0$, and denote $\mathbb{B}_r^n := \{x \in \mathbb{R}^n : \|x\| < r\}$ as the open ball in $\mathbb{R}^n$ of radius r, and $\mathbb{C}_r^n := \{x \in \mathbb{R}^n : \|x\| > r\} = \mathbb{R}^n \backslash \bar{\mathbb{B}}_r^n$ as the complement of a closed ball in $\mathbb{R}^n$ of radius r, respectively. A map $\alpha : [0, \infty) \to [0, \infty)$ belongs to $\mathcal{K}^\infty$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. Next, let A and B be manifolds, and consider a differentiable map $f : A \to B$. For any point $a \in A$, $df(a) : T_aA \to T_{f(a)}B$ denotes the derivative (which is a linear map) of the map f at the point a. $d_if(a_1, \cdots, a_i, \cdots, a_n)$ denotes the derivative of f with respect to its ith entry. If $B = \mathbb{R}$, ∇f denotes the gradient of f, i.e., $\langle \nabla f(a), \delta a\rangle = df(a)\delta a$ for any $\delta a \in T_aA$; If $N = \mathbb{R}$, and for some $r \in \mathbb{R}$, $f_{\leq r} := \{a \in A : f(a) \leq r\}$ denotes the sublevel set of value r of the map f; $f_{\star r}$, for $\star \in \{<, \leq, =, \geq, >\}$, is similarly defined. Finally, we define $\log_{v_0} \in \mathcal{K}^\infty$ as (normally, $\log_{v_0}$ stands for the logarithm in base v_0 for some $v_0 > 0$, which is not, however, the meaning of $\log_{v_0}$ in this paper.) the map given by $\log_{v_0}(v) := v_0 \log\left(1 + \frac{v}{v_0}\right)$ for some $v_0 > 0$, and, as such, $\log'_{v_0}(v) = \frac{1}{1+\frac{v}{v_0}} > 0$; when $v_0 = \infty$, then $\log_{v_0} = \mathrm{id}_{[0,\infty)}$ and $\log'_{v_0} = 1$.

Fig. 1 shows the distribution of forces: $u \in \mathbb{R}^3$ denotes the input force that the UAV can apply and $(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3$ denotes the pair of wind forces applied on each point-mass. w is the wind force applied on the load and W is the wind force applied on the UAV. For the purposes of control design, the wind forces are assumed to be constant (i.e., $\dot{w} = 0_3$ and $\dot{W} = 0_3$), but we provide a simulation where this assumption is violated. Finally, $T(z, u)$ is the tension on the cable, which depends on the state z, the input u (as well as the wind forces).

Combining the system kinematics (velocity equations) with the dynamics (acceleration equations), we have that the system is described by $Z_{w,W} : \mathbb{R}^{12} \times \mathbb{R}^3 \to \mathbb{R}^{12}$ as[3]

$$\dot{z}=Z_{w,W}(z,u):\Leftrightarrow\begin{bmatrix}\dot{p}\\\dot{P}\\\dot{v}\\\dot{V}\end{bmatrix}=\begin{bmatrix}v\\V\\\frac{\bar{T}(z,u)}{m}\frac{P-p}{l}-ge_3+\frac{w}{m}\\\frac{u}{M}-\frac{\bar{T}(z,u)}{M}\frac{P-p}{l}-ge_3+\frac{W}{M}\end{bmatrix},(4)$$

where the tension $\bar{T} : \mathbb{R}^{12} \times \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\bar{T}(z,u) := \left\langle \frac{mu}{m+M} + \frac{mW-Mw}{m+M}, \frac{P-p}{l} \right\rangle + \frac{mM\|V-v\|^2}{(m+M)l}. \quad (5)$$

Note that the set $\mathbb{Z}$ in (3a), hereafter the state space, is invariant for solutions of the vector field $Z_{w,W}$ in (4), for any pair of wind forces $(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3$: this follows by noting that $Z_{w,W}(z, u) \in T_z\mathbb{Z} \Leftrightarrow df(z)Z_{w,W}(z, u) = 0_2$ for any $(z, u) \in \mathbb{Z} \times \mathbb{R}^3$, which actually leads to the expression (5). This conclusion is valid for time-varying wind forces too.

At this point, we state the problem that we wish to solve.

Problem 1: Let $\mathbb{R} \ni t \mapsto p_\star(t) \in \mathbb{R}^3$ be some given desired position trajectory, and consider the (open-loop) vector field $Z_{w,W}$ in (4), for some *unknown (by the controller)* wind forces $(w, W) \in \mathbb{R}^3 \times \mathbb{R}^3$. Design a control law $\mathbb{R} \times \mathbb{Z} \ni (t, z) \mapsto u^{cl}(t, z) \in \mathbb{R}^3$ such that $\lim_{t\to+\infty}(p(t) - p_\star(t)) = 0_3$ along the trajectory of $\dot{z}(t) = Z_{w,W}(z(t), u^{cl}(t, z(t)))$ with $z(0) \in \mathbb{Z}$.

Remark 3.1: We assume that the wind force applied on the load does not match the weight of the load; i.e., $(w, W) \in (\mathbb{R}^3 \backslash \{mge_3\}) \times \mathbb{R}^3$. If $w = mge_3$, then the load is buoyant in the air, and this makes it impossible to stabilize (with a continuous control law) the load around any point in space.□

Throughout the paper, we impose the following constraints on the desired trajectory,

$$\inf_{t\in\mathbb{R}} \|mp_\star^{(2)}(t) + mge_3 - w\| > 0, \quad (6a)$$

$$p_\star \in \mathcal{C}^5(\mathbb{R}), \sup_{t\in\mathbb{R}} \|p_\star^{(i)}(t)\| < \infty \text{ for } i \in \{2, 3, 4, 5\}. \quad (6b)$$

We say that a desired trajectory is *feasible* if (6a) is satisfied, and *not feasible* otherwise. In particular, if the wind applied on the load does not match the load's weight (cf. Remark 3.1), then trajectories with *small* accelerations are feasible: i.e., (6a) is satisfied if $\sup_{t\in\mathbb{R}} \|p_\star^{(2)}(t)\| \leq \left\|ge_3 - \frac{w}{m}\right\| \neq 0$. As a special case, we thus have that constant-speed trajectories are feasible. The condition on the left in (6b) is required, because we rely on the position trajectory and its first five derivatives (velocity, acceleration, jerk, snap and crackle) in our discussion; and the condition on the right in (6b) is required later when studying the stability and attractivity of the equilibrium trajectory.

[3]We index the vector field with the wind forces w and W to emphasize the fact that it depends on those unknown forces, while the control law, which we design later, does not.

IV. Dynamical Properties of Slung-Load System

In this section, we describe two important properties of the slung-load system. First, we show in Section IV-A that the system is differentially flat with respect to the load's linear position. Next, we introduce a change of coordinates in Section IV-B that yields a canonical form for describing any slung-load system. This form highlights the cascaded structure of the dynamics and is particularly well suited for our control design purposes. We build on these developments in Section IV-C to refine the problem statement.

A. Differential Flatness of Slung-Load System

We show that the system is differentially flat, cf. [21] if one assumes the wind forces (w, W) are known. This assumption allows us to formally introduce the equilibria (which depend on the wind forces) and is not detrimental to the practicality of the proposed control law, since our design does not rely on the knowledge of neither the equilibria nor the wind forces.

Proposition 4.1: The slung-load system is differentially flat with respect to the load's linear position, if (w, W) are known.

Proof: Recall $p_\star$ in Problem 1. If we require that $p(t) = p_\star(t)$ for all time instants $t \in \mathbb{R}$, then we can determine uniquely[4] the whole system trajectory $\mathbb{R} \ni t \mapsto z(t) \in \mathbb{Z}$. With this in mind, and given a feasible trajectory $p_\star$, define the pairs $(z_{\star+}, u_{\star+})$ and $(z_{\star-}, u_{\star-})$, with $z_{\star\pm} : \mathbb{R} \to \mathbb{Z}$ given by

$$\begin{bmatrix}p_\star(t)\\P_{\star\pm}(t)\\v_\star(t)\\V_{\star\pm}(t)\end{bmatrix} := \underbrace{\begin{bmatrix}p_\star^{(0)}(t)\\p_\star^{(0)}(t) \pm l\frac{a(t)}{\|a(t)\|}\\p_\star^{(1)}(t)\\p_\star^{(1)}(t) \pm l\Pi\left(\frac{a(t)}{\|a(t)\|}\right)\frac{mp_\star^{(3)}(t)}{\|a(t)\|}\end{bmatrix}}_{\text{with } a(t)\equiv mp_\star^{(2)}(t)+mge_3-w}, \quad (7a)$$

and $u_{\star\pm} : \mathbb{R} \to \mathbb{R}^3$ given by

$$u_{\star\pm}(t) := MV_{\star\pm}^{(1)}(t) + mv_\star^{(1)}(t) + (M+m)ge_3 - (W+w). (7b)$$

It is now clear from the definition of $(z_{\star\pm}, u_{\star\pm})$ why we require (6) to be satisfied. Condition (6a) guarantees that the unit vector $\frac{a(t)}{\|a(t)\|}$ with $a(t) = mp_\star^{(2)}(t) + mge_3 - w$ is well defined for any time instant $t \in \mathbb{R}$; while condition (6b) guarantees that $(z_{\star\pm}, u_{\star\pm})$ is well-defined and continuous on $\mathbb{R}$. It is simple to verify that $t \mapsto z_{\star\pm}(t)$ satisfies $\dot{z}_{\star\pm}(t) = Z_{w,W}(z_{\star\pm}(t), u_{\star\pm}(t))$ for all $t \in \mathbb{R}$, which concludes the proof. ■

Note that $(z_{\star\pm}, u_{\star\pm})$ can be thought as the two equilibria options that guarantee that the load tracks the desired position trajectory. However, given the tension function in (5), it follows that $\bar{T}(z_{\star\pm}, u_{\star\pm}) = \pm\|mp_\star^{(2)}(t) + mge_3 - w\|$, which means that $(z_{\star+}, u_{\star+})$ is feasible (cable is taut), while $(z_{\star-}, u_{\star-})$ is not feasible (cable is slack).We use differential flatness, and in particular $(z_{\star+}, u_{\star+})$, to refine the problem statement: instead of requiring $p(t) - p_\star(t) \to 0_3$, cf. Problem 1, we will require $z(t) - z_{\star+}(t) \to 0_{12}$.

B. Canonical Form of Slung-Load System

Here we introduce a change of coordinates, illustrated in Fig. 2, that puts the slung-load system in canonical form. The

[4]We obtain two solutions: however, only one is feasible, which is the one that satisfies the requirement that the cable remains taut.

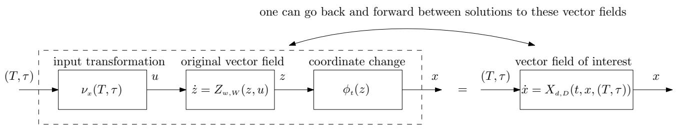


Fig. 2: Change of coordinates that allows us to obtain the vector field $X_{d,D}$ in (13), for which the controller design is simpler.

resulting form of the equations is particularly useful in the controller design stage. Consider the state space

$$\mathbb{X} := \{x=(e,\nu,n,\omega) \in (\mathbb{R}^3)^{\times 4} : \langle n,n\rangle = 1, \langle n,\omega\rangle = 0\}. \tag{8}$$

Given a time instant $t \in \mathbb{R}$, consider the change of coordinates $\phi_t : \mathbb{Z} \to \mathbb{X}$ and $\phi_t^{-1} : \mathbb{X} \to \mathbb{Z}$ defined as

$$\phi_t(z) := \begin{bmatrix} p - p_\star^{(0)}(t) \\ v - p_\star^{(1)}(t) \\ \frac{P-p}{l} \\ \mathcal{S}\left(\frac{P-p}{l}\right)\frac{V-v}{l} \end{bmatrix}, \phi_t^{-1}(x) := \begin{bmatrix} e + p_\star^{(0)}(t) \\ e + ln \\ \nu + p_\star^{(1)}(t) \\ \nu + l\mathcal{S}(\omega)n \end{bmatrix}. \tag{9}$$

The maps ϕ_t and ϕ_t^{-1} are analytic, and it is easy to verify $\phi_t \circ \phi_t^{-1} = \mathrm{id}_{\mathbb{X}}$ and $\phi_t^{-1} \circ \phi_t = \mathrm{id}_{\mathbb{Z}}$. As such ϕ_t and ϕ_t^{-1} are diffeomorphisms. Given $x \in \mathbb{X}$, consider the input transformation $\nu_x : \mathbb{R} \times T_n\mathbb{S}^2 \to \mathbb{R}^3$,

$$\nu_x(T,\tau) := ((m+M)T - Ml\langle\omega,\omega\rangle)\, n - Ml\mathcal{S}(n)\,\tau, \tag{10}$$

where T and τ are new inputs. Consider the notation

$$d := \frac{w}{m} \text{ and } D := \frac{M}{M+m}\left(\frac{W}{M} - \frac{w}{m}\right), \tag{11}$$

$$\Phi(n) := \frac{1}{l}\frac{M+m}{M}\mathcal{S}(n) \in \mathbb{R}^{3\times 3}, \tag{12}$$

where we emphasize that $d, D = 0_3$ when $w, W = 0_3$, and that d, D have the physical dimensions of linear accelerations.

The vector field (4) in the new coordinates and with the new inputs takes the form

$$\dot x = \underbrace{\left(\frac{d}{dt}\phi_t(z) + d\phi_t(z)Z_{w,W}(z,u)\right)|_{z=\phi_t^{-1}(x), u=\nu_x(T,\tau)}}_{=:X_{d,D}(t,x,(T,\tau))} \Leftrightarrow$$

$$\begin{bmatrix} \dot e \\ \dot \nu \\ \dot n \\ \dot \omega \end{bmatrix} = \begin{bmatrix} \nu \\ (T + \langle n, D\rangle)\, n - g(t) + d \\ \mathcal{S}(\omega)\, n \\ \Pi(n)\,(\tau + \Phi(n)D) \end{bmatrix} \tag{13}$$

where $g : \mathbb{R} \to \mathbb{R}^3$, hereafter called *time-varying gravity acceleration*, is defined as

$$g(t) := p_\star^{(2)}(t) + ge_3. \tag{14}$$

The vector field $X_{d,D}$ has a cascaded structure, cf. Fig. 3. This is the canonical vector field for the slung-load system since it does not depend on the system's physical parameters[5]. These parameters need to be known when controlling the vector field of the real physical system (but they do not need to be known when controlling the canonical vector field).

C. Refined Problem Statement

The result on differential flatness of the slung-load system (cf. Proposition 4.1) naturally extends to the system in the

[5]Note that $\Phi(n) := \gamma\mathcal{S}(n)$, with $\gamma = \frac{1}{l}\frac{M+m}{M}$, depends on the physical parameters, thus, formally speaking, two slung-load systems have the same canonical form only if the constant γ is the same for both. In a true canonical form, we would replace $\dot\omega = \Pi(n)(\tau + \Phi(n)D)$ in (13) with $\dot\omega = \Pi(n)(\tau + \mathcal{S}(n)\tilde D)$ where $\tilde D := \gamma D$ is considered as a *third* unknown disturbance (in addition to d and D), which has the physical dimensions of an angular acceleration. However, and for simplicity, we stick to the formulation in (13).

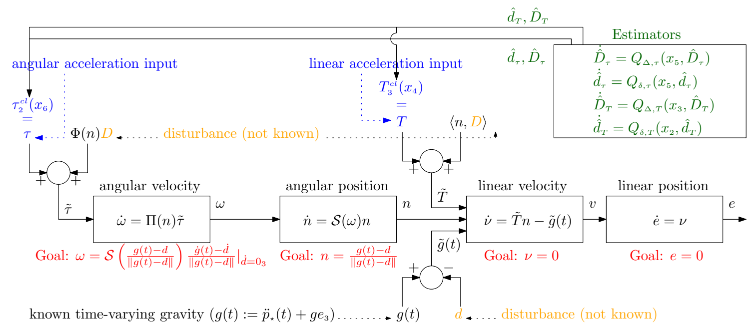


Fig. 3: Cascaded structure of the vector field in (13). Note that if we set $p = 0$ (which is our goal) then the disturbance d propagates backwards: i.e., it propagates to n and ω (part of the state), and it propagates to τ (part of the input), even though τ is not immediately affected by d.

new coordinates. For that reason, given the definition of the equilibrium trajectories and inputs $t \mapsto (z_{\star\pm}(t), u_{\star\pm}(t))$ in (7), we can also define the equilibrium trajectories and inputs in the new coordinates, namely

$$t \mapsto \quad x_{\star\pm}(t) \quad := \quad \phi_t(z_{\star\pm}(t)) \quad \overset{(7a),(9)}{:\Leftrightarrow} \tag{15a}$$

$$t \mapsto \begin{bmatrix} e_\star(t) \\ \nu_\star(t) \\ n_{\star\pm}(t) \\ \omega_\star(t) \end{bmatrix} := \begin{bmatrix} 0_3 \\ 0_3 \\ \pm\frac{g(t)-d}{\|g(t)-d\|} \\ \mathcal{S}\left(\frac{g(t)-d}{\|g(t)-d\|}\right)\frac{\dot g(t)-\dot d}{\|g(t)-d\|} \end{bmatrix}|_{\dot d=0_3},$$

and

$$t \mapsto \begin{bmatrix} T_{\star\pm}(t) \\ \tau_{\star\pm}(t) \end{bmatrix} := \begin{bmatrix} \pm\|g(t)-d\| - \left\langle \pm\frac{g(t)-d}{\|g(t)-d\|}, D\right\rangle \\ \omega_\star^{(1)}(t) - \Phi\left(\pm\frac{g(t)-d}{\|g(t)-d\|}\right)D \end{bmatrix}|_{\substack{\dot d=0_3 \\ \ddot d=0_3}} \tag{15b}$$

as illustrated in Fig. 3.

