A Common Framework for Attitude Synchronization of Unit Vectors in Networks with Switching Topology

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***Abstract*— In this paper, we study attitude synchronization for elements in the unit sphere of $\mathbb{R}^3$ and for elements in the $3D$ rotation group, for a network with switching topology. The agents' angular velocities are assumed to be the control inputs, and a switching control law for each agent is devised that guarantees synchronization, provided that all elements are initially contained in a given region, unknown to the network. The control law is decentralized and it does not require a common orientation frame among all agents. We refer to synchronization of unit vectors in $\mathbb{R}^3$ as incomplete synchronization, and of $3D$ rotation matrices as complete synchronization. Our main contribution lies on showing that these two problems can be analyzed under a common framework, where all elements' dynamics are transformed into unit vectors dynamics on a sphere of appropriate dimension.**

I. INTRODUCTION

Decentralized control in a multi-agent environment has been a topic of active research, with applications in large scale robotic systems. Attitude synchronization in satellite formations is one of the relevant applications [1], where the control goal is to guarantee that a network of fully actuated rigid bodies acquire a common attitude. Coordination of underwater vehicles in ocean exploration missions [2], and of unmanned aerial vehicles in aerial exploration missions, may also be casted as attitude synchronization problems.

In the literature, attitude synchronization strategies for elements in the special orthogonal group are found in [3]–[10], which focus on *complete* attitude synchronization; and in [11]–[19], which focus on *incomplete* attitude synchronization. In this paper, we focus on both *complete* and *incomplete* attitude synchronization. We refer to *incomplete* attitude synchronization when the agents are unit vectors in $\mathbb{R}^3$, a space also called $\mathcal{S}^2$; and we refer to *complete* attitude synchronization, when the agents are $3D$ rotation matrices, a space also called $\mathcal{SO}(3)$. Incomplete synchronization represents an interesting practical problem, when the goal among multiple agents is to share a common direction. In flocking, for example, moving along a common direction is a requirement. Also, in a network of satellites, whose antennas are to point in a common direction, incomplete synchronization may be more important than complete.

In [3]–[6], [8]–[10], state dependent control laws for torques are presented which guarantee synchronization for elements in $\mathcal{SO}(3)$, while in [13], [14], [20] state dependent control laws for torques are presented which guarantee synchronization for elements in $\mathcal{S}^2$. In these works, all agents are dynamic and their angular velocities are part of the state of the system, rather than control inputs. In this paper, however, we consider kinematic agents and we design control laws, for the agents' angular velocities, which are not exclusively state dependent, but are also time dependent, with the time dependency encapsulating the switching network topology.

We note that, regarding synchronization in $\mathcal{SO}(3)$, similar results are found in [17], [21]–[24]. In [17], [23]–[25], consensus on non-linear spaces is analyzed with the help of a common weak non-smooth Lyapunov function, i.e., a Lyapunov function which is non-increasing along solutions. Also, in [21], control laws which guarantee synchronization under a switching topology are presented, under the hypothesis of a dwell time between consecutive switches. In our proposed framework, we relax the assumption of a dwell time by providing conditions for synchronization under arbitrary switching. Our approach is based on the construction of a common weak non-smooth Lyapunov function for analyzing synchronization in $\mathcal{S}^n$. In order to handle the non-smoothness of the proposed Lyapunov function, we present an invariance-like result which does not require any dwell time assumptions for the switching dynamics (see also [26]–[29] for invariance like theorems for switched systems). We propose control laws for angular velocities of unit vectors in $\mathbb{R}^3$ and $3D$ rotation matrices that guarantee synchronization for a network of agents with a switching topology. The control laws devised for unit vectors and rotation matrices achieve different goals, and differ in two aspects worth emphasizing. First, controlling rotation matrices requires more measurements when compared with controlling unit vectors; secondly, while controlling rotation matrices requires full actuation, i.e., all body components of the angular velocity need to be controllable, controlling unit vectors does not. Our main contribution compared to the aforementioned literature lies in analyzing both problems under a common framework, in order to allow for a unified stability analysis under arbitrary switching using the same common weak Lyapunov function. Particularly both problems are transformed into synchronization problems in $\mathcal{S}^m$ for an appropriate $m \in \mathbb{N}$. Since rotation matrices can be parametrized by unit quaternions, which are unit vectors in $\mathbb{R}^4$, i.e., $\mathcal{S}^3$, these are chosen for the analysis of the proposed control law. We also note that consensus in $\mathbb{R}^n$ can be casted as a synchronization problem in $\mathcal{S}^n$. Finally, we note that, under our framework, we do not require a dwell time between consecutive switches.

The remainder of this paper is structured as follows. In Section III, we present conditions on the vector fields that

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guarantee convergence to the consensus set. In Section IV, we describe the common framework for analysis of both synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$. In Sections IV-B and IV-A, the control laws for synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$ are presented, respectively, and the agents dynamics are transformed into the common framework. In Section VI, illustrative simulations are presented.

II. Notation

Let $n \in \mathbb{N}$. $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ stands for the identity matrix and $\mathbf{0}_n := (0, \cdots, 0) \in \mathbb{R}^n$ and $\mathbf{1}_n := (1, \cdots, 1) \in \mathbb{R}^n$. When the subscript n is omitted, it is assumed clear from the context. Let $M_{n,n} \subset \mathbb{R}^{n\times n}$ and $\bar{M}_{n,n} \subset \mathbb{R}^{n\times n}$ be the sets of symmetric matrices and antisymmetric matrices, respectively. The map $\mathcal{S} : \mathbb{R}^3 \mapsto \bar{M}_{3,3}$ yields an antisymmetric matrix satisfying $\mathcal{S}(\mathbf{a})\,\mathbf{b} = \mathbf{a} \times \mathbf{b}$, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$; and $\mathcal{S}^{-1} : \bar{M}_{3,3} \to \mathbb{R}^3$ is the inverse map, satisfying $\mathbf{X} = \mathcal{S}(\mathcal{S}^{-1}(\mathbf{X}))$ and $\mathbf{x} = \mathcal{S}^{-1}(\mathcal{S}(\mathbf{x}))$, for any $\mathbf{X} \in \bar{M}_{3,3}$ and $\mathbf{x} \in \mathbb{R}^3$. We denote by $\mathcal{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{x} = 1\}$ the set of unit vectors in $\mathbb{R}^n$ The map $\Pi : \mathcal{S}^{n-1} \mapsto \mathbb{R}^{n\times n}$, defined as $\Pi(\mathbf{x}) = \mathbf{I} - \mathbf{x}\mathbf{x}^T$, yields a matrix that represents the orthogonal projection operator onto the subspace perpendicular to $\mathbf{x} \in \mathcal{S}^{n-1}$. Given a set $\Omega \subseteq \mathbb{R}^n$, we denote by $\Omega^\epsilon = \{\mathbf{x} \in \mathbb{R}^n : \exists \mathbf{z} \in \Omega : \|\mathbf{x} - \mathbf{z}\| < \epsilon\}$ the set of all points ϵ-close to the set $\Omega \subseteq \mathbb{R}^n$, and $\mathrm{dist}(\mathbf{x}, \Omega) = \inf_{\mathbf{y}\in\Omega} \|\mathbf{x} - \mathbf{y}\|$. Given $r \geq 0$, we denote $\mathcal{B}(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$ and $\bar{\mathcal{B}}(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq r\}$ as the open and closed balls of radius r and centered around $\mathbf{0}$, respectively. Finally, $|\mathcal{H}|$ stands for the cardinality of the set $\mathcal{H}$.