We can then restate Problem 1 in the new coordinates.

Problem 2: Let $(d, D) \in \mathbb{R}^3 \times \mathbb{R}^3$ be some unknown disturbances, and $g : \mathbb{R} \to \mathbb{R}^3$ be some time-varying gravity acceleration satisfying

$$\inf_{t\in\mathbb{R}} \|g(t) - d\|_{\mathbb{R}^3} > 0, \text{ and} \tag{16a}$$

$$g \in \mathcal{C}^3, \sup_{t\in\mathbb{R}} \|g^{(i)}(t)\|_{\mathbb{R}^3} < \infty \text{ for } i \in \{0,1,2,3\}. \tag{16b}$$

Consider then the vector field $X_{d,D}$ in (13), and the desired trajectory $\mathbb{R} \ni t \mapsto x_{\star+}(t) \in \mathbb{X}$ as defined in (15). Design control laws $(T^{cl}, \tau^{cl}) : \mathbb{R} \times \mathbb{X} \to \mathbb{R} \times \mathbb{R}^3$ such that, along the solution $t \mapsto x(t)$ of $\dot x(t) = X_{d,D}(t, x(t), (T^{cl}(t, x(t)), \tau^{cl}(t, x(t))))$ with $x(0) \in \mathbb{X}_0 \subset \mathbb{X}$, it follows that $\lim_{t\to\infty} \|x(t) - x_{\star+}(t)\|_{\mathbb{R}^{12}} = 0$ for some dense set of initial conditions $\mathbb{X}_0$.

Note that the conditions in (16) are equivalent to the conditions in (6), which guarantee well-posedness of the equilibrium trajectories. Also, we do not expect to have $\mathbb{X}_0 = \mathbb{X}$ because there are two equilibrium trajectories. Finally note that (16a), from a controller design perspective is not useful, since it depends on the unknown disturbance d. This leads us to further refine our problem statement as follows.

Problem 3: Let the *unknown* disturbance $d \in \mathbb{R}^3$ be upper bounded in norm by some *known* $\bar d \geq 0$, i.e., $\|d\|_{\mathbb{R}^3} \leq \bar d$. We seek to solve Problem 2 with (16a) replaced by

$$\inf_{t\in\mathbb{R}} \|g(t)\|_{\mathbb{R}^3} > \bar d. \tag{16c}$$

Condition (16c) implies (16a), and is hence more restrictive on the set of trajectories that the load can track. This conservativeness is unavoidable since d is unknown.

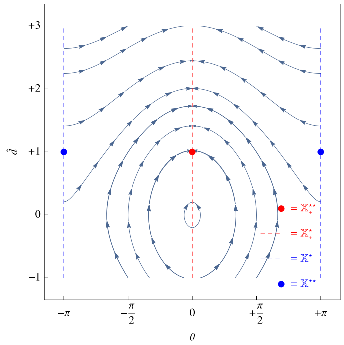


Fig. 4: Phase plot for system described in Example 5.1 with $d = 1$. Illustration of equilibria sets $\mathbb{X}_\pm^\star$ and $\mathbb{X}_\pm^{\star\star}$: $\mathbb{X}_+^{\star\star}$ is stable, but not attractive; there is a subset of $\mathbb{X}_+^\star$ which is attractive; $\mathbb{X}_-^\star$ and $\mathbb{X}_-^{\star\star}$ are neither stable nor attractive; there is a subset of $\mathbb{X}_-^\star$ which is attractive. This illustrates the different stability and attractivity properties of the equilibria sets that emerge when attempting to accomplish slung-load transportation with unknown wind forces. We also refer to Remark 5.6 for some further comments on the stability and attractivity of the latter sets, i.e., $\mathbb{X}_\pm^\star$ and $\mathbb{X}_\pm^{\star\star}$.

V. Stability and Attractivity of Equilibria Sets

In solving Problem 3, we follow a backstepping procedure. At the end of each step of this, we wish to infer the stability and attractivity properties of the respective equilibria sets. The main result presented in this section (cf. Theorem 5.5) serves exactly this purpose (simpler versions of the result could be used for the first steps of the backstepping procedure, but its complete version is necessary in the final steps). Let us provide an illustrative example, which sheds intuition into the definitions we present in this section.

Example 5.1: Consider $\dot{\theta} = \omega + \sin(\theta)d$, where one may think of $\theta \in \mathbb{R}$ as an angular position one wishes to stabilize at $\{\ldots, -2\pi, 0, +2\pi, \ldots\}$; think of $d \in \mathbb{R}$ as an unknown disturbance; and think of $\omega \in \mathbb{R}$ as an input angular velocity. Consider then the (controlled) system

$$\begin{bmatrix}\dot{\theta}\\ \dot{\hat{d}}\end{bmatrix} = \begin{bmatrix}-\sin(\theta) + \sin(\theta)(d - \hat{d})\\ \sin(\theta)^2\end{bmatrix}, \begin{bmatrix}\theta(0) \in \mathbb{R}\\ \hat{d}(0) \in \mathbb{R}\end{bmatrix}, \quad (17)$$

where one may think of $\hat{d} \in \mathbb{R}$ as an estimator for the unknown disturbance $d \in \mathbb{R}$, whose update-law constitutes the dynamics of the internal state of the dynamic control law $\omega = -\sin(\theta)(1 + \hat{d})$. Consider the Lyapunov function $V(\theta, \hat{d}) := 1 - \cos(\theta) + \frac{1}{2}(d - \hat{d})^2 \geq 0$ and its derivative $W(\theta, \hat{d}) := d_1 V(\theta, \hat{d})\dot{\theta} + d_2 V(\theta, \hat{d})\dot{\hat{d}} = -\sin(\theta)^2 \leq 0$, and define $\mathbb{X}_\pm^\star := \{(\theta, \hat{d}) \in \mathbb{R} \times \mathbb{R} : \cos(\theta) = \pm 1\}$ and $\mathbb{X}_\pm^{\star\star} := \{(\theta, \hat{d}) \in \mathbb{R} \times \mathbb{R} : \cos(\theta) = \pm 1 \text{ and } \hat{d} = d\}$, cf. Fig. 4. It then follows that $\mathbb{X}_+^{\star\star}$ is stable, that part of $\mathbb{X}_+^\star$ is attractive, and that $\mathbb{X}_-^\star$ and $\mathbb{X}_-^{\star\star}$ are neither stable nor attractive. The latter can be proven following the steps discussed next. □

Given manifolds $\mathbb{X}, \mathbb{E}$, consider a system in $\mathbb{X}$,

$$\dot{x}(t) = X(x(t), e(t)) \text{ with } x(t_0) \in \mathbb{X} \text{ and } t_0 \in \mathbb{R}, \quad (18)$$

where $X : \mathbb{X} \times \mathbb{E} \ni (x, e) \mapsto X(x, e) \in T_x\mathbb{X}$ is some smooth vector field and $e : \mathbb{R} \to \mathbb{E}$ is some exogenous input (the vector field is time-varying, and its dependence on a time instance t comes encapsulated in the form of the exogenous input $e(t)$).

Definition 1: Consider a smooth map $f : \mathbb{X} \to \mathbb{R}^n$, and the vector field X in (18). If $\dot{f}(x) = df(x)X(x, e)$ is independent of $e \in \mathbb{E}$ for every $x \in \mathbb{X}$, we say $f \in \mathcal{C}_X^1(\mathbb{R}^n)$ and define

$$d_X^1 f : \mathbb{X} \to \mathbb{R}^n, d_X^1 f(x) := df(x)X(x, e) \quad \text{(independent of } e\text{)}.$$

Moreover, note that $d_X f : \mathbb{X} \to \mathbb{R}^n$ is itself a smooth map. Finally, we say that $f \in \mathcal{C}_X^k(\mathbb{R}^n)$ if $d_X f \in \mathcal{C}_X^{k-1}(\mathbb{R}^n)$, for $k \in \{2, 3, \ldots\}$ (i.e., the kth time derivative of f is independent of e), and we define $d_X^k f$ accordingly. □

The previous definition means that $f \in \mathcal{C}_X^1(\mathbb{R}^n)$ if the time derivative of $t \mapsto f(x(t))$, along solutions of (18), is independent of the exogenous input. As a simple example, consider the vector field $(\dot{x}_1, \dot{x}_2) = (x_2, x_1 + e) =: X(x, e)$ and the function $f(x) := x_1$: then $f \in \mathcal{C}_X^1(\mathbb{R})$ and $d_X^1 f(x) = x_2$. The next result makes use of the previous definition.

Proposition 5.2: (Barbalat's Lemma) Let $x : [t_0, +\infty) \to \mathbb{X}$ be a complete solution of (18), which remains in a compact subset of $\mathbb{X}$, and where the exogenous input $e : [t_0, +\infty) \to \mathbb{E}$ also remains in a compact subset of $\mathbb{E}$. Finally, consider a map $f \in \mathcal{C}_X^1(\mathbb{R}^n)$ for which the limit $\lim_{t\to+\infty} f(x(t))$ exists. Then, $\lim_{t\to+\infty} \dot{f}(x(t)) = \lim_{t\to+\infty} d_X^1 f(x(t)) = 0_n$.

Proof: The proof follows from an application of Barbalat's lemma [31, Lemma 4.2], which can be invoked if one concludes that $[t_0, +\infty) \ni t \mapsto \dot{f}(x(t)) \in \mathbb{R}^n$ is uniformly continuous. In particular, that is the case if $[t_0, +\infty) \ni t \mapsto \ddot{f}(x(t)) \in \mathbb{R}^n$ is bounded, which is the case, since

$$\begin{aligned}\sup_{t\geq t_0} \|\ddot{f}(x(t))\| &= \sup_{t\geq t_0} \left\|\frac{d}{dt} d_X^1 f(x(t))\right\| \\ &= \sup_{t\geq t_0} \|d(d_X^1 f)(x(t))X(x(t), e(t))\| \\ &\leq \sup_{\substack{x\in\text{compact subset of }\mathbb{X}\\ e\in\text{compact subset of }\mathbb{E}}} \|d(d_X^1 f)(x)X(x, e)\| < \infty,\end{aligned}$$

where the final inequality follows since $d_X^1 f \in \mathcal{C}^1(\mathbb{X})$ and since $X \in \mathcal{C}^0(\mathbb{X} \times \mathbb{E})$. ■

Now, let there be a candidate Lyapunov function

$$V : \mathbb{X} \to [0, +\infty), \quad (19)$$

such that $V \in \mathcal{C}_X^1(\mathbb{R})$ and with a non-positive derivative along solutions of (18), i.e.,

$$W : \mathbb{X} \to (-\infty, 0], W(x) := d_X^1 V(x) \leq 0. \quad (20)$$

In addition, let there exist a function $w \in \mathcal{C}_X^k(\mathbb{R}^n)$, for some positive integers n and k, such that[6]

$$W(x) = 0 \Rightarrow w(x) = 0_n. \quad (21a)$$

Then, let there be two disjoint sets $\mathbb{X}_+^\star, \mathbb{X}_-^\star \subset \mathbb{X}$,[7] such that

$$\mathbb{X}_+^\star \cup \mathbb{X}_-^\star \supseteq \left\{ x \in \mathbb{X} : \left\{ \begin{aligned} \dot{V} &= d_X^1 V(x) = 0 \\ w^{(1)} &= d_X^1 w(x) = 0_n \\ &\vdots \\ w^{(k)} &= d_X^k w(x) = 0_n \end{aligned} \right. \right\}, \quad (21b)$$

and such that

$$X(x, e) \in \mathbb{X}_+^\star \text{ for all } (x, e) \in \mathbb{X}_+^\star \times \mathbb{E}, \quad (22a)$$

$$X(x, e) \in \mathbb{X}_-^\star \text{ for all } (x, e) \in \mathbb{X}_-^\star \times \mathbb{E}. \quad (22b)$$

The idea, as illustrated in Fig. 4, is that $\mathbb{X}_+^\star$ and $\mathbb{X}_-^\star$ correspond to two disjoint equilibria sets. Finally, let there be two sets

[6] Note that $\lim_{t\to+\infty} W(x(t)) = 0 \not\Rightarrow \lim_{t\to+\infty} w(x(t)) = 0_n$, but if $t \mapsto x(t)$ lies in a compact set then $\lim_{t\to+\infty} W(x(t)) = 0 \Rightarrow \lim_{t\to+\infty} w(x(t)) = 0_n$.

[7] By disjoint, we mean that $\inf_{a\in A, b\in B} \text{dist}(a, b) > 0$; i.e., if the sets A and B are not compact, then they cannot "approach each other".

$\mathbb{X}_+^{\star\star} \subseteq \mathbb{X}_+^\star$ and $\mathbb{X}_-^{\star\star} \subseteq \mathbb{X}_-^\star$, such that

$$\begin{cases} V(x) = 0 \Leftrightarrow x \in \mathbb{X}_+^{\star\star} \\ V(x) > 0 \text{ for all } x \in \mathbb{X}_+^\star \backslash \mathbb{X}_+^{\star\star} \end{cases}, \tag{23a}$$

$$\begin{cases} V(x) =: V_-^\star \text{ for all } x \in \mathbb{X}_-^{\star\star} \\ V(x) > V_-^\star \text{ for all } x \in \mathbb{X}_-^\star \backslash \mathbb{X}_-^{\star\star} \end{cases}, \tag{23b}$$

for some positive $V_-^\star > 0$. Again, we refer the reader to Fig. 4 and Example 5.1, for an intuition on the sets $\mathbb{X}_+^{\star\star}$ and $\mathbb{X}_-^{\star\star}$.

We are interested in inferring stability of the set $\mathbb{X}_+^{\star\star}$, which begs the question on whether the Lyapunov function V can be used for that purpose. The next result, whose proof can be found in [32, Proposition 2.2], sheds some light into it.

Proposition 5.3: Let *(1)* M be a manifold; *(2)* $V : M \to [0,\infty)$ be a continuous map, and denote $M^\star := \{m \in M : V(m) = 0\}$ as the set where V vanishes; *(3)* for any $V_0 \geq 0$, the sub-level set $V_{\leq V_0}$ is a compact subset of M. Then, there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha\left(\text{dist}_M(m, M^\star)\right) \leq V(m)$ for each $m \in M$.

We are also interested in inferring attractivity of the set $\mathbb{X}_+^\star$, and the next remark sheds some light into the latter.

Remark 5.4: (21a) and (21b) provide conditions for establishing attractivity of the set $\mathbb{X}_+^\star \cup \mathbb{X}_-^\star$. If we can infer that V, w_1, …, w_k all converge to some constants, then, by invoking Barbalat's lemma, we can conclude that $\dot{V}$, $\dot{w}_1$, …, $\dot{w}_k$ vanish asymptotically; if a solution is, in addition, trapped in a compact set, then the latter suffices to conclude that a solution approaches $\mathbb{X}_+^\star \cup \mathbb{X}_-^\star$. Also, (23b) implies that if the Lyapunov function is ever "below" the threshold $V_-^\star$, then a solution cannot converge to the set $\mathbb{X}_-^\star$, since the Lyapunov function is non-increasing along any solution. □

With all the above in mind, we can then state our main theorem, which we invoke several times later in this paper.

Theorem 5.5: Consider the system with the vector field X in (18), the Lyapunov function V in (19), and the sets $\mathbb{X}_\pm^\star, \mathbb{X}_\pm^{\star\star}$, satisfying the conditions listed in (20)–(23). Finally,

- let $e : \mathbb{R} \to \{\text{compact subset of } \mathbb{E}\}$;
- let $U \subseteq \mathbb{X}$ be some invariant subset w.r.t. (18);
- let $V_{\leq V_0} \cap U$ form a compact subset of $\mathbb{X}$ for any $V_0 \geq 0$.

For brevity, denote $\tilde{\mathbb{X}} := \mathbb{X} \cap U$, $\tilde{\mathbb{X}}_\pm^\star := \mathbb{X}_\pm^\star \cap U$ and $\tilde{\mathbb{X}}_\pm^{\star\star} := \mathbb{X}_\pm^{\star\star} \cap U$. Finally, consider the differential equation

$$\dot{x}(t) = X(x(t), e(t)) \text{ with } x(t_0) := x_0 \in \tilde{\mathbb{X}},$$

for some $t_0 \in \mathbb{R}$. Then, (below, $V_-^\star$ is as described in (23b))

1) there exists a unique and complete solution $[t_0, +\infty) \ni t \mapsto x(t) \in \tilde{\mathbb{X}}$ to the differential equation above;
2) the sets $\mathbb{X}_+^\star$ and $\mathbb{X}_-^\star$ are invariant;
3) the set $\tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star$ is globally attractive, i.e., $\lim_{t\to+\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star) = 0,$ for all $(t_0, x_0) \in \mathbb{R} \times \tilde{\mathbb{X}}$;
4) the set $\tilde{\mathbb{X}}_+^{\star\star}$ is stable;
5) for any $\epsilon \in (0, V_-^\star]$, the set $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$ is attractive, i.e., $\lim_{t\to\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}) = 0$ for all $(t_0, x_0) \in \mathbb{R} \times V_{\leq\epsilon}$;
6) the sets $\tilde{\mathbb{X}}_-^{\star\star}$ and $\tilde{\mathbb{X}}_-^\star$ are neither stable nor attractive;
7) if $\mathbb{X}_\pm^\star = \mathbb{X}_\pm^{\star\star}$, the set $\tilde{\mathbb{X}}_+^\star$ is (locally) asymptotically stable and $\lim_{t\to+\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star) = 0$ for all $(t_0, x_0) \in \mathbb{R} \times V_{<V_-^\star}$; and the set $\mathbb{X}_-^\star$ is unstable.

Proof: (1) Define $V_0 := V(x_0) \in [0, \infty)$, and note that $V_{\leq V_0}$ is positively invariant since $d_X^1 V$ is non-positive. Thus, $V_{\leq V_0} \cap U$ defines a positively invariant compact subset of $\mathbb{X}$. Since e is contained in a compact subset of $\mathbb{E}$, and since the vector field is Lipschitz continuous ($X \in \mathcal{C}^1(\mathbb{X} \times \mathbb{E})$), the first conclusion follows immediately.

(2) This follows immediately from (22a) and (22b), and the fact that $X \in \mathcal{C}^1(\mathbb{X} \times \mathbb{E})$.

(3) To prove that the set $\tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star$ is globally attractive, recall (21b) and consider the solution $[t_0, +\infty) \ni t \mapsto x(t) \in \tilde{\mathbb{X}}$. Then, note that *(a)* the solution is contained in $V_{\leq V_0} \cap U$, which is a compact subset of $\mathbb{X}$; *(b_0)* since V is lower bounded, and since $\dot{V}(x(t)) = W(x(t)) \leq 0$, it follows that $\lim_{t\to+\infty} V(x(t))$ exists; *(c_0)* we can then invoke Proposition 5.2 (Barbalat's lemma) to conclude that $\lim_{t\to+\infty} d_X^1 V(x(t)) = \lim_{t\to+\infty} W(x(t)) = 0$. *($b_1$)* recall (21a); combining *(a)* (containment in compact subset) and *(c_0)* ($\lim_{t\to+\infty} W(x(t)) = 0$) it follows that $\lim_{t\to+\infty} w(x(t)) = 0_n$; *($c_1$)* we can then invoke Proposition 5.2 (Barbalat's lemma) to conclude that $\lim_{t\to+\infty} w^{(1)}(x(t)) = \lim_{t\to+\infty} d_X^1 w(x(t)) = 0_n$. We then "repeat" the later steps (*(b_1)* and *(c_1)*) several times, culminating in the ones that follow: *(b_k)* recall (21a); combining *(a)* (containment in compact subset) with *(c_0)*, *(c_1)*, …, *(c_{k-1})* it follows that $\lim_{t\to+\infty} w^{(k-1)}(x(t)) = 0_n$; *($c_k$)* we can then invoke Proposition 5.2 (Barbalat's lemma) to conclude that $\lim_{t\to+\infty} w^{(k)}(x(t)) = \lim_{t\to+\infty} d_X^k w(x(t)) = 0_n$. *(d)* finally, combining *(a)* (containment in compact subset) with *(c_0)*, *(c_1)*, …, *(c_k)*, it follows that $\lim_{t\to+\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star) = 0$.

(4) To prove that the set $\tilde{\mathbb{X}}_+^{\star\star}$ is stable, note that $\mathbb{X}_+^{\star\star} = V_{\leq 0}$ (i.e., $\mathbb{X}_+^{\star\star}$ corresponds to the sublevel set of value 0 of the Lyapunov function V) and where we emphasize that V is non-negative and continuous. We can then invoke Proposition 5.3, with $M = \tilde{\mathbb{X}}$, $M^\star = \tilde{\mathbb{X}}_+^{\star\star}$ and $V = V|_{\tilde{\mathbb{X}}}$, to conclude that there exists $\alpha \in \mathcal{K}^\infty$ such that $\alpha\left(\text{dist}_M(m, M^\star)\right) \leq V(m)$ for all $m \in M$. (That is, $\{x \in \tilde{\mathbb{X}} : V(x) \leq \epsilon\}$ defines a neighborhood around the equilibrium set $\tilde{\mathbb{X}}_+^{\star\star}$, which coincides with the latter iff $\epsilon = 0$, and which is positively invariant for any $\epsilon \geq 0$.) This suffices to conclude that the set $\tilde{\mathbb{X}}_+^{\star\star}$ is stable.

(5) We know from *(3)* that the set $\tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star$ is globally attractive, and we also know that $V(x) \geq V_-^\star$ for $x \in \mathbb{X}_-^\star$. This suffices to conclude that $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$ is attractive for any $\epsilon \in (0, V_-^\star]$: i.e., for all $(t_0, x_0) \in \mathbb{R} \times V_{\leq\epsilon}$, $\lim_{t\to+\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}) = 0$. (Technically speaking, $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$ is not attractive since we can pick $x_0 \in \tilde{\mathbb{X}}_+^\star \cap V_{>\epsilon}$, arbitrarily close to $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$, whose solution does not necessarily approach $\tilde{\mathbb{X}}_+^\star \cap V_{<\epsilon}$.)

(6) If we prove that $\tilde{\mathbb{X}}_-^{\star\star}$ is neither stable nor attractive, then we prove also that $\tilde{\mathbb{X}}_-^\star$ is neither stable nor attractive (since $\tilde{\mathbb{X}}_-^{\star\star} \subseteq \tilde{\mathbb{X}}_-^\star$). The idea is to find an initial condition, arbitrarily close to the set $\tilde{\mathbb{X}}_-^{\star\star}$, such that the corresponding complete solution does not stay arbitrarily close to the set (not stable) and it does not approach the set (not attractive). Consider the state space $\mathbb{X}$ which can be partitioned in two parts, namely $\mathbb{X} = V_{<V_-^\star} \cup V_{\geq V_-^\star}$. Moreover, recall from (23b) that $V(x) = V_-^\star$ for any $x \in \mathbb{X}_-^{\star\star}$. Pick then $x_0 \in V_{<V_-^\star}$ and note that we can pick x_0 arbitrarily close to the set $\tilde{\mathbb{X}}_-^{\star\star} \subset V_{=V_-^\star}$. Since $\dot{V}(x(t)) \leq 0$, it follows immediately that $\tilde{\mathbb{X}}_-^{\star\star}$ is not attractive (we found an initial condition arbitrarily close to $\tilde{\mathbb{X}}_-^{\star\star}$ whose corresponding solution does not approach this set). Moreover,

we know from *(3)* that $\tilde{\mathbb{X}}_+^\star \cup \tilde{\mathbb{X}}_-^\star$ is globally attractive, and thus, given that $x_0 \in V_{<V_-^\star}$, it follows that $\lim_{t\to+\infty} \text{dist}(x(t), \tilde{\mathbb{X}}_+^\star) = 0$; since $\mathbb{X}_+^\star$ and $\mathbb{X}_-^\star$ are disjoint sets (by assumption), it then follows that $\tilde{\mathbb{X}}_-^{\star\star}$ is unstable (we found an initial condition arbitrarily close to $\tilde{\mathbb{X}}_-^{\star\star}$ whose corresponding solution does not stay arbitrarily close to this set).

(7) If $\mathbb{X}_+^\star = \mathbb{X}_+^{\star\star}$, it suffices to combine *(4)* and *(5)* to conclude that $\mathbb{X}_\pm^\star$ is asymptotically stable. The proof for the indicated subset of the region of attraction is found in the proof of *(5)*. Lack of stability of $\mathbb{X}_-^\star$ follows from *(6)*. ■

Remark 5.6: We cannot prove stability of the set $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$ because the Lyapunov function V does not vanish at all points of that set (it only vanishes at $\tilde{\mathbb{X}}_+^{\star\star}$); and, as such, sub-level sets of that V (which would be positively invariant) do not define neighborhoods around the set $\tilde{\mathbb{X}}_+^\star$. In fact there are subsets of $\tilde{\mathbb{X}}_+^\star$ which may be unstable, as illustrated in Fig. 4 (one may argue, *by visual inspection*, that $\tilde{\mathbb{X}}_+^\star \cap V_{\leq\epsilon}$ is stable, but this conclusion cannot be made by using V). The fact that $\tilde{\mathbb{X}}_-^\star$ is neither stable nor attractive does not mean that its region of attraction is a set of zero measure, as shown in Fig. 4. □

VI. Smooth Update Law for Disturbance Removal

In our design, cf. Fig. 3, we rely on four disturbance estimators. One can observe from (15) that only the disturbance $d \in \mathbb{R}^3$ impacts the equilibrium trajectory, while the disturbance $D \in \mathbb{R}^3$ does not (D is an input-additive disturbance, while d is not). Since $d \in \mathbb{R}^3$ is not known, if we replace it with an estimate $\hat{d} \in \mathbb{R}^3$, which is updated according to some update-law, then such law must satisfy two important criteria: *(1)* the estimate $\hat{d} = \hat{d}^{(0)}(t)$ must remain in some ball of pre-specified radius, so that $\inf_{t\in\mathbb{R}} \|g(t) - \hat{d}\| > 0$ (and thus, so that $\frac{g(t)-\hat{d}}{\|g(t)-\hat{d}\|}$ is well defined); *(2)* the update-law $\hat{d}^{(1)}$ must be at least $\mathcal{C}^1$ in its domain, so that both the state and input are continuous (that is, we need $t \mapsto \hat{d}^{(0)}(t), \hat{d}^{(1)}(t), \hat{d}^{(2)}(t)$ to be continuous). Similar issues have been tackled in [23], [24], which have relied on sufficiently smooth update-laws [33], while in this paper, we improve on those, by considering a smooth update-law instead, which we address next.

Definition 2 (Projection operator): Let *(1)* $q \in \mathbb{R}^n$ be an unknown disturbance, and $\bar{q} \in \mathbb{R}$ be a known upper-bound on its norm, i.e., $\|q\| \leq \bar{q}$; *(2)* $\hat{q} \in \mathbb{R}^n$ be an estimate of the unknown disturbance, and $\bar{\hat{q}} \in \mathbb{R}$ be a pre-specified upper-bound on its norm, i.e., $\|\hat{q}\| < \bar{\hat{q}}$, where $\bar{\hat{q}} > \bar{q}$. We assume that a projection operator exists, of the form

$$\mathbb{R}^n \times \mathbb{R}^n \ni (\hat{\mathfrak{q}}^1, \hat{q}) \mapsto \text{Proj}_{\bar{q},\bar{\hat{q}}}(\hat{\mathfrak{q}}^1, \hat{q}) \in \mathbb{R}^n, \tag{27a}$$

which satisfies the following properties:

1) for any $\hat{\mathfrak{q}}^1 : \mathbb{R} \to \mathbb{R}^n$ and for some $r \in (\bar{q}, \bar{\hat{q}})$,
$$\bar{\mathbb{B}}_r^n \text{ is invariant for } \dot{\hat{q}}(t) = \text{Proj}_{\bar{q},\bar{\hat{q}}}(\hat{\mathfrak{q}}^1(t), \hat{q}(t)). \tag{27b}$$
2) for all $q \in \bar{\mathbb{B}}_{\bar{q}}^n$ and for all $(\hat{\mathfrak{q}}^1, \hat{q}) \in \mathbb{R}^n \times \mathbb{B}_{\bar{\hat{q}}}^n$,
$$-\langle q - \hat{q}, \text{Proj}_{\bar{q},\bar{\hat{q}}}(\hat{\mathfrak{q}}^1, \hat{q}) - \hat{\mathfrak{q}}^1 \rangle \leq 0. \tag{27c}$$
3) $\text{Proj}_{\bar{q},\bar{\hat{q}}}$ is smooth, i.e., it belongs to $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{B}_{\bar{\hat{q}}}^n)$. □

Loosely speaking, $\text{Proj}_{\bar{q},\bar{\hat{q}}}(\hat{\mathfrak{q}}^1, \hat{q})$ accepts $\hat{\mathfrak{q}}^1$ from a standard update-law, and modifies $\hat{\mathfrak{q}}^1$ if the estimate $\hat{q}$ exits the ball $\bar{\mathbb{B}}_{\bar{q}}^n$ (within which the unknown disturbance q is known to belong to), while ensuring the estimate $\hat{q}$ remains in the pre-specified ball $\mathbb{B}_{\bar{\hat{q}}}^n$ (which is bigger than $\bar{\mathbb{B}}_{\bar{q}}^n$).

Remark 6.1: The specific form of the projector function is irrelevant in what follows, provided that the conditions in (27) are satisfied. For completeness, we present here the chosen projector: it is inspired by the one proposed in [33] but, unlike it, in this paper it is smooth and given by

$$\text{Proj}_{\bar{q},\bar{\hat{q}}}(\hat{\mathfrak{q}}^1, \hat{q}) := \hat{\mathfrak{q}}^1 - f\left(\frac{\|\hat{q}\|^2 - \bar{q}^2}{\bar{\hat{q}}^2 - \bar{q}^2}\right) \frac{\langle \frac{\hat{q}}{\bar{\hat{q}}}, \hat{\mathfrak{q}}^1 \rangle + \sqrt{\langle \frac{\hat{q}}{\bar{\hat{q}}}, \hat{\mathfrak{q}}^1 \rangle^2 + \delta^2}}{2} \frac{\hat{q}}{\bar{\hat{q}}},$$

for some $\delta > 0$, and where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function given by $f(x) := 0$ if $x \leq 0$ and $f(x) := e^{\frac{x-1}{x}}$ if $x > 0$. □

With the concept of projection operator introduced, we now describe the procedure for designing an update-law. The strategy we follow is to close the loop assuming a disturbance q is known, and leading to a closed-loop vector field X_q, accompanied by a Lyapunov function V_q and equilibria sets $\mathbb{X}_{\pm,p}^\star$ (recall the discussion in the previous section, and let $\mathbb{X}_{\pm,p}^{\star\star} = \mathbb{X}_{\pm,p}^\star$). All the closed-loop vector fields we consider in this paper are affine in the disturbance, i.e., $X_q = X_{\hat{q}} + E(q - \hat{q})$ (where E is some linear map), in which case the procedure we describe next can be applied. Consider then the system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{q}} \end{bmatrix} = \begin{bmatrix} X_{\hat{q}}(x, q) \\ 0_n \end{bmatrix} + \begin{bmatrix} E(x)(q - \hat{q}) \\ Q(x, \hat{q}) \end{bmatrix} \tag{28}$$

where $q \in \mathbb{R}^n$ is an unknown disturbance, $\hat{q} \in \mathbb{R}^n$ its estimate, $E(x)$ is a linear map (matrix) from $\mathbb{R}^n$ to $T_x\mathbb{X}$, and Q is the update-law associated to the estimator. With the projection operator in Definition 2 in mind, consider then the update-law Q, accompanied by the Lyapunov function $\mathcal{V}$ and its derivative $\mathcal{W}$ along (28), given by

$$\begin{aligned} Q(x, q) &:= \text{Proj}_{\bar{q},\bar{\hat{q}}}\big(\overbrace{k \log'_{\bar{V}}(V_{\hat{q}}(x)) E(x)^T \nabla V_{\hat{q}}(x)}^{=:\tilde{Q}(x,\hat{q})}, \hat{q}\big), \\ \mathcal{V}(x, \hat{q}) &:= \log_{\bar{V}}(V_{\hat{q}}(x)) + \frac{1}{2k}\|q - \hat{q}\|^2, \\ \mathcal{W}(x, \hat{q}) &:= \underbrace{\log'_{\bar{V}}(V_{\hat{q}}(x)) W_{\hat{q}}(x)}_{\leq 0 \because (20)} \underbrace{- \frac{\langle q - \hat{q}, Q(x, \hat{q}) - \tilde{Q}(x, \hat{q})\rangle}{k}}_{\leq 0 \because (27c)}. \end{aligned} \tag{29}$$

where $k > 0$ is a positive gain (integral gain); where $\bar{V} > 0$ is a positive constant and with the function $\log_{\bar{V}}$ as introduced in the Notation; and where $V_{\hat{q}}$ is the Lyapunov function associated to the vector field $X_{\hat{q}}$.