III. Preliminaries

Let $n, N \in \mathbb{N}$ and consider $\mathbf{x} := (\mathbf{x}_1, \cdots, \mathbf{x}_N) \in (\mathbb{R}^n)^N$, with N as the number of agents, n as the dimension of the space the agents belong to and $\mathbf{x}$ as the state of the system. We consider a switched system with dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)) =: \mathbf{f}_{\sigma(t)}(\mathbf{x}(t)), \mathbf{x}(0) \in \mathbb{R}^{nN}, \tag{1}$$

where, $\mathbf{f}_{\sigma(t)}(\mathbf{x}) := (\mathbf{f}_{1,\sigma(t)}(\mathbf{x}), \cdots, \mathbf{f}_{N,\sigma(t)}(\mathbf{x})) \in (\mathbb{R}^n)^N$, with $\mathbf{f}_{i,\sigma(t)}(\cdot)$ as the vector field of of agent $i \in \{1, \cdots, N\} =: \mathcal{N}$, at time $t \geq 0$. We denote by $\mathcal{T}$ the increasing sequence of the switching time instants, and by $\sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \cdots, p\} =: \mathcal{P}$ the time dependent switching signal, piecewise constant and continuous from the right. Since $\mathcal{T}$ is a countable set, then the switching signal $\sigma(\cdot)$ is continuous almost everywhere. Additionally, $\{\mathbf{f}_p(\cdot)\}_{p\in\mathcal{P}}$ are assumed to be locally Lipschitz continuous, and thus, since $|\mathcal{P}| < \infty$, $\mathbf{f}(t, \mathbf{x}) = \mathbf{f}_{\sigma(t)}(\mathbf{x})$ is locally Lipschitz continuous on $\mathbf{x}$ for each switching signal $\sigma(t)$. Thus the conditions of Theorem 54 in [30] are satisfied, which implies that the initial value problem, $\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}_{\sigma(\tau)}(\mathbf{x}(\tau))d\tau$, has a unique solution $\mathbf{x} : [0, T_{\max}) \mapsto \mathbb{R}^{nN}$, which is an absolutely continuous function defined on the maximal right interval $[0, T_{\max})$ [30, p.471].

A. Assumptions on $\mathbf{f}_{i,p}$

We now present conditions on the vector fields $\{\mathbf{f}_p(\cdot)\}_{p\in\mathcal{P}}$, which, if satisfied, guarantee that $\mathbf{x}(\cdot)$ in (1) converges to the set $\mathcal{C} = \{(\mathbf{x}_1, \cdots, \mathbf{x}_N) \in (\mathbb{R}^n)^N : \mathbf{x}_1 = \cdots = \mathbf{x}_N\}$, corresponding to the consensus set. This means that $\lim_{t\to\infty}(\mathbf{x}(t) - \mathbf{1}_N \otimes \mathbf{x}^\infty(t)) = 0$, for some $\mathbf{x}^\infty : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$, i.e., the result presented in this section does not provide any information on whether $\mathbf{x}_i(\cdot)$, for all $i \in \mathcal{N}$, converge to a constant vector or a time-varying vector.

In later sections, when studying synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$, given the proposed control laws, we verify that the dynamics of all agents satisfy these conditions, allowing us to refer to the results in this section.

For brevity, we denote, for every $i \in \mathcal{N}$, $\mathbf{v}_i^{\max}(\mathbf{x}) := \max_{p\in\mathcal{P}} \mathbf{x}_i^T\mathbf{f}_{i,p}(\mathbf{x})$, which quantifies an upper bound on the time derivative of $\frac{1}{2}\|\mathbf{x}_i\|^2$ along a solution of (1), since $\frac{d}{dt}\frac{1}{2}\mathbf{x}_i^T(t)\mathbf{x}_i(t) = \mathbf{x}_i^T(t)\mathbf{f}_{i,\sigma(t)}(\mathbf{x}(t)) \leq \mathbf{v}_i^{\max}(\mathbf{x}(t))$, for all times where the derivative is well defined.

Next, let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_N)$, and denote $\mathcal{H}(\mathbf{x}) = \{i \in \mathcal{N} : i = \arg\max_{j\in\mathcal{N}}(\|\mathbf{x}_j\|)\}$ as the set of indexes $i \in \mathcal{N}$, for which $\|\mathbf{x}_i\|$ is larger than or equal to all other $\|\mathbf{x}_j\|$. Given $\mathbf{x} \notin \mathcal{C}$, assume that $\mathbf{v}_i^{\max}(\mathbf{x}) \leq 0$ for all $i \in \mathcal{H}(\mathbf{x})$ and that

$$\forall p \in \mathcal{P} \exists k \in \mathcal{H}(\mathbf{x}) : \mathbf{x}_k^T\mathbf{f}_{k,p}(\mathbf{x}) < 0. \tag{2}$$

Condition (2) states that, for every $p \in \mathcal{P}$, and within the set $\{\frac{1}{2}\|\mathbf{x}_i\|^2\}_{i\in\mathcal{H}(\mathbf{x})}$, we can always find an element whose time derivative along the solution of (1), if well defined, is negative. Then, denote $\mathcal{H}^\star(\mathbf{x}, p) = \{i \in \mathcal{H}(\mathbf{x}) : \mathbf{x}_i^T\mathbf{f}_{i,p}(\mathbf{x}) < 0\}$, which due to (2) is a non-empty subset of $\mathcal{H}(\mathbf{x})$, i.e., $\emptyset \neq \mathcal{H}^\star(\mathbf{x}, p) \subseteq \mathcal{H}(\mathbf{x})$ for all $\mathbf{x} \notin \mathcal{C}$ and all $p \in \mathcal{P}$. Finally, define $\mathbf{v}^{\max}(\mathbf{x}) := \max_{p\in\mathcal{P}} \max_{k\in\mathcal{H}^\star(\mathbf{x},p)} \mathbf{x}_i^T\mathbf{f}_{i,p}(\mathbf{x})$.