The purpose of the projection function has already been explained, so let us explain next the purpose of the function $\log_{\bar{V}}$ with a simple example: suppose, for example, that $V_{\hat{q}}(x) = \frac{1}{2}x^2$, that we wish to steer x to the origin, and let $\dot{\hat{q}} = \log'_{\bar{V}}(V_{\hat{q}}(x)) \nabla V_{\hat{q}}(x)$; if $\bar{V} < \infty$, then $\dot{\hat{q}} = x(1 + \frac{V_{\hat{q}}(x)}{\bar{V}})^2$, which is bounded and vanishes as $|x| \to \infty$; while, if $\bar{V} = \infty$, then $\dot{\hat{q}} = x$, which is unbounded. That is, if $\bar{V} = \infty$, when x is far away from the origin, then the estimator $\hat{q}$ changes too quickly (and it saturates immediately if a projection operator is used); on the contrary, if $\bar{V} < \infty$, when x is far away from the origin (i.e., when $V_{\hat{q}}(x) \gg \bar{V}$), the estimate $\hat{q}$ remains constant, and it only "starts working" when x is close enough to the origin (i.e., when $V_{\hat{q}}(x) \ll \bar{V}$). The effect of $\bar{V}$ is illustrated in the simulations at the end of this paper. Also, there are circumstances when $\bar{V} < \infty$ is not a valid option. If we pick $\bar{V} = \infty$, then $\log_{\bar{V}} = \text{id}_{[0,\infty)}$ and $\log'_{\bar{V}}(\cdot) = 1$, in which case (29) reads as

$$Q(x, q) := \text{Proj}_{\bar{q},\bar{\hat{q}}}\big(kE(x)^T \nabla V_{\hat{q}}(x), \hat{q}\big),$$

$$\mathcal{V}(x,\hat{q}) := V_{\hat{q}}(x) + \frac{1}{2k}\|q-\hat{q}\|^2, \qquad (30)$$
$$\mathcal{W}(x,\hat{q}) := W_{\hat{q}}(x) - \frac{1}{k}\langle q-\hat{q}, Q(x,\hat{q}) - \tilde{Q}(x,\hat{q})\rangle.$$

The difference between (29) and (30) lies in the fact that Q in (29) requires full knowledge of the Lyapunov function, while Q in (30) does not, which is critical when the Lyapunov function depends on some "unknown quantity". For example, let d be some unknown quantity, $V_{\hat{q}}(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2-d)^2$ and $E(x) = (1,0) \in \mathbb{R}^{2\times 1}$: then $E(x)^T\nabla V_{\hat{q}}(x) = x_1$ does not depend on d, which means the update-law in (30) can be implemented, while the update-law in (29) cannot. We define the equilibria sets

$$\mathbb{Y}^\star_\pm = \{(x,\hat{q}) : x \in \mathbb{X}^\star_{\pm,\hat{q}}, E(x)(\hat{q}-q) = 0 \in T_x\mathbb{X}\},$$
$$\mathbb{Y}^{\star\star}_\pm = \{(x,\hat{q}) : x \in \mathbb{X}^\star_{\pm,p}, \hat{q} = q\},$$

whose stability and attractivity we study using Theorem 5.5 (to invoke it, we need to introduce the maps, which are specific to each step of the backstepping procedure discussed next).

VII. Backstepping Design

We follow a 6-step backstepping procedure, motivating the necessity and relevance of each step. The presentation at the end of each step, with the exception of step 1, always proceeds by stating two propositions and one theorem: the first proposition describes the set where the Lyapunov function vanishes and the set where its derivative vanishes; the second proposition establishes compactness of sublevel sets of the Lyapunov function; finally, the theorem establishes stability and attractivity properties of the equilibria sets. The technical analysis at each step builds upon the results obtained in the previous step. We present a detailed overview of the design in the extended version [30]. Here, the section titles are meant to function as a roadmap of our design.

Before proceeding, we introduce some constants: *(1)* we pick $\underline{g} > 0$ such that $\underline{g} < \inf_{t\in\mathbb{R}} \|g(t)\|$; *(2)* the disturbance $d \in \mathbb{R}^3$ is bounded by some known upper bound $\bar{d} > 0$[8]; and we pick $\bar{\bar{d}}$ such that $\bar{\bar{d}} > \bar{d}$, which will be the upper bound on the estimator we design for d; *(3)* we pick a constant $\bar{u} > 0$ such that $\underline{g} - (\bar{u} + \bar{\bar{d}}) > 0$[9]; *(4)* the disturbance $D \in \mathbb{R}^3$ also is bounded by some known upper bound $\bar{D} > 0$; and we pick $\bar{\bar{D}}$ such that $\bar{\bar{D}} > \bar{D}$, which will be the upper bound on the estimator we design for D; *(5)* the vector field $X_{d,D}$ in (13) depends on time through the exogenous input $t \mapsto g(t)$; as we construct the control law, and close the loop, the vector field will depend on several derivatives $t \mapsto g^{(0)}(t), \ldots, g^{(k)}(t)$ of the exogenous input; we include those derivatives as part of the state by introducing $g^0, \ldots, g^k$, and requiring

$$g^i(t_0) := g^{(i)}(t_0). \qquad (31)$$

A. Step 1: Control with thrust and angular position

Consider the state, state space and vector field

$$x_1 \in \mathbb{X}_1 :\Leftrightarrow (e,\nu,g^0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{C}^3_{\underline{g}}, \qquad (32)$$

$$\dot{x}_1 = X_{1,d,D}(x_1,(T,n,g^1)) :\Leftrightarrow \begin{bmatrix}\dot{e}\\ \dot{\nu}\\ \dot{g}^0\end{bmatrix} = \begin{bmatrix}\nu\\ (T+\langle n,D\rangle)\,n - g^0 + d\\ g^1\end{bmatrix}$$

where e,ν are the linear position, velocity in (8); g^0, g^1 stand for the 0th, 1st derivative of the time-varying gravity acceleration $t \mapsto g(t)$ in (14) (recall that $\mathbb{C}^3_{\underline{g}} := \{g \in \mathbb{R}^3 : \|g\| > \underline{g}\}$); and where the goal is to design a control law for the thrust T and the angular position n, assuming that the disturbances d and D are known, and such that $(e,\nu) \to (0_3, 0_3)$ (recall (15a)). For that purpose, denote the equilibrium set by

$$\mathbb{X}^\star_1 := \{x_1 \in \mathbb{X}_1 : e = 0_3 \text{ and } \nu = 0_3\}. \qquad (33)$$

Next, we list the tools we need to solve this problem.

Assumption 7.1: Consider a double integrator system $(\dot{e},\dot{\nu}) = (\nu, u)$, with e,ν,u as the position, the velocity, the acceleration input; and recall the constant $\bar{u}$ introduced at the beginning of this section. We assume we have available

$$u_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \ni (e,\nu) \mapsto u_{di}(e,\nu) \in \mathbb{B}^3_{\bar{u}}, \qquad (34a)$$
$$V_{di} : \mathbb{R}^3 \times \mathbb{R}^3 \ni (e,\nu) \mapsto V_{di}(e,\nu) \in [0,\infty), \qquad (34b)$$

where u_{di} is a bounded control law ($\mathbb{B}^3_{\bar{u}} := \{u \in \mathbb{R}^3 : \|u\| < \bar{u}\}$) equipped with a Lyapunov function V_{di}, and such that *(1)* $u_{di}(0_3,0_3) = 0_3$ and $V_{di}(e,\nu) = 0 \Leftrightarrow (e,\nu) = (0_3,0_3)$; *(2)* $u_{di}, V_{di} \in \mathcal{C}^\infty(\mathbb{R}^3\times\mathbb{R}^3)$; *(3)* any sub-level set of V_{di} defines a compact set in $\mathbb{R}^3\times\mathbb{R}^3$ (see Proposition 5.3); *(4)* "$\dot{V}_{di}(e,\nu)$" $= W_{di}(e,\nu) := d_1V_{di}(e,\nu)\nu + d_2V_{di}(e,\nu)u_{di}(e,\nu) < 0$ for all $(e,\nu) \in (\mathbb{R}^3\times\mathbb{R}^3)\backslash\{(0_3,0_3)\}$. □

We emphasize that the description that follows in the next backstepping steps is agnostic to the specific form of the functions u_{di}, V_{di} in (34): we only require that they satisfy the conditions in Assumption 7.1. However, in the next remark, we present functions that satisfy those conditions.

Remark 7.2: Let $k_p, k_d, \sigma_p, \sigma_d$ be positive numbers (proportional and derivative gains and saturations), and let $\text{sat}_\sigma(x) := \sigma(\sigma^2 + \|x\|^2)^{-\frac{1}{2}}x$. Consider then the double integrator control law u_{di}, with $\bar{u} = k_p\sigma_p + k_d\sigma_d$, defined as $u_{di}(e,\nu) := -k_p\text{sat}_{\sigma_p}(e) - k_d\text{sat}_{\sigma_d}(\nu)$, and the Lyapunov function V_{di} defined as $V_{di}(e,\nu) := k_p\sigma_p\left(\sqrt{\langle e,e\rangle + \sigma_p^2} - \sigma_p\right) + \beta\langle\text{sat}_{\sigma_p}(e), \text{sat}_{\sigma_d}(\nu)\rangle + \frac{\langle\nu,\nu\rangle}{2}$, for some positive number $\beta < k_d\left(1 + k_d^2(4k_p)^{-1}\right)^{-1}$ (gain that guarantees that V_{di} has compact sub-level sets, and that $\dot{V}_{di} = W_{di}$ is negative definite). The functions u_{di} and V_{di} satisfy the conditions in Assumption 7.1 [30]. □

Based on (34), let us then define what we label as the desired three dimensional acceleration, i.e.,

$$T^{3d} : \mathbb{X}_1 \times \mathbb{B}^3_{\bar{\bar{d}}} \to \mathbb{C}^3_\epsilon, \text{ where } \epsilon := \underline{g} - (\bar{u} + \bar{\bar{d}}) > 0 \qquad (35)$$
$$T^{3d}(x_1,d) := u_{di}(e,\nu) + g^0 - d,$$

where the meaning of the constants $\underline{g}, \bar{u}, \bar{\bar{d}}$ was discussed at the beginning of this section. Note that T^{3d} never vanishes in its domain, and thus we can define the unit vector $\frac{T^{3d}(x_1,d)}{\|T^{3d}(x_1,d)\|}$ for any $(x_1,d) \in \mathbb{X}_1 \times \mathbb{B}^3_{\bar{\bar{d}}}$. With the latter in mind, if we pick the thrust control law $T^{cl} : \mathbb{X}_1 \times \mathbb{B}^3_{\bar{\bar{d}}} \times \mathbb{R}^3 \to \mathbb{R}$ as

$$T^{cl}(x_1,d,D) := \|s\| - \left\langle \frac{s}{\|s\|}, D\right\rangle|_{s=T^{3d}(x_1,d)}, \qquad (36a)$$

[8] From the physical system (see (11)), we know that $d = \frac{w}{m}$; thus, one needs to know that the wind force on the load $w \in \mathbb{R}^3$ is bounded in norm by some known upper bound $\bar{\bar{w}} > 0$, in which case, $\bar{d} = \frac{\bar{w}}{m}$

[9] The choice of constants $\underline{g}, \bar{\bar{d}}, \bar{u}$ is always feasible if $\inf_{t\in\mathbb{R}}\|g(t)\| - \bar{d} > 0$, as required in Problem 3.

and the angular position control law $n^{cl} : \mathbb{X}_1 \times \mathbb{B}^3_{\bar{d}} \to \mathbb{S}^2$ as

$$n^{cl}(x_1, d) := \frac{T^{3d}(x_1, d)}{\|T^{3d}(x_1, d)\|}, \tag{36b}$$

it follows immediately that composing the vector field $X_{1,d,D}$ in (32) with the control laws T^{cl} and n^{cl} in (36) yields

$$\dot{x}_1 = \underbrace{X_{1,d,D}(x_1, (T^{cl}(x_1, d, D), n^{cl}(x_1, d), g^1))}_{=:X_1^{cl}(x_1, g^1) \text{ (independent of } d \text{ and } D)} :\Leftrightarrow$$

$$\begin{bmatrix} \dot{e} \\ \dot{\nu} \\ \dot{g}^0 \end{bmatrix} = \begin{bmatrix} \nu \\ u_{di}(e, \nu) \\ g^1 \end{bmatrix}. \tag{37}$$

Since we are in the first step (and to have a coherent notation among all the sections/steps that follow), we define

$$\mathbb{X}_1 \ni x_1 \mapsto V_1(x_1) := V_{di}(e, \nu), W_1(x_1) := W_{di}(e, \nu),$$

as the Lyapunov function and its derivative at the end of step 1. Stability and global attractivity of the equilibria set $\mathbb{X}_1^\star$ in (33) can be easily inferred from V_1 and W_1 (by invoking the conditions in Assumption 7.1). Next, we introduce a Proposition, which shall be invoked in the next step.

Proposition 7.3: Consider the time-varying gravity acceleration $t \mapsto g(t)$ in (14), and let us define the set $U_1 := \{x_1 \in \mathbb{X}_1 : \inf_{t \in \mathbb{R}} \|g^{(0)}(t)\| \leq \|g^0\| \leq \sup_{t \in \mathbb{R}} \|g^{(0)}(t)\|\}$, where by assumption $\underline{g} < \inf_{t \in \mathbb{R}} \|g^{(0)}(t)\|$ and $\sup_{t \in \mathbb{R}} \|g^{(0)}(t)\| < \infty$. If (31) holds, then U_1 is an invariant set. Consider also a sub-level set of V_1, i.e., $(V_1)_{\leq V_0}$ for some non-negative V_0. Then $(V_1)_{\leq V_0} \cap U_1$ defines a compact subset of the state space $\mathbb{X}_1$.

B. Step 2: Control with thrust and angular velocity

In the second step, we lift the assumption that we control the angular position n and control the angular velocity ω instead. Consider then the state, state space and vector field

$$x_2 \in \mathbb{X}_2 :\Leftrightarrow (x_1, n, g^1) \in \mathbb{X}_1 \times \mathbb{S}^2 \times \mathbb{R}^3, \tag{38}$$

$$\dot{x}_2 = X_{2,d,D}(x_2, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} = \begin{bmatrix} X_{1,d,D}(x_1, (T, n, g^1)) \\ \mathcal{S}(\omega) n \\ g^2 \end{bmatrix},$$

with $x_1, \mathbb{X}_1, X_1$ described in the previous step; where g^2 stands for the 2nd derivative of the time-varying gravity acceleration $t \mapsto g(t)$ in (14); and where the goal is to design a control law for the thrust T and the angular velocity ω, assuming that the disturbances d and D are known, and such that $(e, \nu, n) \to \left(0_3, 0_3, \pm \frac{g^0 - d}{\|g^0 - d\|}\right)$ (recall (15a)). For that purpose, denote the equilibria sets by

$$\mathbb{X}_{2\pm}^\star := \left\{ x_2 \in \mathbb{X}_2 : x_1 \in \mathbb{X}_1^\star \text{ and } n = \pm \frac{g^0 - d}{\|g^0 - d\|} \right\}, \tag{39}$$

as the two disjoint equilibria sets: later we show that $\mathbb{X}_{2+}^\star$ is asymptotically stable, while $\mathbb{X}_{2-}^\star$ is unstable.[10]