From the definition of $\mathcal{H}^\star(\mathbf{x}, p)$ it follows that $\mathbf{v}^{\max}(\mathbf{x}) < 0$ and that $\mathbf{v}^{\max}(\mathbf{x})$ quantifies an upper bound on the derivative of $\frac{1}{2}\|\mathbf{x}_i\|^2$ along a solution of (1), at the time instant $t \geq 0$, and for $i \in \mathcal{H}^\star(\mathbf{x}, \sigma(t))$. Based on this consideration, it can be shown that the time derivative of $V(\mathbf{x}) = \max_{i\in\mathcal{N}} \frac{1}{2}\mathbf{x}_i^T\mathbf{x}_i$ along a solution of (1) is bounded a.e. by $\mathbf{v}^{\max}(\mathbf{x}) < 0$, for all $\mathbf{x} \notin \mathcal{C}$ (proof of Theorem 2 in [31]). The latter property is central for the proof of the following theorem, which constitutes the main result of this section.

Theorem 1: Consider the system (1) and assume that for certain $r > 0$ and all $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_N) \in \mathcal{B}(r)^N$ the following hold:

1) when $\mathbf{x} \notin \mathcal{C}$, a) $\mathbf{v}_i^{\max}(\mathbf{x}) \leq 0 \forall i \in \mathcal{H}(\mathbf{x})$,
   b) $\forall p \in \mathcal{P} \exists k \in \mathcal{H}(\mathbf{x}) : \mathbf{x}_k^T\mathbf{f}_{k,p}(\mathbf{x}) < 0$,
2) when $\mathbf{x} \in \mathcal{C}$, $\mathbf{x}^T\mathbf{f}_p(\mathbf{x}) = \mathbf{0} \forall p \in \mathcal{P}$,

Then for each initial condition $\mathbf{x}(0) \in \mathcal{B}(r)^N$, the set $\bar{\mathcal{B}}(r_0)^N$, with $r_0 = \max_{i\in\mathcal{N}} \|\mathbf{x}_i(0)\| < r$, is positively invariant and $\mathbf{x}(\cdot)$ converges asymptotically to $\Omega = \bar{\mathcal{B}}(r_0)^N \cap \mathcal{C}$. Moreover, let $V : (\mathbb{R}^n)^N \mapsto \mathbb{R}_{\geq 0}$ be defined as $V(\mathbf{x}) = \max_{i\in\mathcal{N}} \frac{1}{2}\mathbf{x}_i^T\mathbf{x}_i$. Then, there exists a constant $V^\infty \in [0, V(\mathbf{x}(0))]$ such that $\lim_{t\to\infty} V(\mathbf{x}(t)) = V^\infty$.

A proof can be found in [31]. Briefly, it is shown that $\dot{V}(\mathbf{x}(t))$ exists almost everywhere and it is almost always upper bounded by $\mathbf{v}^{\max}(\mathbf{x}(t)) \leq 0$. It then follows that Ω_0 is positively invariant, and thus, from compactness of Ω_0, that the solution of (1) is defined for all positive times, namely $T_{\max} = \infty$ (Proposition C.3.6 in [30]). Asymptotic convergence to $\Omega = \Omega_0 \cap \mathcal{C}$ can then be proved by contradiction.

IV. Synchronization

In the next subsections, we define synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$. We then present the proposed control laws for the angular velocities in both cases, and the dynamics for the individual agents, either in $\mathcal{S}^2$ or $\mathcal{SO}(3)$, that follow from the chosen control laws. We then rewrite these in a form that allows for studying synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$ under a common framework. We note that consensus in $\mathbb{R}^n$ can also be casted as a synchronization problem in $\mathcal{S}^n$, for any $n \in \mathbb{N}$; however, due to paper length constraints, we omit this derivation, and refer to [31]. Next, we give some preliminary useful definitions.

Definition 1: Given $n \geq 1$, $\alpha \in [0, \pi]$ and $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$, the open α-cone $\mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})$ is defined as $\mathcal{C}(\alpha, \bar{\boldsymbol{\nu}}) = \{\boldsymbol{\nu} \in \mathcal{S}^n : \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} > \cos(\alpha)\}$, representing the set of unit vectors that are α close to the unit vector $\bar{\boldsymbol{\nu}}$. Similarly, we define the closed α-cone $\bar{\mathcal{C}}(\alpha, \bar{\boldsymbol{\nu}}) = \{\boldsymbol{\nu} \in \mathcal{S}^n : \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} \geq \cos(\alpha)\}$.
In Fig. 1, we illustrate an open α-cone, for $n = 3$, where three unit vectors $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ and $\boldsymbol{\nu}_3$ are contained in the open $30°$-cone formed by the unit vector $\bar{\boldsymbol{\nu}}$. Consider a group of unit vectors $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) \in (\mathcal{S}^n)^N$, for some $N, n \geq 1$. We say that $\boldsymbol{\nu}$ belongs to an open (closed) α-cone, for some $\alpha \in [0, \pi]$, if $\exists \bar{\boldsymbol{\nu}} \in \mathcal{S}^n : \boldsymbol{\nu} \in \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^N (\bar{\mathcal{C}}(\alpha, \bar{\boldsymbol{\nu}})^N)$. We say that $\boldsymbol{\nu}$ is synchronized if $\boldsymbol{\nu}_1 = \cdots = \boldsymbol{\nu}_N$.