Because the angular position n is no longer an input (but rather part of the state), we then pick a control law for the thrust T such that we minimize the error between the desired vector field designed in the previous step (X_1^{cl} in (37)) and the current one: that is

$$\inf_{T \in \mathbb{R}} \|X_{1,d,D}(x_1, (T, n, g^1)) - X_1^{cl}(x_1, g^1)\|_{\mathbb{R}^9} =$$

$$= \inf_{T \in \mathbb{R}} \|(T + \langle n, D \rangle) n - T^{3d}(x_1, d)\|_{\mathbb{R}^3}, \tag{40a}$$

where T^{3d} was designed in the previous step, in (35). Motivated by (40a), we define the thrust control law $T_1^{cl} : \mathbb{X}_2 \times \mathbb{B}^3_{\bar{d}} \times \mathbb{R}^3 \to \mathbb{R}$ as[11]

$$T_1^{cl}(x_2, d, D) := \langle n, T^{3d}(x_1, d) - D \rangle. \tag{40b}$$

Composing the linear acceleration ($\dot{\nu}$ in (32)) with the proposed thrust control law yields

$$\begin{aligned} \dot{\nu} &= ((T + \langle n, D \rangle) n - g^0 + d) |_{T = T_1^{cl}(x_2, d, D)} \\ &= u(e, \nu) - \|T^{3d}(x_1, d)\| \Pi(n) n^{cl}(x_1, d), \end{aligned} \tag{41}$$

and, thus, it follows that composing the vector field $X_{1,d,D}$ in (32) with the control law T_1^{cl} in (40b) yields

$$\begin{aligned} \dot{x}_1 &= \underbrace{X_{1,d,D}(x_1, (T, n, g^1))|_{T = T_1^{cl}(x_2, d, D)}}_{=:\tilde{X}_1(x_2, d) \text{ (independent of } D)} \\ &= X_1^{cl}(x_1, g^1) \underbrace{- \|T^{3d}(x_1, d)\| (e_2^3 \otimes \Pi(n)) n^{cl}(x_1, d)}_{\text{error}}, \end{aligned} \tag{42}$$

where we emphasize the independence of $\tilde{X}_1$ in (42) with respect to the disturbance D ($e_2^3 = (0, 1, 0)$ is the second canonical basis vector in $\mathbb{R}^3$, as described in the Notation). Then, composing the vector field $X_{2,d,D}$ in (38) with the control law T_1^{cl} in (40b) yields

$$\begin{aligned} \dot{x}_2 &= X_{2,d,D}(x_2, (T_1^{cl}(x_2, d, D), \omega, g^2)) :\Leftrightarrow \\ \begin{bmatrix} \dot{x}_1 \\ \dot{n} \\ \dot{g}^1 \end{bmatrix} &= \underbrace{\begin{bmatrix} X_1^{cl}(x_1, g^1) \\ 0_3 \\ 0_3 \end{bmatrix}}_{\text{step 1}} + \underbrace{\begin{bmatrix} \star(e_2^3 \otimes \Pi(n)) n^{cl}(x_1, d) \\ \mathcal{S}(\omega) n \\ g^2 \end{bmatrix}}_{\text{where } \star = -\|T^{3d}(x_1, d)\|}. \end{aligned} \tag{43}$$

We are thus in conditions of applying a backstepping step. Let k_θ, γ_θ, $\bar{V}_1$ be positive gains, and let us then choose the angular velocity control law

$$\omega_1^{cl} : \mathbb{X}_2 \times \mathbb{B}_{\bar{d}} \times \mathbb{R}^3 \ni (x_2, d, \dot{d}) \mapsto \omega_1^{cl}(x_2, d, \dot{d}) \in T_n \mathbb{S}^2$$

$$\omega_1^{cl}(x_2, d, \dot{d}) := -k_\theta \mathcal{S}(n^{cl}(x_1, d)) n + \tag{44a}$$

$$\Pi(n) \mathcal{S}(n^{cl}(x_1, d)) \frac{d_1 T^{3d}(x_1, d) \tilde{X}_1(x_2, d) + d_2 T^{3d}(x_1, d) \dot{d}}{\|T^{3d}(x_1, d)\|} + \tag{44b}$$

$$\gamma_\theta^{-1} \|T^{3d}(x_1, d)\| (e_2^3 \otimes \mathcal{S}(n))^T \log'_{\bar{V}_1}(V_1(x_1)) \nabla V_1(x_1), \tag{44c}$$

and where we included the term $\dot{d}$ (even though it is zero) just to emphasize its importance: indeed, once we replace d with its estimate (which is not constant and whose dynamics we design) that term cannot be neglected. We note that $d_2 T^{3d}(x_1, \hat{d}_T) = -I_3$ (see (35)), but we kept it in (44b) just for the sake of clarity. Let us provide a brief description on the terms in (44) which, altogether, steer the angular position n to the desired angular position $n^{cl}(x_1, d)$: (44a) acts as a proportional feedback term; (44b) is a feedforward term; and (44c) is a backstepping term.

We then have that the closed-loop dynamics, at the end of step 2, are given by

$$\dot{x}_2 = \underbrace{X_{2,d,D}(x_2, (T, \omega, g^2))|_{T = T_1^{cl}(x_2, d, D), \omega = \omega_1^{cl}(x_2, d, 0_3)}}_{=:X_{2,d}^{cl}(x_2, g^2) \text{ (independent of } D)}, \tag{45}$$

where we emphasize that they depend on the disturbance d but not on D (that is the case, because the thrust input cancels the

[10] $\mathbb{X}_{2\pm}^\star$ depends on the unknown disturbance d (it does not depend on the disturbance D), and we should highlight that by denoting it instead as $\mathbb{X}_{2\pm,d}^\star$ (recall Section VI): however, we adopt the former notation to simplify the exposition.

[11] The thrust control law goes through three iterations: that is, we define T_1^{cl}, T_2^{cl}, and T_3^{cl}; and where a successor control law is constructed on top of a predecessor control law.

effect of the disturbance D). We can then define the Lyapunov function $V_{2,d} : \mathbb{X}_2 \to [0, \infty)$, and its derivative $W_{2,d} : \mathbb{X}_2 \to (-\infty, 0]$ along the vector field $X^{cl}_{2,d}$ in (45), as

$$V_{2,d}(x_2) := \log_{\bar{V}_1}(V_1(x_1)) + \gamma_\theta (1 - \langle n, n^{cl}(x_1, d)\rangle), \quad (46)$$

$$W_{2,d}(x_2) := \underbrace{\log'_{\bar{V}_1}(V_1(x_1))W_1(x_1)}_{\leq 0} \underbrace{-k_\theta \gamma_\theta \|\mathcal{S}(n)\, n^{cl}(x_1, d)\|^2}_{\leq 0}, \quad (47)$$

where we emphasize that $V_{2,d}, W_{2,d}$ depend on the disturbance d (but not on D). The purpose of the constant $\bar{V}_1$ is similar to the purpose of $\bar{V}$ in Section VI: the backstepping term only starts working when $V_1(x_1) \ll \bar{V}_1$; and, when $V_1(x_1) \gg \bar{V}_1$, the proportional and feedforward terms dominate.

We wish to invoke Theorem 5.5, which motivates the introduction of the next two Propositions: the first is concerned with checking conditions (21) and (23), and the second is concerned with checking the second and third bullets in Theorem 5.5.

Proposition 7.4: Consider the functions $V_{2,d}, W_{2,d}$ in (46)–(47), and the sets $\mathbb{X}^\star_{2\pm}$ in (39). It holds that

$$V_{2,d}(x_2) = 0 \Leftrightarrow x_2 \in \mathbb{X}^\star_{2+}, \quad (48a)$$

$$V_{2,d}(x_2) = 2\gamma_\theta =: V^\star_{2-} \text{ for all } x_2 \in \mathbb{X}^\star_{2-}, \quad (48b)$$

$$\mathbb{X}^\star_{2+} \cup \mathbb{X}^\star_{2-} = \{x_2 \in \mathbb{X}_2 : W_{2,d}(x_2) = 0\}. \quad (48c)$$

Proposition 7.5: Define the set $U_2 := \{x_2 \in \mathbb{X}_2 : x_1 \in U_1 \text{ and } \|g^1\| \leq \sup_{t\in\mathbb{R}} \|g^{(1)}(t)\|\}$, with U_1 as described in Proposition 7.3. If (31) holds, then U_2 defines an invariant set. Moreover, $(V_{2,d})_{\leq V_0} \cap U_2$ defines a compact subset of $\mathbb{X}_2$ for any sub-level set $(V_{2,d})_{\leq V_0}$.

The proofs for both previous Propositions are straightforward, and, thus, omitted here[12]. All the conditions in Theorem 5.5 are satisfied, which allows us to state the following result.

Theorem 7.6: Consider the vector field $X^{cl}_{2,d}$ in (45), the Lyapunov function $V_{2,d}$ in (46), the sets $\mathbb{X}^\star_{2\pm}$ in (39), and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). Finally, consider the differential equation $\dot{x}_2(t) = X^{cl}_{2,d}(x_2(t), g^{(2)}(t))$ with $x_2(t_0) \in \tilde{\mathbb{X}}_2$. Then, all the consequences stated in Theorem 5.5 follow.

C. Step 3: Step 2, with unknown disturbance d

In the third step, we lift the assumption that the disturbance d is known in the thrust and angular velocity control laws, and we replace it with an estimate $\hat{d}_T$. In this step, it will become clear why a second estimator ($\hat{d}_\tau$, introduced in step 6) for the disturbance d is necessary. Consider then the state, state space and vector field

$$x_3 \in \mathbb{X}_3 :\Leftrightarrow (x_2, \hat{d}_T) \in \mathbb{X}_2 \times \mathbb{B}^3_{\bar{\bar{d}}}, \quad (49)$$

$$\dot{x}_3 = X_{3,d,D}(x_2, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{\hat{d}}_T \end{bmatrix} = \begin{bmatrix} X_{2,d,D}(x_2, (T, \omega, g^2)) \\ Q_{\delta,T}(x_2, \hat{d}_T) \end{bmatrix},$$

with $x_2, \mathbb{X}_2, X_2$ described in the previous step; where $\hat{d}_T$ stands for the estimator of the disturbance d; and where the goal is to reuse the thrust and angular velocity control laws from the previous step, and design an update law $Q_{\delta,T}$ for the estimator $\hat{d}_T$, such that $(e, \nu, n) \to \left(0_3, 0_3, \pm\frac{g^0 - d}{\|g^0 - d\|}\right)$ (recall (15a)), and such that the estimator $\hat{d}_T$ remains in the ball $\mathbb{B}^3_{\bar{\bar{d}}}$ (this is important, as the control laws are not defined if the estimator exits this domain), cf. Section VI. For that purpose, denote the equilibria sets by

$$\mathbb{X}^\star_{3\pm} := \{x_3 \in \mathbb{X}_3 : x_2 \in \mathbb{X}^\star_{2\pm} \text{ and } \hat{d}_T = d\} \quad (50)$$

as two disjoint equilibria sets ($\mathbb{X}^\star_{2\pm}$ in (39)).

Because the disturbance d is no longer known by the controller, we must replace it by its estimate $\hat{d}_T$ in the thrust control law (T^{cl}_1 in (40b)) and in the angular velocity control law (ω^{cl}_1 in (44)); i.e., we define the new thrust control law $T^{cl}_2 : \mathbb{X}_3 \times \mathbb{R}^3 \to \mathbb{R}$ and the new angular velocity control law $\omega^{cl}_2 : \mathbb{X}_3 \to \mathbb{R}^3$ as

$$T^{cl}_2(x_3, D) := T^{cl}_1(x_2, \hat{d}_T, D), \quad (51a)$$

$$\omega^{cl}_2(x_3) := \omega^{cl}_1(x_2, \hat{d}_T, \dot{\hat{d}}_T) \text{ with } \dot{\hat{d}}_T = P_{\delta,T}(x_2, \hat{d}_T). \quad (51b)$$

Remark 7.7: Note that the angular velocity control law ω^{cl}_2 in (51b) depends on $\dot{\hat{d}}_T = P_{\delta,T}(x_2, \hat{d}_T)$, and, ultimately, one wants $\omega = \omega^{cl}_2(x_3)$ to hold. This is the reason why one should not design an update-law for $\hat{d}_T$ that depends on the angular velocity ω: if one does so, then one designs a control law for the angular velocity that depends on the angular velocity itself – this leads to an implicit equation ($\omega = \text{function}(\omega)$) which may, or may not, have a solution. This is the reason why a second estimator for the disturbance d is necessary. □

Remark 7.8: Note that the angular velocity control law ω^{cl}_2 in (51b) depends on the estimator dynamics, and thus on the projection function $\text{Proj}_{\bar{d},\bar{\bar{d}}}$ defined in Section VI. In step 5, we will require the partial derivatives of ω^{cl}_2, which then motivates the necessity of the smoothness properties of $\text{Proj}_{\bar{d},\bar{\bar{d}}}$ (which must be twice continuous differentiable in our case). □

Composing the linear acceleration ($\dot{\nu}$ in (32)) with the control law in (51a) yields

$$\begin{aligned} \dot{\nu} &= ((T + \langle n, D\rangle)\, n - g^0 + d)\,|_{T = T^{cl}_2(x_3, D)} \quad (52) \\ &= u(e, \nu) - \Pi(n)\, T^{3d}(x_1, \hat{d}_T) + (d - \hat{d}_T) =: \nu^1(x_3). \end{aligned}$$

and, thus, it follows that composing the vector field $X_{3,d,D}$ in (49) with the control laws in (51a)–(51b) yields

$$\dot{x}_3 = \underbrace{X_{3,d,D}(x_3, (T^{cl}_2(x_3, D), \omega^{cl}_2(x_3), g^2))}_{=: X^{cl}_{3,d}(x_3, g^2) \text{ (independent of } D)} :\Leftrightarrow \quad (53)$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{\hat{d}}_T \end{bmatrix} = \underbrace{\begin{bmatrix} X^{cl}_{2,\hat{d}_T}(x_2, g^2) \\ 0_3 \end{bmatrix}}_{\text{step 2 (independent of } d \text{ and } D)} + \underbrace{\begin{bmatrix} (e^5_2 \otimes I_3)\,(d - \hat{d}_T) \\ Q_{\delta,T}(x_2, \hat{d}_T) \end{bmatrix}}_{\text{top is linear in } d - \hat{d}_T},$$

Since the vector field in (53) fits with (28), we can then proceed with the estimator design procedure described in Section VI. As such, consider the update-law, the Lyapunov function, and its derivative given by

$$Q_{\delta,T}(x_2, \hat{d}_T), V_{3,d}(x_3), W_{3,d}(x_3) \text{ as in (29)}, \quad (54)$$

with $\bar{p} = \bar{d}$, $\bar{\bar{p}} = \bar{\bar{d}}$, $k = k_{\delta,T}$, $\bar{V} = \bar{V}_2$ and $V_{\hat{p}} = V_{2,\hat{d}}$. It follows immediately from (27b) that $\hat{d}_T$ remains in some closed subset of $\mathbb{B}^3_{\bar{\bar{d}}}$, and therefore the control law in (51b) is well-defined (along a solution of $\dot{x}_3 = X^{cl}_{3,d}(x_3, g^2)$).

Remark 7.9: The update-law $Q_{\delta,T}$ in (54) depends on the estimate $\hat{d}_T$ even if the projection operator $\text{Proj}_{\bar{d},\bar{\bar{d}}}$ were omitted. This is related with the fact that the disturbance d is not an input-additive disturbance. □

As in the previous step, we wish to invoke Theorem 5.5, which motivates the introduction of the next two Propositions.

[12] We refer the reader to the proofs of Propositions 7.10 and 7.11, which contain the same arguments that would need to be invoked here.

Proposition 7.10: Consider the functions $V_{3,d}, W_{3,d}$ in (54), the sets $\mathbb{X}^\star_{3\pm}$ in (50), and, for brevity, denote $\cup\tilde{\mathbb{X}}^\star_{3\pm} \equiv \tilde{\mathbb{X}}^\star_{3+} \cup \tilde{\mathbb{X}}^\star_{3-}$. It holds that

$$V_{3,d}(x_3) = 0 \Leftrightarrow x_3 \in \mathbb{X}^\star_{3+}, \quad (55a)$$

$$V_{3,d}(x_3) = \log_{\bar{V}_2}(V^\star_{2-}) =: V^\star_{3-} \text{ for all } x_3 \in \mathbb{X}^\star_{3-}, \quad (55b)$$

$$\cup\tilde{\mathbb{X}}^\star_{3\pm} = \left\{ x_3 \in \mathbb{X}_3 : \begin{cases} \text{“}\dot{V}_{3,d}\text{”} = W_{3,d}(x_3) = 0 \\ \text{“}\dot{\nu}\text{”} = \nu^1(x_3) = 0_3 \end{cases} \right\}, \quad (55c)$$

with $V^\star_{2-}$ and v^1 defined in (48b) and (52), respectively.

Proof: Verifying (55a)–(55b) is simple, and we verify only (55c): *(1)* from (29), $W_{3,d}(x_3) = 0$ implies that $W_{2,d}(x_2) = 0$; *(2)* from (47), $W_{2,d}(x_2) = 0$ implies that $(e, \nu) = (0_3, 0_3)$ and that $n = \pm\frac{g^0 - \hat{d}_T}{\|g^0 - \hat{d}_T\|}$; *(3)* combining *(1)–(2)*, it then follows that “$\dot{\nu}$” $= d - \hat{d}_T = 0_3 \Leftrightarrow \hat{d}_T = d$; *(4)* combining *(2)–(3)*, it then follows that $n = \pm\frac{g^0 - d}{\|g^0 - d\|}$. ■

Proposition 7.11: Define the set $U_3 := \{x_3 \in \mathbb{X}_3 : x_2 \in U_2, \hat{d}_T \in \bar{\mathbb{B}}^3_r \text{ for a } r \in (\bar{d}, \bar{\bar{d}})\}$, with U_2 as described in Proposition 7.5. If (31) holds, then U_3 defines an invariant set. Moreover, $(V_{3,d})_{\leq V_0} \cap U_3$ defines a compact subset of the state space $\mathbb{X}_3$, for any sub-level set $(V_{3,d})_{\leq V_0}$.
The proof of this result follows from (27b).