In the next subsections, we always consider a group of N agents, indexed by the set $\mathcal{N} = \{1, \cdots, N\}$, operating in either $\mathcal{S}^2$, $\mathcal{SO}(3)$ or $\mathbb{R}^n$. The agents' network is modeled as a time varying digraph, $\mathcal{G}(\sigma(\cdot)) = \{\mathcal{N}, \mathcal{E}(\sigma(\cdot))\}$, with $\mathcal{N}$ as the time invariant vertices' set containing the team members; with $\sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \cdots, p\} =: \mathcal{P}$ as the switching signal; and with $\mathcal{G}(q)$ and $\mathcal{E}(q)$ as the graph and edges' sets corresponding to the switching signal $q \in \mathcal{P}$ (and where $|\mathcal{P}| \leq 2^{N(N-1)}$ since $2^{N(N-1)}$ provides an upper bound on the number of possible digraphs). We also denote $\mathcal{N}_i(q) \subset \mathcal{N}$ as the neighbor set of agent $i \in \mathcal{N}$ for the switching signal $q \in \mathcal{P}$.

In order to perform analysis under a common framework, we transform all problems's dynamics into a standard form, namely that presented below. Given $n \geq 1$, we denote $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^n)^N$ as the state of a group of unit vectors in $\mathcal{S}^n$, which evolves according to

$$\dot{\boldsymbol{\nu}}(t) = \tilde{\mathbf{f}}_{\sigma(t)}(\boldsymbol{\nu}(t)) = (\tilde{\mathbf{f}}_{1,\sigma(t)}(\boldsymbol{\nu}(t)), \cdots, \tilde{\mathbf{f}}_{N,\sigma(t)}(\boldsymbol{\nu}(t))), \quad (3)$$

with $\boldsymbol{\nu}(0) \in (\mathcal{S}^n)^N$ and $\tilde{\mathbf{f}}_{i,\sigma(t)} : (\mathcal{S}^n)^N \mapsto \mathbb{R}^{n+1}$ defined as

$$\tilde{\mathbf{f}}_{i,\sigma(t)}(\boldsymbol{\nu}) = \sum\nolimits_{j \in \mathcal{N}_i(\sigma(t))} \tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j, \quad (4)$$

for all $i \in \mathcal{N}$; i.e., $\dot{\boldsymbol{\nu}}_i(t) = \tilde{\mathbf{f}}_{i,\sigma(t)}(\boldsymbol{\nu}(t))$. Notice that $\boldsymbol{\nu}_i^T \tilde{\mathbf{f}}_{i,p}(\boldsymbol{\nu}) = 0$ for all $i \in \mathcal{N}$, $p \in \mathcal{P}$ and $\boldsymbol{\nu} \in (\mathcal{S}^n)^N$, which implies that the set $(\mathcal{S}^n)^N$ is positively invariant with respect to (3). The system (3)-(4) is the standard form all problems are transformed into: for synchronization in $\mathcal{S}^2$, $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^2)^N$; for synchronization in $\mathcal{SO}(3)$, $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^3)^N$; and for consensus in $\mathbb{R}^n$, $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^n)^N$.

The functions $\tilde{w}_{ij} : \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^2 \mapsto \mathbb{R}_{\geq 0}$ in (4) are continuous weight functions, for some $\alpha \in [\frac{\pi}{2}, \pi]$ and $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$. Thus, given $(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \in \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^2$, $\tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j)$ is the weight agent i assigns to the deviation between itself and its neighbor j, for all $i, j \in \mathcal{N}$ (and where we emphasize that the agents are within the same cone). All functions $\tilde{w}_{ij}(\cdot, \cdot)$ are assumed to satisfy the following condition,

$$\tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) > 0 \forall (\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \in \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^2 \text{ with } \boldsymbol{\nu}_i^T \boldsymbol{\nu}_j \neq 1. \quad (5)$$

Thus, from continuity, it follows that the weight between two neighbors is zero if and only if they are synchronized, though the weight may be arbitrarily small when the neighbors are arbitrarily close to each other or when the neighbors are close the boundaries of the domain of the weight functions.

The dependency of the dynamics (3)-(4) on time comes from the time varying network graph, and more specifically, the time varying neighbor set of each agent, as specified in (4).

Although the results of the paper remain valid under arbitrary switching, in practical cases the switching instants of each agent's control law cannot accumulate to a certain time value. In order to formulate this observation as an assumption, we adopt Definition 2.3. in [28], and say that the switching signal $\sigma(\cdot)$ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of $\sigma(\cdot)$ in any open finite interval $(t_1, t_2) \subset \mathbb{R}_{\geq 0}$ is upper bounded by $N_0 + \frac{t_2 - t_1}{\tau_D}$.

Assumption 2: For each agent $i \in \mathcal{N}$, we assume its neighbor set $\mathcal{N}_i(\sigma(\cdot))$ switches with a dwell-time τ_D and independently from all other agents in the network.
As explained in more detail in the next sections, each agent $i \in \mathcal{N}$ is in charge of providing the input that follows from composing the proposed control laws with the measurements made at each time instant. Thus, requiring an agent's neighbor set to switch with a dwell time guarantees that the agent's input does not experience infinite many discontinuities in any time interval of finite length. In fact, if Assumption 2 is satisfied, then $\sigma(\cdot)$ has an average dwell time, and therefore it follows that the set of switching times of the network dynamics $\mathcal{T}$ has zero measure in $\mathbb{R}_{\geq 0}$, a necessary condition in order to invoke the results from Section III-A. A proof for the next Proposition is found in [31].

Proposition 3: If each agent's $i \in \mathcal{N}$ neighbor set $\mathcal{N}_i(\sigma(\cdot))$ switches with a dwell-time τ_D^i, then the network dynamics (3) has a switching signal with average dwell time $\tau_D = \frac{1}{N} \min_{i \in \mathcal{N}} \tau_D^i$ and chatter bound $N_0 = N$.