Remark 7.12: Note the subtlety: a sub-level set $(V_{3,d})_{\leq V_0}$ guarantees that $\hat{d}_T$ belongs to a compact subset of $\mathbb{R}^3$, but it does not guarantee that $\hat{d}_T$ belongs to a compact subset of $\mathbb{B}^3_{\bar{\bar{d}}}$ (which is what is important, to guarantee that solutions do not exit the domain of the functions we designed thus far). That property is satisfied because the update-law, which makes use of a projection operator, satisfies (27b). □

All the conditions required by Theorem 5.5 are satisfied, which allows us to state the following result.

Theorem 7.13: Consider the vector field $X^{cl}_{3,d}$ in (53), and the Lyapunov function $V_{3,d}$ in (54), the sets $\mathbb{X}^\star_{3\pm}$ in (50), and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). Finally, consider the differential equation $\dot{x}_3(t) = X^{cl}_{3,d}(x_3(t), g^{(2)}(t))$ with $x_3(t_0) \in \tilde{\mathbb{X}}_3$. Then, all the consequences stated in Theorem 5.5 follow.

D. Step 4: Step 3, with unknown disturbance D

In the fourth step, we lift the assumption that the disturbance D is known in the thrust control law, and we replace it with an estimate $\hat{D}_T$. Lifting this assumption can only be done at this point, because the update-law for $\hat{D}_T$ depends on the estimator $\hat{d}_T$ (which was only introduced in the previous step): this further explains why removing/estimating the disturbances in one single step is not possible. Consider then the state, state space and vector field

$$x_4 \in \mathbb{X}_4 :\Leftrightarrow (x_3, \hat{D}_T) \in \mathbb{X}_3 \times \mathbb{R}^3, \quad (56)$$

$$\dot{x}_4 = X_{4,d,D}(x_4, (T, \omega, g^2)) :\Leftrightarrow \begin{bmatrix} \dot{x}_3 \\ \dot{\hat{D}}_T \end{bmatrix} = \begin{bmatrix} X_{3,d,D}(x_3, (T, \omega, g^2)) \\ Q_{\Delta,T}(x_3, \hat{D}_T) \end{bmatrix},$$

with $x_3, \mathbb{X}_3, X_3$ described in the previous step; where $\hat{D}_T$ stands for the estimator of the unknown disturbance D; and where the goal is to reuse the thrust and angular velocity control laws from the previous step, and design an update law $Q_{\Delta,T}$ for the estimator $\hat{D}_T$, such that $(e, \nu, n) \to \left(0_3, 0_3, \pm\frac{g^0 - d}{\|g^0 - d\|}\right)$ (recall (15a))[13].

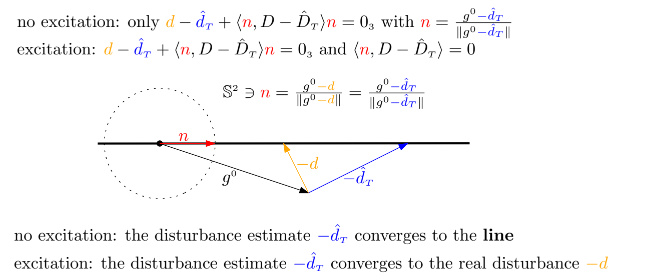


Fig. 5: Illustration of equilibria sets $\mathbb{X}^\star_{4\pm}$ in (57) with and without excitation.

As in the previous steps, we need to define the equilibria sets, but, at this point, the equilibria sets depend on an excitation criterion. Consider then the limit $\omega^\infty_\star \equiv \lim_{t\to+\infty} \omega_\star(t)$, which may, or may not, exist (and where $\omega_\star$ is the desired angular velocity defined in (15a)). If $\omega^\infty_\star = 0_3$, let

$$\mathbb{X}^\star_{4\pm} := \left\{ x_4 \in \mathbb{X}_4 : x_2 \in \mathbb{X}^\star_{2\pm}, d - \hat{d}_T = \langle n, \hat{D}_T - D\rangle n \right\}, \quad (57a)$$

otherwise, let

$$\mathbb{X}^\star_{4\pm} := \left\{ x_4 \in \mathbb{X}_4 : x_2 \in \mathbb{X}^\star_{2\pm}, \hat{d}_T = d, \langle n, D - \hat{D}_T\rangle = 0 \right\} \quad (57b)$$

Finally, denote also

$$\mathbb{X}^{\star\star}_{4\pm} := \left\{ x_4 \in \mathbb{X}_4 : x_2 \in \mathbb{X}^\star_{2\pm}, \hat{d}_T = d, \hat{D}_T = D \right\}. \quad (57c)$$

The sets above are equilibria sets, and they depend on the satisfaction of $\lim_{t\to+\infty} \omega_\star(t) = 0_3$. When the latter condition is satisfied (“no excitation”), the disturbance estimate $\hat{d}_T$ does not necessarily converge to the real unknown disturbance d. On the other hand, when the condition is not satisfied (“excitation”), the disturbance estimate $\hat{d}_T$ does converge to the real unknown disturbance d (the latter ideas are illustrated in Fig. 5). This is in contrast with the previous step, where the disturbance D was assumed known, and where the disturbance estimate $\hat{d}_T$ converges to the real unknown disturbance d regardless of any excitation criterion.

Remark 7.14: Recall the change of disturbances in (11): we may think of the estimators $(\hat{d}_T, \hat{D}_T)$ as being associated to wind estimators $(\hat{w}_T, \hat{W}_T)$. Then, for $n = \frac{mge_3 + w}{\|mge_3 + w\|}$, the condition $d - \hat{d}_T + \langle n, D - \hat{D}_T\rangle n = 0_3$ is satisfied for $(\hat{w}_T, \hat{W}_T) = (w + \alpha n, W - \alpha n)$, for any $\alpha \in \mathbb{R}$. Thus, in a hovering scenario, where there is no excitation, the wind estimators do not necessarily converge to the actual wind forces: that is the case, because the winds *cancel each other* along the cable direction. In a non-hovering scenario, instead of $n = \frac{mge_3 - w}{\|mge_3 - w\|}$, we have that $n(t) = \frac{m(ge_3 + p^{(2)}_\star(t)) - w}{\|m(ge_3 + p^{(2)}_\star(t)) - w\|}$, and provided that $n(t)$ *is always changing*, we can show that the estimators converge to the actual disturbances. □

As opposed to the previous step, the disturbance D is no longer known; moreover, recall that, in the latter step, neither the angular velocity ω^{cl}_2, nor the update-law $Q_{\delta,T}$, depend on D. Thus, it suffices that we amend the thrust control law T^{cl}_2 (see (51a)) by replacing the disturbance D with its estimate $\hat{D}_T$, i.e., we define a new thrust control law as

$$T^{cl}_3 : \mathbb{X}_4 \to \mathbb{R}, T^{cl}_3(x_4) := T^{cl}_2(x_3, \hat{D}_T). \quad (58)$$

Composing $\dot{\nu}$ in (32) with the control law T^{cl}_3 (58) yields

$$\dot{\nu} = ((T + \langle n, D\rangle) n - g^0 + d)|_{T = T^{cl}_3(x_4)} \quad (59)$$

[13] Requiring that $\hat{D}_T$ remains in some ball of pre-specified size is not as for $\hat{d}_T$, as the control laws are well-defined for any $\hat{D}_T$ in $\mathbb{R}^3$.

$$= \underbrace{u(e,\nu) - \Pi(n)\, T^{3d}(x_1, \hat{d}_T) + d - \hat{d}_T + \langle n, D - \hat{D}_T\rangle n}_{=:\nu^1(x_4)}.$$

(We note that ν^1 in (59) is not the same as that in (52), as they have different domains – i.e., the former "depends" on x_4, while the latter "depends" on x_3). For reasons that will be made clear later, we also need to define the second time derivative of the linear velocity (jerk), namely

$$\ddot{\nu} = \underbrace{d\nu^1(x_4) X_{4,d,D}(x_3, (T, \omega, g^2))|_{T=T_3^{cl}(x_4), \omega=\omega_2^{cl}(x_3)}}_{=:\nu^2(x_4) \text{ (independent of } g^2 \text{ because } \nu^1 \text{ does not depend on } g^1)} \quad (60)$$

Therefore composing the vector field $X_{4,d,D}$ in (56) with the thrust control law T_3^{cl} in (58) and the angular velocity control law ω_2^{cl} in (51b) yields

$$\dot{x}_4 = \underbrace{X_{4,d,D}(x_4, (T_3^{cl}(x_4), \omega_2^{cl}(x_3), g^2))}_{=:X_{4,d,D}^{cl}(x_4, g^2)} :\Leftrightarrow \quad (61)$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{\hat{D}}_T \end{bmatrix} = \underbrace{\begin{bmatrix} X_{3,d}^{cl}(x_3, g^2) \\ 0_3 \end{bmatrix}}_{\text{step 3}} + \underbrace{\begin{bmatrix} \langle n, D - \hat{D}_T\rangle (e_2^6 \otimes I_3) n \\ Q_{\Delta,T}(x_3, \hat{D}_T) \end{bmatrix}}_{\text{top is linear in } D - \hat{D}_T}.$$

Since the vector field in (61) fits with (28), we can then proceed with the estimator design procedure described in Section VI. As such, consider the update-law, the Lyapunov function, and its derivative given by

$$Q_{\Delta,T}(x_3, \hat{D}_T), V_{4,d,D}(x_4), W_{4,d,D}(x_4) \text{ as in (29)}, \quad (62)$$

with $\bar{p} = \bar{D}$, $\bar{\bar{p}} = \bar{\bar{D}}$, $k = k_{\Delta,T}$, $\bar{V} = \infty$ and $V_{\hat{p}} = V_{3,d}$. We must pick $\bar{V} = \infty$ as the Lyapunov function $V_{3,d}$ depends on an "unknown quantity". We emphasize that the update-law $Q_{\Delta,T}$ is indeed well-defined and computable, as it is equivalently expressed as

$$Q_{\Delta,T}(x_3, \hat{D}_T) = \text{Proj}_{\bar{D}, \bar{\bar{D}}}(\star, \hat{D}_T) \quad (63)$$

$$\star = k_{\Delta,T} \langle (e_2^6 \otimes I_3) n, \log'_{\bar{V}_2}(V_{2,\hat{d}_T}(x_2)) \nabla V_{2,\hat{d}_T}(x_2)\rangle n,$$

i.e., it does not depend on any "unknown quantity". As in the previous step, we wish to invoke Theorem 5.5, which motivates the introduction of the next two Propositions.

Proposition 7.15: Consider the functions $V_{4,d,D}, W_{4,d,D}$ in (62), the sets $\mathbb{X}_{4\pm}^{\star}, \mathbb{X}_{4\pm}^{\star\star}$ in (57), and, for brevity, denote $\cup\mathbb{X}_{4\pm}^{\star} \equiv \mathbb{X}_{4+}^{\star} \cup \mathbb{X}_{4-}^{\star}$. It holds that

$$\begin{cases} V_{4,d,D}(x_4) = 0 \Leftrightarrow x_4 \in \mathbb{X}_{4+}^{\star\star} \\ V_{4,d,D}(x_4) > 0 \text{ for all } x_4 \in \mathbb{X}_{4+}^{\star} \backslash \mathbb{X}_{4+}^{\star\star} \end{cases}, \quad (64)$$

$$\begin{cases} V_{4,d,D}(x_4) = V_{3-}^{\star} =: V_{4-}^{\star} \text{ for all } x_4 \in \mathbb{X}_{4-}^{\star\star} \\ V_{4,d,D}(x_4) > V_{3-}^{\star} =: V_{4-}^{\star} \text{ for all } x_4 \in \mathbb{X}_{4-}^{\star} \backslash \mathbb{X}_{4-}^{\star\star} \end{cases}, \quad (65)$$

$$\cup\mathbb{X}_{4\pm}^{\star} \supset \left\{ x_4 \in \mathbb{X}_4 : \begin{cases} \text{“}\dot{V}_{4,d}\text{”} = W_{4,d,D}(x_4) = 0 \\ \text{“}\dot{\nu}\text{”} = \nu^1(x_4) = 0_3 \\ \text{“}\ddot{\nu}\text{”} = \nu^2(x_4) = 0_3 \end{cases} \right\}, \quad (66)$$

with $V_{3-}^{\star}$, ν^1 and ν^2 defined in (55b), (59) and (60).
Verifying (64)–(65) is simple, and verifying (66) follows similar steps to those in the proof of Proposition 7.10.

Proposition 7.16: Define the set $U_4 := \{x_4 \in \mathbb{X}_4 : x_3 \in U_3\}$, with U_3 as described in Proposition 7.11. If (31) holds, then U_4 is an invariant set. Moreover, $(V_{4,d,D})_{\le V_0} \cap U_4$ defines a compact subset of $\mathbb{X}_4$ for any sub-level set $(V_{4,d,D})_{\le V_0}$.
All the conditions required by Theorem 5.5 are satisfied, which allows us to state the following result.

Theorem 7.17: Consider the vector field $X_{4,d,D}^{cl}$ in (61), the Lyapunov function $V_{4,d,D}$ in (62), the sets $\mathbb{X}_{4\pm}^{\star}, \mathbb{X}_{4\pm}^{\star\star}$ in (57), and let $t \mapsto g^{(2)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). Finally, consider the differential equation $\dot{x}_4(t) = X_{4,d,D}^{cl}(x_4(t), g^{(2)}(t))$ with $x_4(t_0) \in \tilde{\mathbb{X}}_4$. Then, all the consequences stated in Theorem 5.5 follow.

E. Step 5: Control with thrust and angular acceleration

In the fifth step, we lift the assumption that we control the angular velocity ω and control the angular acceleration τ instead; and we assume, once again, that d and D are known. Consider then the state, state space and vector field

$$x_5 \in \mathbb{X}_5 :\Leftrightarrow (x_4, \omega, g^2) \in \mathbb{X}_4 \times T_n\mathbb{S}^2 \times \mathbb{R}^3, \quad (67)$$

$$\dot{x}_5 = X_{5,d,D}(x_5, (T, \tau, g^3)) :\Leftrightarrow \begin{bmatrix} \dot{x}_4 \\ \dot{\omega} \\ \dot{g}^2 \end{bmatrix} = \begin{bmatrix} X_{4,d,D}(x_4, (T, \omega, g^2)) \\ \Pi(n)(\tau + \Phi(n) D) \\ g^3 \end{bmatrix},$$

with $x_4, \mathbb{X}_4, X_4$ described in the previous step[14]; where g^3 stands for the 3rd derivative of the time-varying gravity acceleration $t \mapsto g(t)$ in (14); and where the goal is to design a control law for the thrust T and angular acceleration τ, assuming that the disturbances d and D are known, and such that $(e, \nu, n, \omega) \to \left(0_3, 0_3, \pm\frac{g^0 - d}{\|g^0 - d\|}, \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{g^1}{\|g^0 - d\|}\right)$ (recall (15a)). For that purpose, consider then the two disjoint equilibria sets given by

$$\mathbb{X}_{5\pm}^{\star} := \left\{ x_5 \in \mathbb{X}_5 : x_4 \in \mathbb{X}_{4\pm}^{\star}, \omega = \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{g^1}{\|g^0 - d\|} \right\}, \quad (68a)$$

$$\mathbb{X}_{5\pm}^{\star\star} := \left\{ x_5 \in \mathbb{X}_5 : x_4 \in \mathbb{X}_{4\pm}^{\star\star}, \omega = \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right)\frac{g^1}{\|g^0 - d\|} \right\}. \quad (68b)$$

This step follows the same spirit as that of the second step. We construct the angular acceleration control law

$$\tau_1^{cl} : \mathbb{X}_5 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

$$\tau_1^{cl}(x_5, d, D) := -k_\omega(\omega - \omega^{cl}(x_3)) \quad (69a)$$

$$+ \Pi(n)\, d\omega^{cl}(x_3) X_{3,d,D}(x_3, (T_3^{cl}(x_4, \omega, g^2))) \quad (69b)$$

$$+ \gamma_\omega^{-1} \Pi(n)(-e_4^5 \otimes \mathcal{S}(n))^T \log'_{\bar{V}_2}(V_{2,\hat{d}_T}(x_2)) \nabla V_{2,\hat{d}_T}(x_2) \quad (69c)$$

$$- \Pi(n)\Phi(n) D, \quad (69d)$$

composed of a proportional term in (69a); of a feedforward term in (69b); of a backstepping term in (69c); and of a disturbance cancellation term in (69d). We emphasize that the control law τ_1^{cl} depends on both disturbances d and D: it depends on d because of the feed-forward term, and it depends on D because of the same feed-forward term and because of disturbance cancellation term.