A. Synchronization in $\mathcal{SO}(3)$

Here, we consider a group rotation matrices in $\mathcal{SO}(3) = \{\mathcal{R} \in \mathbb{R}^{3\times 3} : \mathcal{R}^T \mathcal{R} = \mathbf{I}, \mathcal{R}\mathcal{R}^T = \mathbf{I}, \det(\mathcal{R}) = 1\}$. If, at a time instant t, agent $i \in \mathcal{N}$ is aware of the relative attitude between itself and another agent j, then $j \in \mathcal{N}_i(\sigma(t))$. The goal of attitude synchronization is that all agents share the same *complete* orientation, i.e., that $\mathcal{R}_1 = \cdots = \mathcal{R}_N$. Notice that agent i is not aware of $\mathcal{R}_i$, since this is specified in an unknown inertial orientation frame; instead agent i is aware of the relative attitude between its orientation and its neighbor's j own orientation, i.e., $\mathcal{R}_i^T \mathcal{R}_j \in \mathcal{SO}(3)$. Measuring $\mathcal{R}_i^T \mathcal{R}_j$ corresponds to agent i being able to measure the projection of agent's j body axis onto its own body axis, and it can be measured by agent i without the need of a common inertial orientation between agents i and j.

A rotation matrix $\mathcal{R}_i : \mathbb{R}_{\geq 0} \mapsto \mathcal{SO}(3)$ evolves according to

$$\dot{\mathcal{R}}_i(t) = \mathcal{R}_i(t) \mathcal{S}(\boldsymbol{\omega}_i(t)), \quad (6)$$

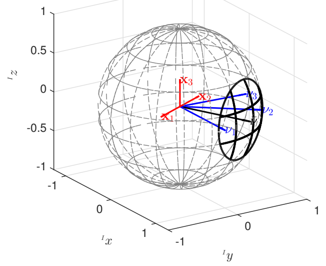
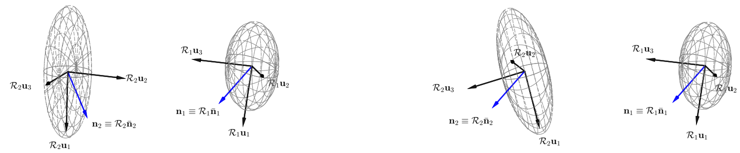


Fig. 1. Three unit vectors, $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ and $\boldsymbol{\nu}_3$, in $\mathbb{R}^3$ contained in open $30°$-cone formed by unit vector $\bar{\boldsymbol{\nu}}$. The quantities $\mathbf{x}_i = Q_{\bar{\nu}}\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i) = \Pi(\bar{\boldsymbol{\nu}})\,\boldsymbol{\nu}_i$ – see Def. 4 – are also presented.



(a) Two agents not synchronized, i.e., $\mathbf{n}_1 \neq \mathbf{n}_2$ with $\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2 = 3^{-\frac{1}{2}}[1\,1\,1]^T$.

(b) Two agents synchronized, i.e, $\mathbf{n}_1 = \mathbf{n}_2$ with $\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2 = 3^{-\frac{1}{2}}[1\,1\,1]^T$.

Fig. 2. In incomplete synchronization, all agents $i = \{1, \cdots, N\}$, align the unit vectors $\mathbf{n}_i := \mathcal{R}_i\bar{\mathbf{n}}_i$, where $\bar{\mathbf{n}}_i$ is fixed in rigid body i ($\mathbf{u}_1$,$\mathbf{u}_2$ and $\mathbf{u}_3$ stand for the canonical basis vectors in $\mathbb{R}^3$).

where $\boldsymbol{\omega}_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ is the body-framed angular velocity, and that can be actuated, for all $i \in \mathcal{N}$.

Problem 1: Given a group of rotation matrices with dynamics (6), design distributed control laws for their angular velocities that guarantee convergence to a synchronized network, in the absence of a common inertial orientation frame.

Definition 2: Denote $\theta : \mathcal{SO}(3) \times \mathcal{SO}(3) \mapsto [0, \pi]$ as the angular displacement between two rotation matrices $\mathcal{R}_i$ and $\mathcal{R}_j$, defined as $\theta(\mathcal{R}_i, \mathcal{R}_j) = \arccos\left(\frac{\mathrm{tr}(\mathcal{R}_i^T\mathcal{R}_j)-1}{2}\right)$.

For each agent $i \in \mathcal{N}$, we propose the control law $\boldsymbol{\omega}_i^{cl} : \mathbb{R}_{\geq 0} \times \mathcal{SO}(3)^N \mapsto \mathbb{R}^3$ defined as

$$\boldsymbol{\omega}_i^{cl}(t, \boldsymbol{\mathcal{R}}) = \sum_{j\in\mathcal{N}_i(\sigma(t))} g_{ij}(\theta(\mathbf{I}, \mathcal{R}_i^T\mathcal{R}_j))\mathcal{S}^{-1}\left(\frac{\mathcal{R}_i^T\mathcal{R}_j - \mathcal{R}_j^T\mathcal{R}_i}{2}\right) \quad (7)$$

where $\boldsymbol{\mathcal{R}} = (\mathcal{R}_i^T\mathcal{R}_1, \cdots, \mathcal{R}_i^T\mathcal{R}_N) \in \mathcal{SO}(3)^N$; and where $g_{ij} : [0, \pi] \mapsto \mathbb{R}_{\geq 0}$ is continuous and satisfies $g_{ij}(\theta) > 0$ for $\theta \in (0, \pi]$ (and thus $g_{ij}(0) \geq 0$), corresponding to a weight agent i assigns to the error between itself and its neighbor j, for $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$. Notice that the control law (7) depends only on the relative orientation measurements, and thus a common orientation frame among agents is not necessary.

Since we will study synchronization in $\mathcal{S}^2$ and $\mathcal{SO}(3)$ under a common framework, we now perform a change of coordinates that serves only the purpose of analysis, while the implemented control law is still that in (7). We consider now the unit quaternion $\mathbf{q}_i \in \mathcal{S}^3$ as a parametrization of $\mathcal{R}_i$, for each $i \in \mathcal{N}$, and denote $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N) \in (\mathcal{S}^3)^N$. Details on the next derivations are found in [31]. For this parameterization, the control law (7) can be rewritten as (for brevity, denote $s_{ij} := \arccos(2(\mathbf{q}_i^T\mathbf{q}_j)^2 - 1)$)

$$\boldsymbol{\omega}_i^{cl}(t, \mathbf{q}) = \sum_{j\in\mathcal{N}_i(\sigma(t))} 2\mathbf{q}_i^T\mathbf{q}_j g_{ij}(s_{ij})\begin{bmatrix}\mathbf{I}_3 & \mathbf{0}\end{bmatrix} Q(\mathbf{q}_i^\star)\mathbf{q}_j, \quad (8)$$

where $\mathbf{q}_i^\star$ denotes the conjugate quaternion of $\mathbf{q}_i$. Notice that $\boldsymbol{\omega}_i^{cl}(t, \mathbf{q}) = \boldsymbol{\omega}_i^{cl}(t, \tilde{\mathbf{I}}\mathbf{q})$ for all $\tilde{\mathbf{I}} \in \mathrm{diag}(\pm 1, \cdots, \pm 1) \otimes \mathbf{I}_4 \in \mathbb{R}^{4N\times 4N}$, which is a consequence of the fact that $\mathcal{S}^3$ is a double cover of $\mathcal{SO}(3)$; i.e., if $\mathbf{q}_i \in \mathcal{S}^3$ parametrizes $\mathcal{R}_i \in \mathcal{SO}(3)$ then so does $-\mathbf{q}_i \in \mathcal{S}^3$ [32]. If $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N) \in \mathcal{C}(\frac{\pi}{4}, \bar{\mathbf{q}})$, for some $\bar{\mathbf{q}} \in \mathcal{S}^3$, then $\mathbf{q}_i^T\mathbf{q}_j > 0$ for all $i, j \in \mathcal{N}$ [31]. Thus, if we denote $\tilde{g}_{ij}(\mathbf{q}_i, \mathbf{q}_j) := \mathbf{q}_i^T\mathbf{q}_j g_{ij}(\arccos(2(\mathbf{q}_i^T\mathbf{q}_j)^2 - 1))$, this satisfies (5) for $\alpha = \frac{\pi}{4}$ and for $\bar{\boldsymbol{\nu}} = \bar{\mathbf{q}} \in \mathcal{S}^3$. Then the dynamics of each agent, parametrized by unit quaternions, when composed with the proposed control law (8), can be described by $\dot{\mathbf{q}}_i(t) = \tilde{\mathbf{f}}_{i,\sigma(t)}(\mathbf{q}(t))$, where

$$\tilde{\mathbf{f}}_{i,\sigma(t)}(\mathbf{q}) = \tfrac{1}{2}Q(\mathbf{q}_i)\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\boldsymbol{\omega}_i^{cl}(t, \mathbf{q}) = \sum_{j\in\mathcal{N}_i(\sigma(t))} \tilde{g}_{ij}(\mathbf{q}_i, \mathbf{q}_j)\Pi(\mathbf{q}_i)\,\mathbf{q}_j.$$

We have then casted this problem in the form (3)-(4) with $\boldsymbol{\nu} = \mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N) \in (\mathcal{S}^3)^N$.

Remark 4: If $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N) \in \mathcal{C}(\frac{\pi}{4}, \bar{\mathbf{q}})$, for some $\bar{\mathbf{q}} \in \mathcal{S}^3$, then, by definition, $\bar{\mathbf{q}}^T\mathbf{q}_i > \cos\left(\frac{\pi}{4}\right)$ for all $i \in \mathcal{N}$. Thus, it follows that there exists a rotation matrix $\bar{\mathcal{R}}$, parametrized by $\bar{\mathbf{q}}$, such that $\theta(\bar{\mathcal{R}}, \mathcal{R}_i) = \arccos(2(\bar{\mathbf{q}}^T\mathbf{q}_i)^2 - 1) \leq \frac{\pi}{2}$ for all $i \in \mathcal{N}$, i.e., all rotation matrices are $\frac{\pi}{2}$ *close* to $\bar{\mathcal{R}}$.

B. Synchronization in $\mathcal{S}^2$

Here, we consider a group of unit vectors in $\mathbb{R}^3$, i.e., $\mathcal{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}^T\mathbf{x} = 1\}$. In what follows, let $i \in \mathcal{N}$. If, at a time instant t, agent i is aware of the relative attitude between itself and another agent j, then $j \in \mathcal{N}_i(\sigma(t))$. Each agent $i \in \mathcal{N}$ has its own orientation frame (w.r.t. an unknown inertial orientation frame), represented by $\mathcal{R}_i \in \mathcal{SO}(3)$. Let $\mathbf{n}_i \in \mathcal{S}^2$ be a direction along agent's i orientation, i.e., $\mathbf{n}_i := \mathcal{R}_i\bar{\mathbf{n}}_i$, where $\bar{\mathbf{n}}_i \in \mathcal{S}^2$ is a constant unit vector that is known by the agent and its neighbors. The goal of attitude synchronization in $\mathcal{S}^2$ is not that all agents share the same *complete* orientation, i.e., that $\mathcal{R}_1 = \cdots = \mathcal{R}_N$, but rather that all agents share the same orientation along a specific direction, i.e., that $\mathbf{n}_1 = \cdots = \mathbf{n}_N$. Figure 2 illustrates the concept of incomplete synchronization for two agents. For example, in Figs. 2(a) and 2(b), $\bar{\mathbf{n}}_1 = \bar{\mathbf{n}}_2 = \frac{1}{\sqrt{3}}(1, 1, 1)$, and in Fig. 2(b) the synchronized network satisfies $\mathcal{R}_1\bar{\mathbf{n}}_1 = \mathcal{R}_2\bar{\mathbf{n}}_2$ (while $\mathcal{R}_1 \neq \mathcal{R}_2$). Notice that agent i is not aware of $\mathbf{n}_i$ (since this is specified in an unknown inertial orientation frame); instead, agent i is aware of its direction $\bar{\mathbf{n}}_i$ and its neighbors directions $\{\bar{\mathbf{n}}_j\}_{j\in\mathcal{N}_i}$ – fixed in the respective body orientation frames; additionally, agent i is aware of the relative attitude between its orientation and its neighbors' own directions, i.e., agent i can measure $\mathcal{R}_i^T\mathbf{n}_j = \mathcal{R}_i^T\mathcal{R}_j\bar{\mathbf{n}}_j$. A rotation matrix $\mathcal{R}_i : \mathbb{R}_{\geq 0} \mapsto \mathcal{SO}(3)$ evolves according to (6), and thus the unit vector $\mathbf{n}_i(\cdot) = \mathcal{R}_i(\cdot)\bar{\mathbf{n}}_i$ evolves according to

$$\dot{\mathbf{n}}_i(t) = \mathcal{S}\left(\mathcal{R}_i(t)\boldsymbol{\omega}_i(t)\right)\mathbf{n}_i(t), \quad (9)$$

where $\boldsymbol{\omega}_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ is the body-framed angular velocity, which can be actuated, for all $i \in \mathcal{N}$ (recall that $\bar{\mathbf{n}}_i$ is constant).