We then have that the closed-loop dynamics, at the end of step 5, are given by

$$\dot{x}_5 = \underbrace{X_{5,d,D}(x_5, (T, \tau, g^3))|_{T=T_3^{cl}(x_4), \tau=\tau_1^{cl}(x_5,d,D)}}_{=:X_{5,d,D}^{cl}(x_5, g^3)}, \quad (70)$$

where we emphasize that they depend on both d and D. We can then define the Lyapunov function $V_{5,d,D} : \mathbb{X}_5 \to [0, \infty)$, and its derivative $W_{5,d,D} : \mathbb{X}_5 \to (-\infty, 0]$ along the vector field $X_{5,d,D}^{cl}$ in (70), as

$$V_{5,d,D}(x_5) := V_{4,d,D}(x_4) + \gamma_\omega \frac{1}{2}\|\omega - \omega_2^{cl}(x_3)\|^2, \quad (71a)$$

[14]Technically speaking, the set $\mathbb{X}_5$ in (67) cannot be expressed as a Cartesian product. The correct formulation is $\{(p, v, g^0, n, g^1, \hat{d}_T, \hat{D}_T, \omega, g^2) \in (\mathbb{R}^3)^{\times 9} : g^0 \in \mathbb{C}^3_{\underline{g}}, n \in \mathbb{S}^2, \omega \in T_n\mathbb{S}^2\}$.

$$W_{5,d,D}(x_5) := W_{4,d,D}(x_4) - k_\omega \gamma_\omega \|\omega - \omega_2^{cl}(x_3)\|^2 \leq 0. \quad (71b)$$

At this point we could state similar conclusions to those provided in Propositions 7.15 and 7.16, and in Theorem 7.17.

F. Step 6: Step 5, with unknown disturbances d and D

In the sixth and final step, we lift the assumption that the disturbances d, D are known in the angular acceleration control law, and replace them with corresponding estimators $\hat{d}_\tau, \hat{D}_\tau$. Consider then the state, state space and vector field

$$x_6 \in \mathbb{X}_6 :\Leftrightarrow (x_5, \hat{d}_\tau, \hat{D}_\tau) \in \mathbb{X}_5 \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (72)$$

$$\dot{x}_6 = X_{6,d,D}(x_6, (T, \tau, g^3)) :\Leftrightarrow \begin{bmatrix} \dot{x}_5 \\ \dot{\hat{d}}_\tau \\ \dot{\hat{D}}_\tau \end{bmatrix} = \begin{bmatrix} X_{5,d,D}(x_5, (T, \tau, g^3)) \\ Q_{\delta,\tau}(x_5, \hat{d}_T) \\ Q_{\Delta,\tau}(x_5, \hat{D}_T) \end{bmatrix},$$

with $x_5, \mathbb{X}_5, X_5$ described in the previous step; and where the goal is to reuse the thrust and angular acceleration control laws from the previous step, and design update laws for the estimators, such that (recall (15a)) $(e, \nu, n, \omega) \to \left(0_3, 0_3, \frac{\pm(g^0 - d)}{\|g^0 - d\|}, \mathcal{S}\left(\frac{g^0 - d}{\|g^0 - d\|}\right) \frac{g^1}{\|g^0 - d\|}\right)$. For that purpose, consider then the two disjoint equilibria sets

$$\mathbb{X}^\star_{6\pm} := \left\{x_6 \in \mathbb{X}_6 : x_5 \in \mathbb{X}^\star_{5\pm}\right\}, \quad (73a)$$

$$\mathbb{X}^{\star\star}_{6\pm} := \left\{x_6 \in \mathbb{X}_6 : x_5 \in \mathbb{X}^{\star\star}_{5\pm}, d = \hat{d}_\tau, D = \hat{D}_\tau\right\}. \quad (73b)$$

As opposed to the previous step, the disturbances d and D are not known, and their knowledge was required by the angular acceleration control law τ_1^{cl} in (69). As such, we replace those by their estimates, i.e., we define the new angular acceleration control law $\tau_2^{cl} : \mathbb{X}_6 \to \mathbb{R}^3$ given by

$$\tau_2^{cl}(x_6) := \tau_1^{cl}(x_5, \hat{d}_\tau, \hat{D}_\tau), \quad (74)$$

where we emphasize that τ_1^{cl} in (69) is affine with respect to (d, D). The control law τ_2^{cl} in (74) then leads to the closed-loop vector field

$$\dot{x}_6 = X_{6,d,D}(x_6, (T_3^{cl}(x_4), \tau_2^{cl}(x_6), g^3)) =: X^{cl}_{6,d,D}(x_6, g^3) \quad (75)$$

$$\begin{bmatrix} \dot{x}_5 \\ \dot{\hat{d}}_\tau \\ \dot{\hat{D}}_\tau \end{bmatrix} = \underbrace{\begin{bmatrix} X^{cl}_{5,d,D}(x_5, g^3) \\ 0_3 \\ 0_3 \end{bmatrix}}_{\text{previous step}} + \underbrace{\begin{bmatrix} (e_8^9 \otimes I_3) \sum E_\alpha(x_3)(a - \hat{a}_\tau) \\ Q_{\delta,\tau}(x_5, \hat{d}_\tau) \\ Q_{\Delta,\tau}(x_5, \hat{D}_\tau) \end{bmatrix}}_{\text{top is linear in } (d - \hat{d}_\tau) \text{ and } (D - \hat{D}_\tau)},$$

for some $E_\delta(x_3), E_\Delta(x_3) \in \mathbb{R}^{3\times 3}$ (which we omit here for brevity), and where $\sum \equiv \sum_{(a,\alpha)=(d,\delta) \text{ and } (a,\alpha)=(D,\Delta)}$. Since the vector field in (75) fits with (28), we can then proceed with the estimator design procedure described in Section VI. As such, consider the update-laws, the Lyapunov function, and its derivative given by

$$Q_{\alpha,\tau}(x_5, \hat{a}_\tau), V_{6,d,D}(x_6), W_{6,d,D}(x_6) \text{ as in (30)}, \quad (76)$$

with $\bar{p} = \bar{a}$, $\bar{\bar{p}} = \bar{\bar{a}}$, $k_\alpha = k_{\alpha,T}$, $\bar{V} = \infty$ and $V_{\hat{p}} = V_{4,d,D}$. We must pick $\bar{V} = \infty$ as the Lyapunov function $V_{4,d,D}$ depends on unknown quantities. We emphasize that the update-laws $Q_{\delta,\tau}, Q_{\Delta,\tau}$ are indeed well-defined and computable, as they are equivalently expressed as

$$Q_{\alpha,\tau}(x_5, \hat{a}_\tau) = \text{Proj}_{\bar{a},\bar{\bar{a}}}\left(k_{\alpha,\tau} E_\alpha(x_3)^T \gamma_\omega(\omega - \omega^{cl}(x_3)), \hat{a}_\tau\right).$$

At this point we could state similar conclusions to those provided in Propositions 7.15 and 7.16, and in Theorem 7.17. For brevity, we state only the theorem.

Theorem 7.18: Consider the vector field $X^{cl}_{6,d,D}$ in (75), the Lyapunov function $V_{6,d,D}$ in (76), the sets $\mathbb{X}^\star_{6\pm}, \mathbb{X}^{\star\star}_{6\pm}$ in (73), and let $t \mapsto g^{(3)}(t)$ be contained in a compact subset (of $\mathbb{R}^3$). Finally, consider the differential equation $\dot{x}_6(t) = X^{cl}_{6,d,D}(x_6(t), g^{(3)}(t))$ with $x_6(t_0) \in \tilde{\mathbb{X}}_6$. Then, all the consequences stated in Theorem 5.5 follow.

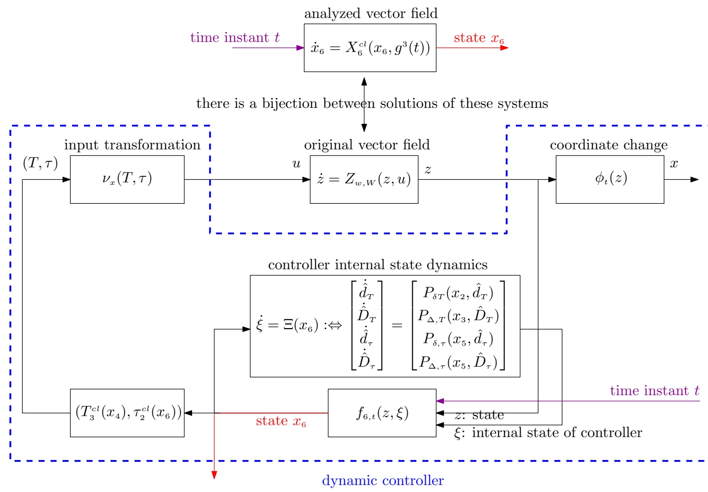


Fig. 6: Dynamic controller in "original coordinates", and, for each time instant, there is a bijection between $(z(t), \xi(t))$ and $x_6(t)$, where the solution for $t \mapsto x_6(t)$ has been analyzed in Theorem 7.18.

The complete control strategy is shown in Fig. 3. For an underactuated UAV, one may lift the full-actuation assumption: one would need to add four more additional steps (two steps to address the UAV attitude dynamics, and two more steps to design estimators after each of the previous two steps).

G. Complete strategy

We note the constructed controller dynamic, and as such it has internal states of its own. With this in mind, denote

$$\xi \in \Xi :\Leftrightarrow (\hat{d}_T, \hat{D}_T, \hat{d}_\tau, \hat{D}_\tau) \in \mathbb{B}^3_{\bar{\bar{d}}} \times \mathbb{B}^3_{\bar{\bar{D}}} \times \mathbb{B}^3_{\bar{\bar{d}}} \times \mathbb{B}^3_{\bar{\bar{D}}},$$

with ξ as the collection of all the internal states of the dynamic controller, and Ξ as the domain where ξ belongs to. Given any time instant $t \in \mathbb{R}$, consider then the map $f_{6,t} : \mathbb{Z} \times \Xi \to \mathbb{X}_6$ that constructs the state x_6 (see (72), ..., (32)), defined as

$$f_{6,t}(z, \xi) := \underbrace{(e, \nu, g^0, n, g^1, \hat{d}_T, \hat{D}_T, \omega, g^2, \hat{d}_\tau, \hat{D}_\tau)}_{\substack{(e,\nu,n,\omega) = \phi_t(z) \text{ and } (\hat{d}_T, \hat{D}_T, \hat{d}_\tau, \hat{D}_\tau) = \xi \\ \text{and } g^i = g^{(i)}(t) \text{ for } i \in \{0,1,2\}}}. \quad (77)$$

The map $f_{6,t}$, for a given time instant t, takes a physical state z and a controller internal state ξ, and constructs a state $x_6 = f_{6,t}(z, \xi)$ (we emphasize that this map is a bijection). With the thrust control law T_3^{cl} in (58) and the angular acceleration control law τ_2^{cl} in (74) in mind, we can then define the complete control law to be applied on the slung-load system, namely $u^{cl} : \mathbb{R} \times \mathbb{Z} \times \Xi \to \mathbb{R}^3$ given by

$$u^{cl}(t, z, \xi) := \nu_{\phi_t(z)}(T_3^{cl}(x_4), \tau_2^{cl}(x_6))|_{x_6 = f_{6,t}(z,\xi)}, \quad (78)$$

with the input transformation $\nu_\cdot$ defined in (10), the change of coordinates map ϕ_t defined in (9), and the map $f_{6,t}$ defined in (77). The map u^{cl} takes a time instant t, a physical state z and a controller internal state ξ, and constructs a three-dimensional force $u = u^{cl}(t, z, \xi)$ that the UAV must apply.

Next, we introduce a constant $V_6^\star$ that allows us to construct a sublevel set where the tension in the cable is guaranteed to be positive. Note then that the tension in (5), when composed with the proposed control law in (78), is given by $\bar{T}(z, u^{cl}(t, z, \xi))$

which is equivalently expressed as

$$m\Bigg(\underbrace{\|T^{3d}(x_1,\hat{d}_T)\|}_{\geq\epsilon}\underbrace{\langle n,n^{cl}(x_1,\hat{d}_T)\rangle}_{=1\text{ if }V_6=0}+\underbrace{\langle n,D-\hat{D}_T\rangle}_{=0\text{ if }V_6=0}\Bigg),(79)$$

and thus the tension is positive ($\geq\ m\epsilon$) when the desired trajectory is being tracked ($V_6=0$), and, by continuity, it must remain positive when some tracking error exists ($V_6<V_6^\star$). Briefly, the tension is a function of the state x_6, and thus we can find a lower bound on the tension on a sub-level set of the Lyapunov function V_6, i.e., $\underline{\text{Tension}}(v):=\min_{x_6\in(V_6)_{\leq v}}\text{Tension}(x_6)$ (and where we remind that $\underline{\text{Tension}}(0)\geq m\epsilon>0$). With the latter in mind, we can then define $V_6^\star:=\min_{\underline{\text{Tension}}(v)\leq 0}v$. The complete control strategy is shown in Fig. 6, and, next, we present our final result, which provides a solution to Problem 3.

Theorem 7.19: Given some desired position trajectory $p_\star:\mathbb{R}\to\mathbb{R}^3$ (such that the conditions in Problem 3 are satisfied), consider: *(1)* the slung-load vector field $Z_{w,W}$ as defined in (4) for some, *unknown by the controller*, wind forces $w,W\in\mathbb{R}^3$; *(2)* the control law u^{cl} as defined in (78); *(3)* and the estimator dynamics Ξ as defined in the diagram in Fig 6. Then,

$$\begin{bmatrix}\dot{z}(t)\\ \dot{\xi}(t)\end{bmatrix}=\begin{bmatrix}Z_{w,W}(z(t),u^{cl}(t,z(t),\xi(t)))\\ \Xi(f_{6,t}(z(t),\xi(t)))\end{bmatrix},\begin{bmatrix}z(t_0)\in\mathbb{Z}\\ \xi(t_0)\in\Xi\end{bmatrix}.\quad(80)$$

Consider also the map $f_{6,t}$ in (77), the Lyapunov function $V_{6,d,D}$ in (76) and the constant $V_6^\star$. Then, for all initial conditions $(t_0,z_0,\xi_0)\in\mathbb{R}\times\mathbb{Z}\times\Xi$ such that

$$x_{6,0}:=f_{6,t_0}(z_0,\xi_0)\in(V_{6,d,D})_{<V_6^\star},\quad(81)$$

it follows that (recall the equilibrium solution $z_{\star+}$ in (7a), and the equilibria set $X_{6+}^\star$ in (73)) *(1)* there exists a unique and complete solution $[t_0,\infty)\ni t\mapsto(z(t),\xi(t))\in\mathbb{Z}\times\Xi$; *(2)* $\lim_{\text{dist}(x_{6,0},X_{6+}^\star)\to 0}\sup_{t\geq t_0}\|z(t)-z_{\star+}(t)\|=0$ (stability of $z_{\star+}$); *(3)* $\lim_{t\to+\infty}\|z(t)-z_{\star+}(t)\|=0$ (attractivity of $z_{\star+}$); *(4)* $\inf_{t\geq t_0}T(z(t),u^{cl}(t,z(t),\xi(t)))>0$ (cable always taut).

Proof: Studying the solution to (80) is equivalent to studying a solution to (75), where $x_6(t):=f_{6,t}(z(t),\xi(t))$ (for any time instant t, there is a bijection between $(z(t),\xi(t))$ and $x_6(t)$). With the latter in mind, items *(1)*, *(2)* and *(3)* in the Theorem follow from Theorem 7.18. Item *(4)* follows from the fact that any compact subset of the sub-level set $(V_{6,d,D})_{<V_6^\star}$ is positively invariant, and the fact that the tension is strictly positive inside that compact set. ■

VIII. Simulations

Here, we provide simulations that validate our convergence results and also test the robustness of the proposed strategy.