Problem 2: Given a group of unit vectors with dynamics (9), design distributed control laws for their angular velocities that guarantee convergence to a synchronized network, in the absence of a common inertial orientation frame.

Definition 3: Denote $\theta : \mathcal{S}^2 \times \mathcal{S}^2 \mapsto [0, \pi]$ as the angular displacement between two unit vectors $\mathbf{n}_i$ and $\mathbf{n}_j$, defined as $\theta(\mathbf{n}_i, \mathbf{n}_j) = \arccos(\mathbf{n}_i^T \mathbf{n}_j)$.

For each agent $i \in \mathcal{N}$, we propose the control law $\boldsymbol{\omega}_i^{cl} : \mathbb{R}_{\geq 0} \times (\mathcal{S}^2)^N \mapsto \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y}^T \bar{\mathbf{n}}_i = 0\}$ defined as

$$\boldsymbol{\omega}_i^{cl}(t, \mathbf{n}) = \sum_{j \in \mathcal{N}_i(\sigma(t))} g_{ij}(\theta(\bar{\mathbf{n}}_i, \mathcal{R}_i^T \mathbf{n}_j)) \mathcal{S}(\bar{\mathbf{n}}_i) \mathcal{R}_i^T \mathbf{n}_j, \quad (10)$$

where $\mathbf{n} = (\mathcal{R}_i^T \mathbf{n}_1, \cdots, \mathcal{R}_i^T \mathbf{n}_N) \in (\mathcal{S}^2)^N$; and where $g_{ij} : [0, \pi] \mapsto \mathbb{R}_{\geq 0}$ is continuous and it satisfies $g_{ij}(\theta) > 0$ for $\theta \in (0, \pi]$ (and thus $g_{ij}(0) \geq 0$), corresponding to a weight agent i assigns to the error between itself and its neighbor j, for $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$. The control law (10) depends only on the relative orientation measurements: given $\mathcal{R}_i, \mathcal{R}_j \in \mathcal{SO}(3)$ and $\bar{\mathbf{n}}_j \in \mathcal{S}^2$, $\mathcal{R}_i^T \mathcal{R}_j \bar{\mathbf{n}}_j$ corresponds to the projection of $\bar{\mathbf{n}}_j$ onto the body orientation frame of agent i; moreover, (10) is orthogonal to $\bar{\mathbf{n}}_i$, which means full angular velocity control is not necessary, i.e. we only need to control the angular velocity along the two directions which are orthogonal to $\bar{\mathbf{n}}_i$. Notice that $\theta(\mathbf{n}_i, \mathbf{n}_j) = \theta(\bar{\mathbf{n}}_i, \mathcal{R}_i^T \mathbf{n}_j)$ for any $\mathbf{n}_i, \mathbf{n}_j \in \mathcal{S}^2$ and $\mathcal{R}_i \in \mathcal{SO}(3)$, with $\bar{\mathbf{n}}_i = \mathcal{R}_i^T \mathbf{n}_i$. Thus, if we denote $\tilde{g}_{ij}(\mathbf{n}_i, \mathbf{n}_j) := g_{ij}(\theta(\mathbf{n}_i, \mathbf{n}_j))$, this satisfies (5) for any $\alpha \in [0, \pi]$ and any $\bar{\boldsymbol{\nu}} \in \mathcal{S}^2$. Also, the dynamics (9), when composed with the proposed law (10), are described by $\dot{\mathbf{n}}_i(t) = \tilde{\mathbf{f}}_{i,\sigma(t)}(\mathbf{n}(t))$, where $(\mathbf{n} = (\mathbf{n}_1, \cdots, \mathbf{n}_N))$

$$\tilde{\mathbf{f}}_{i,\sigma(t)}(\mathbf{n}) = \sum_{j \in \mathcal{N}_i(\sigma(t))} \tilde{g}_{ij}(\mathbf{n}_i, \mathbf{n}_j) \Pi(\mathbf{n}_i) \mathbf{n}_j.$$

We have thus casted this problem in the form (3)-(4) with $\boldsymbol{\nu} = \mathbf{n} = (\mathbf{n}_1, \cdots, \mathbf{n}_N) \in (\mathcal{S}^2)^N$.

V. Analysis

In this section, we analyze the solutions of (3)-(4), and show that given a wide set of initial conditions, asymptotic synchronization is guaranteed. Specifically, asymptotic synchronization is guaranteed if all unit vectors are initially contained in an open $\alpha^\star$-cone, i.e. if $\exists \bar{\boldsymbol{\nu}} \in \mathcal{S}^n : \boldsymbol{\nu}(0) \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$, where $\alpha^\star = \frac{\pi}{2}$ for synchronization in $\mathcal{S}^2$ and consensus in $\mathbb{R}^n$, and $\alpha^\star = \frac{\pi}{4}$ for synchronization in $\mathcal{SO}(3)$.

Remark 5: If $\boldsymbol{\nu}(0) \in \mathcal{C}(\alpha, \bar{\boldsymbol{\nu}})^N$ for some $\alpha \in [0, \alpha^\star)$ and some $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$, then there exist $n+1$ linearly independent unit vectors $\{\bar{\boldsymbol{\nu}}_k \in \mathcal{S}^n\}_{k \in \{1, \cdots, n+1\}}$ such $\boldsymbol{\nu}(0) \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}}_k)^N \forall k \in \{1, \cdots, n+1\}$. Thus, if $\boldsymbol{\nu}(0)$ is contained in an open α cone, then there exist other bigger cones that contain $\boldsymbol{\nu}(0)$; as such, the choice of $\bar{\boldsymbol{\nu}}$ is not unique.

Next, we introduce a coordinate transformation that we exploit in order to cast the dynamics (4) into a form that satisfies the conditions of Theorem 1. In particular, given some $\bar{\boldsymbol{\nu}} \in \mathcal{S}^2$ we consider the projection of the cone $\mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})$ to the plane in $\mathbb{R}^{n+1}$ orthogonal to $\bar{\boldsymbol{\nu}}$ and containing zero, and then map this plane isometrically to $\mathbb{R}^n$.