In the simulation, the system has physical constants $M=1.1$ kg, $m=0.4$ kg, $l=1.1$ m, and $g=9.81$ m/s/s; and the wind forces are $W=0.1Mg\frac{s}{\|s\|}$ N with $s=(2,2,1)$, and $w=0.1mg\frac{s}{\|s\|}$ N with $s=(1,0,-1)$ (wind forces corresponding to 10% of bodies' weights). The load is required to track the trajectory $p_\star:\mathbb{R}\to\mathbb{R}^3$ defined as

$$p_\star(t):=R_1(25^\circ)R_2(25^\circ)\left(\frac{r\left(\sin(2\omega t),\cos(\omega t),0\right)}{\sin(\omega t)^2+1}+he_3\right)$$

where $R_i(\alpha)$ stands for a positive rotation around the ith axis by an angle α, $r=2$ m, $h=0.5$ m, and $\omega=\frac{2\pi}{12}$ Hz (period of 12 s), which corresponds to an eight-like path in a tilted plane – in particular, it follows that $\inf_{t\in\mathbb{R}}\|ge_3+p_\star^{(2)}(t)\|\approx 9.0$ m/s/s. For the simulations, we let the initial condition be $(p(0),P(0),v(0),V(0))=(7\,1_3,7\,1_3+le_3,0_3,0_3)$ and $(\hat{d}_T(0),\hat{D}_T(0),\hat{d}_\tau(0),\hat{D}_\tau(0))=(0_3,0_3,0_3,0_3)$. For the controller parameters we take $\bar{u}\approx 0.72$ m/s/s, $\bar{\hat{d}}=1.5>\bar{d}=1.2\geq\|d\|\approx 0.98$ m/s/s, $\bar{\hat{D}}=1.8>\bar{D}=1.5\geq\|D\|\approx 1.21$ m/s/s, $\bar{V}_1=1.5$ and $\bar{V}_2=1.5$. In particular, note that the condition $\inf_{t\in\mathbb{R}}\|g(t)\|-(\bar{u}+\bar{\hat{d}}_T)>0$ is satisfied.

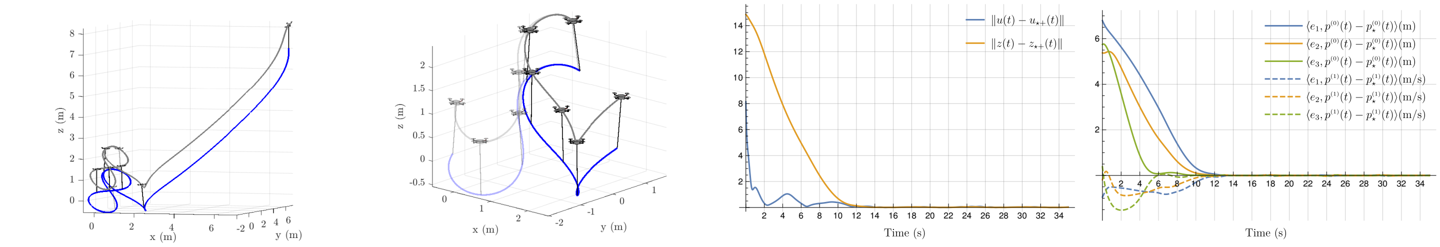
We present several simulations. *(1)* Default/baseline simulation, where all conditions assumed in the paper are respected. *(2)* We assume the model is incorrectly known by the controller (model mismatch), and we implement the controller with $m_{\text{controller}}=1.1m_{\text{model}}$ and $l_{\text{controller}}=1.1l_{\text{model}}$; because the model is not known we increase (by a factor of ≈ 2) the norms on the estimators, i.e., $\bar{\hat{d}}=2.7>\bar{d}=2.4$ m/s/s, $\bar{\hat{D}}=3.3>\bar{D}=3.0$ m/s/s. *(3)* We take $\omega\in\{\frac{2\pi}{12},\frac{2\pi}{8},\frac{2\pi}{6}\}$ Hz (period of 12, 8 or 6 s), and we investigate the effect of the excitation criterion on the convergence of the estimators. *(4)* We take $(p(0),P(0),v(0),V(0))=(20\,1_3,20\,1_3+le_3,5\,1_3,5\,1_3)$, we take $\bar{V}_1=\bar{V}_2\in\{2,20,200\}$, and we observe the effect on the disturbance estimators. *(5)* We let the winds be non-constant, i.e., $W=0.1Mg\frac{s}{\|s\|}$ N with $s=(2+0.1\cos(t),2+0.1\sin(t),1)$, and $w=0.1mg\frac{s}{\|s\|}$ N with $s=(1+0.1\cos(t),0+0.1\sin(t),-1)$, and we investigate the tracking error.

We next comment on these simulations, starting with the baseline *(1)*. Figs. 7a and 7b illustrate the trajectories of the dynamics in the environment, while Figs. 7c and 7d show that the state and input (z,u) converge to the equilibrium state and input $(z_{\star+},u_{\star+})$ defined in (7). Fig. 7e shows the inputs coming from the control law in (78), as well as the tension on the cable (indicating that it remains taut). Finally, Figs. 7f and 7g show the estimators $\hat{d}_T,\hat{D}_T,\hat{d}_\tau,\hat{D}_\tau$, which do not converge to the values of the real disturbances. We note that tracking is always accomplished even if the estimators do not converge to the real disturbances. The caveat is that trajectories where the estimators do not match the real disturbances are attractive but not necessarily stable, while the trajectories where the estimators match the real disturbances are.

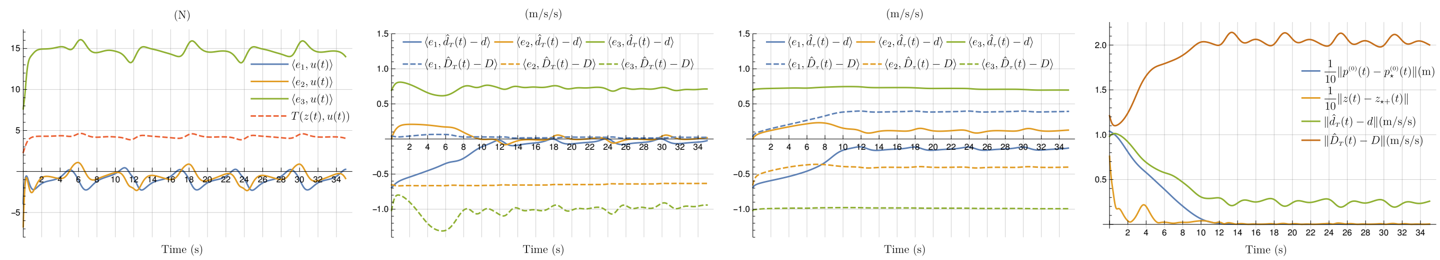
Regarding *(2)*: Fig. 7h shows that, despite the model mismatch, the state and input still converge to their desired trajectories, owing to the robustness added by the estimators. Regarding *(3)*: Fig. 7i shows that, the bigger the excitation is, the faster the estimators (we show only $\hat{d}_T$ and $\hat{D}_T$) converge to the real disturbances (the excitation criterion is understood as the average of $t\mapsto\omega_\star(t)$, and $\frac{1}{T}\int_0^T\|\omega_\star(t)\|dt\in\{0.06,0.19,0.45\}$ Hz for $T\in\{12,8,6\}$ s). Regarding *(4)*: Fig. 7j shows that, the bigger $\bar{V}_1,\bar{V}_2$ are, the quicker the estimators tend to saturate for "large" initial conditions – this illustrates the importance of the function $\log_\cdot$ in (29) and that $\bar{V}_1,\bar{V}_2$ determine when the estimators (integral-action) should start working. Regarding *(5)*: Fig. 7k shows that, tracking still takes places for non-constant winds, owing to the robustness added by the estimators, which rather than settling down, try to estimate the time-varying winds.

IX. Conclusions

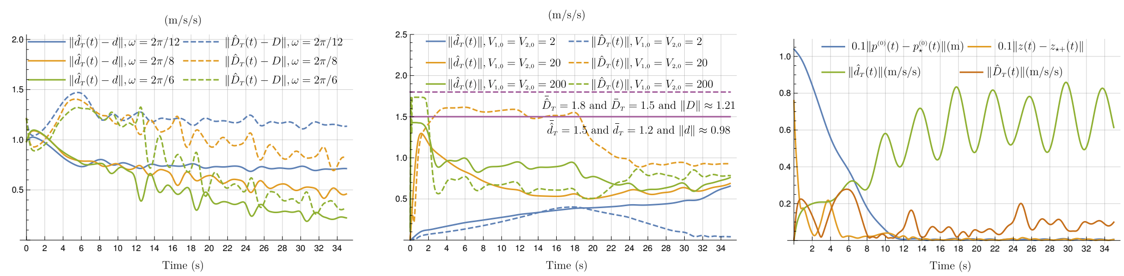
We have proposed a dynamic controller for position tracking of a point-mass load, attached to an aerial vehicle by means



(a) Complete trajectory: load in blue; UAV in gray. (b) 5 sec – 12 sec: real system in opaque; desired system in transparent. (c) Errors to equilibrium state and input trajectories (d) Position and velocity tracking errors



(e) Inputs to slung-load system, and cable tension. (f) Disturbance estimates $\hat{d}_T$ and $\hat{D}_T$ (g) Disturbance estimates $\hat{d}_\tau$ and $\hat{D}_\tau$ (h) Sim *(2)* (model mismatch): position and state tracking errors, and disturbance estimates errors.



(i) Sim *(3)* (excitation vs converge of estimators): disturbance estimates errors. (j) Sim *(4)* (purpose of $\log_{\bar{V}}$): Disturbance estimate $\hat{d}_T$ (k) Sim *(5)* (time-varying winds): position and state tracking errors, and disturbance estimates $\hat{d}_T$ and $\hat{D}_T$.

Fig. 7: Simulations (baseline simulation and simulations under conditions *(2)* to *(5)*).

of a cable, and where both the load and the aerial vehicle are subject to unknown wind forces. We have found a canonical system, which all slung-load systems can be converted to, and designed a stabilizing controller for this canonical form. By imposing conditions on the desired position trajectory and on the wind on the load, we have guaranteed that a well-defined equilibrium trajectory exists. This has allowed us to design a control law following a backstepping procedure, which contains four estimators (each of the two wind disturbances has two separate effects) and which guarantees that the equilibrium state trajectory is asymptotically tracked. Finally, we have established that the designed controller guarantees that the cable remains taut, for a certain set of initial conditions. Simulations show that the proposed controller is robust to model mismatches and that tracking is still accomplished for time-varying winds. Future work will study and test the effects of measurement noise, state estimation, unmodeled aerodynamic forces, and delays in actuation; we will also consider other methods for the design of the estimators and explore whether the use of fewer estimators would be sufficient.

References

[1] M. Bernard and K. Kondak, "Generic slung load transportation system using small size helicopters," in *IEEE International Conference on Robotics and Automation*, May 2009, pp. 3258–3264.

[2] I. Palunko, P. Cruz, and R. Fierro, "Agile load transportation: Safe and efficient load manipulation with aerial robots," *IEEE Robotics Automation Magazine*, vol. 19, no. 3, pp. 69–79, Sept 2012.

[3] I. Palunko, R. Fierro, and P. Cruz, "Trajectory generation for swing-free maneuvers of a quadrotor with suspended payload: A dynamic programming approach," in *IEEE International Conference on Robotics and Automation*, May 2012, pp. 2691–2697.

[4] M. M. Nicotra, E. Garone, R. Naldi, and L. Marconi, "Nested saturation control of an UAV carrying a suspended load," in *American Control Conference*, June 2014, pp. 3585–3590.

[5] M. E. Guerrero, D. A. Mercado, R. Lozano, and C. D. García, "Passivity based control for a quadrotor UAV transporting a cable-suspended payload with minimum swing," in *IEEE Conference on Decision and Control*, Dec 2015, pp. 6718–6723.

[6] T. Lee, K. Sreenath, and V. Kumar, "Geometric control of cooperating multiple quadrotor UAVs with a suspended payload," in *IEEE Conference on Decision and Control*, Dec 2013, pp. 5510–5515.

[7] F. A. Goodarzi, D. Lee, and T. Lee, "Geometric stabilization of a quadrotor UAV with a payload connected by flexible cable," in *American Control Conference*, June 2014, pp. 4925–4930.

[8] S. Dai, T. Lee, and D. S. Bernstein, "Adaptive control of a quadrotor UAV transporting a cable-suspended load with unknown mass," in *IEEE Conference on Decision and Control*, Dec 2014, pp. 6149–6154.

[9] P. O. Pereira, M. Herzog, and D. V. Dimarogonas, in *Med. Conference on Control and Automation*.

[10] M. Bisgaard, A. la Cour-Harbo, and J. D. Bendtsen, “Adaptive control system for autonomous helicopter slung load operations,” *Control Engineering Practice*, vol. 18, no. 7, pp. 800 – 811, 2010, special Issue on Aerial Robotics. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0967066110000341

[11] N. Michael, J. Fink, and V. Kumar, “Cooperative manipulation and transportation with aerial robots,” *Autonomous Robots*, vol. 30, no. 1, pp. 73–86, Jan 2011. [Online]. Available: https://doi.org/10.1007/s10514-010-9205-0

[12] G. Wu and K. Sreenath, “Geometric control of multiple quadrotors transporting a rigid-body load,” in *IEEE Conference on Decision and Control*, Dec 2014, pp. 6141–6148.

[13] P. O. Pereira and D. V. Dimarogonas, “Control framework for slung load transportation with two aerial vehicles,” in *IEEE Conference on Decision and Control*, Dec 2017, pp. 4254–4259.

[14] S. J. Lee and H. J. Kim, “Autonomous swing-angle estimation for stable slung-load flight of multi-rotor UAVs,” in *IEEE International Conference on Robotics and Automation*, May 2017, pp. 4576–4581.

[15] S. Tang, V. Wüest, and V. Kumar, “Aggressive flight with suspended payloads using vision-based control,” *IEEE Robotics and Automation Letters*, vol. 3, no. 2, pp. 1152–1159, April 2018.

[16] K. Sreenath, N. Michael, and V. Kumar, “Trajectory generation and control of a quadrotor with a cable-suspended load - A differentially-flat hybrid system,” in *IEEE International Conference on Robotics and Automation*, May 2013, pp. 4888–4895.

[17] K. Klausen, T. I. Fossen, and T. A. Johansen, “Nonlinear control with swing damping of a multirotor uav with suspended load,” *Journal of Intelligent & Robotic Systems*, vol. 88, no. 2, pp. 379–394, Dec 2017. [Online]. Available: https://doi.org/10.1007/s10846-017-0509-6

[18] M. E. Guerrero-Sánchez, D. A. Mercado-Ravell, R. Lozano, and C. D. García-Beltrán, “Swing-attenuation for a quadrotor transporting a cable-suspended payload,” *ISA Transactions*, vol. 68, pp. 433 – 449, 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0019057817301635

[19] M. Hua, T. Hamel, P. Morin, and C. Samson, “A control approach for thrust-propelled underactuated vehicles and its application to vtol drones,” *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1837–1853, Aug 2009.

[20] R. Naldi, M. Furci, R. G. Sanfelice, and L. Marconi, “Robust global trajectory tracking for underactuated vtol aerial vehicles using inner-outer loop control paradigms,” *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 97–112, Jan 2017.

[21] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, “Flatness and defect of non-linear systems: introductory theory and examples,” *International Journal of Control*, vol. 61, no. 6, pp. 1327–1361, 1995. [Online]. Available: https://doi.org/10.1080/00207179508921959

[22] P. O. Pereira and D. V. Dimarogonas, “Lyapunov-based generic controller design for thrust-propelled underactuated systems,” in *European Control Conference*, June 2016, pp. 594–599.

[23] A. Roberts and A. Tayebi, “Adaptive position tracking of VTOL UAVs,” *IEEE Transactions on Robotics*, vol. 27, no. 1, pp. 129–142, Feb 2011.

[24] D. Cabecinhas, R. Cunha, and C. Silvestre, “A nonlinear quadrotor trajectory tracking controller with disturbance rejection,” *Control Engineering Practice*, vol. 26, pp. 1 – 10, 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0967066114000343

[25] E. Frazzoli, M. A. Dahleh, and E. Feron, “Trajectory tracking control design for autonomous helicopters using a backstepping algorithm,” in *American Control Conference*, June 2000, pp. 4102–4107.

[26] R. Mahony and T. Hamel, “Robust trajectory tracking for a scale model autonomous helicopter,” *International Journal of Robust and Nonlinear Control*, vol. 14, no. 12, pp. 1035–1059, 2004.

[27] M. Tognon and A. Franchi, “Omnidirectional aerial vehicles with unidirectional thrusters: Theory, optimal design, and control,” *IEEE Robotics and Automation Letters*, vol. 3, no. 3, pp. 2277–2282, July 2018.

[28] S. Park, J. Her, J. Kim, and D. Lee, “Design, modeling and control of omni-directional aerial robot,” in *2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, Oct 2016, pp. 1570–1575.

[29] D. Brescianini and R. D’Andrea, “Design, modeling and control of an omni-directional aerial vehicle,” in *2016 IEEE International Conference on Robotics and Automation*, May 2016, pp. 3261–3266.

[30] P. O. Pereira, J. Cortés, and D. V. Dimarogonas, “Extended paper and companion mathematica files used in obtaining the manuscript’s results,” in *https://github.com/potaopereira/aerial-slung-load-position-tracking-under-unknown-wind-forces*, 2019.

[31] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Prentice Hall, 1991.

[32] I. Karafyllis and Z.-P. Jiang, *Stability and Stabilization of Nonlinear Systems*, 1st ed. Springer-Verlag London, 2011.

[33] Z. Cai, M. S. de Queiroz, and D. M. Dawson, “A sufficiently smooth projection operator,” *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 135–139, Jan 2006.

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