Definition 4: Let $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$ and $Q_{\bar{\nu}} \in \mathbb{R}^{(n+1) \times n}$ such that $\bar{\boldsymbol{\nu}}$ and the columns of $Q_{\bar{\nu}}$ form an orthonormal basis of $\mathbb{R}^{n+1}$, and consider the diffeomorphism $\mathbf{h}_{\bar{\nu}} : \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}}) \mapsto \mathcal{B}(1)$ (see Notation for the definition of $\mathcal{B}(\cdot)$), defined as

$$\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i) = Q_{\bar{\nu}}^T \boldsymbol{\nu}_i. \quad (11)$$

Its inverse $\mathbf{h}_{\bar{\nu}}^{-1} : \mathcal{B}(1) \mapsto \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})$ is given by $\mathbf{h}_{\bar{\nu}}^{-1}(\mathbf{x}_i) = \sqrt{1 - \|\mathbf{x}_i\|^2} \bar{\boldsymbol{\nu}} + Q_{\bar{\nu}} \mathbf{x}_i$. Denote also $\mathbf{H}_{\bar{\nu}} : \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})^N \mapsto \mathcal{B}(1)^N$, defined as $\mathbf{H}_{\bar{\nu}}(\boldsymbol{\nu}) = (\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_1), \cdots, \mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_N))$, with $\mathbf{H}_{\bar{\nu}}^{-1}(\cdot)$ defined similarly. Figure 1 illustrates the mapping (11).

Proposition 6: Consider $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$ and $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j \in \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})$. Then, $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} = \sqrt{1 - \|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu})\|^2} > 0$ and the following implications hold: $\|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i)\| > \|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_j)\| \Leftrightarrow 0 < \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i < \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_j$, and $\|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i)\| \geq \|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_j)\| \Leftrightarrow 0 < \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i \leq \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_j$.

Proof: Since $\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}) = Q_{\bar{\nu}} \boldsymbol{\nu}$, then $\|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu})\|^2 = \boldsymbol{\nu}^T Q_{\bar{\nu}} Q_{\bar{\nu}}^T \boldsymbol{\nu} = \boldsymbol{\nu}^T \Pi(\bar{\boldsymbol{\nu}}) \boldsymbol{\nu} = 1 - (\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu})^2$ (notice that $Q_{\bar{\nu}} Q_{\bar{\nu}}^T = \Pi(\bar{\boldsymbol{\nu}})$). Since $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} > 0$, for any $\boldsymbol{\nu} \in \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})$, it follows that $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} = \sqrt{1 - \|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu})\|^2}$. The implications in the Proposition follow since $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu} = \sqrt{1 - \|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu})\|^2}$ is decreasing with $\|\mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu})\|$, and since $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j \in \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})$. ■

Consider now the solution $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^n)^N$ of (3)-(4) with $\boldsymbol{\nu}(0) \in \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})^N$ for some $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$, which, as will be shown in Theorem 9, remains in $\mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})^N$ for all $t \geq 0$; and define $\mathbf{x} : \mathbb{R}_{\geq 0} \mapsto (\mathbb{R}^n)^N$ as $\mathbf{x}(t) = \mathbf{H}_{\bar{\nu}}(\boldsymbol{\nu}(t))$. Then, based on the transformation introduced in Definition 4, it follows that $\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t))$, where

$$\mathbf{f}_{\sigma(t)}(\mathbf{x}) = \frac{\partial \mathbf{H}_{\bar{\nu}}(\boldsymbol{\nu})}{\partial \boldsymbol{\nu}} \tilde{\mathbf{f}}_{\sigma(t)}(\boldsymbol{\nu})|_{\boldsymbol{\nu} = \mathbf{H}_{\bar{\nu}}^{-1}(\mathbf{x})}, \quad (12)$$

and $\dot{\mathbf{x}}_i(t) = \mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i(t))$ evolves according to $\dot{\mathbf{x}}_i(t) = \mathbf{f}_{i,\sigma(t)}(\mathbf{x}(t))$, where

$$\mathbf{f}_{i,\sigma(t)}(\mathbf{x}) = \frac{\partial \mathbf{h}_{\bar{\nu}}(\boldsymbol{\nu}_i)}{\partial \boldsymbol{\nu}_i} \tilde{\mathbf{f}}_{i,\sigma(t)}(\boldsymbol{\nu})|_{\boldsymbol{\nu} = \mathbf{H}_{\bar{\nu}}^{-1}(\mathbf{x})} \quad (13)$$

It follows from (13) that, for $\boldsymbol{\nu} \in \mathcal{C}(\frac{\pi}{2}, \bar{\boldsymbol{\nu}})^N$ with $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$, $\mathbf{x} = \mathbf{H}_{\bar{\nu}}(\boldsymbol{\nu}) \in \mathcal{B}(1)^N$ and any $p \in \mathcal{P}$,

$$\begin{aligned}\mathbf{x}_i^T \mathbf{f}_{i,p}(\mathbf{x}) &\overset{(13)}{=} \boldsymbol{\nu}_i^T Q_{\bar{\nu}} Q_{\bar{\nu}}^T \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j \\ &= \boldsymbol{\nu}_i^T \Pi(\bar{\boldsymbol{\nu}}) \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j \\ &= -\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i \sum_{j \in \mathcal{N}_i(p)} \tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \bar{\boldsymbol{\nu}}^T \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j. \quad (14)\end{aligned}$$

The following result provides certain properties that are exploited in determining the sign of (14).

Proposition 7: Consider three unit vectors $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \bar{\boldsymbol{\nu}} \in \mathcal{S}^n$, satisfying $0 < \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_1 \leq \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_2$. Then *(a)* $\bar{\boldsymbol{\nu}}^T \Pi(\boldsymbol{\nu}_1) \boldsymbol{\nu}_2 = 0$ iff $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_2$, and *(b)* $\bar{\boldsymbol{\nu}}^T \Pi(\boldsymbol{\nu}_1) \boldsymbol{\nu}_2 > 0$ iff $\boldsymbol{\nu}_2 \neq \boldsymbol{\nu}_1$.

The proof of Proposition 7 is found in [31], and it is omitted here for brevity. We show next, by combining Propositions 6 and 7 and exploiting (14), that the conditions of Theorem 1 are satisfied for the dynamics (12)-(13).

Proposition 8: Consider the vector field as defined in (12)-(13) for certain $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$ and assume that the switching signal $\sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \cdots, p\}$ encodes only connected network graphs. Then the vector field (12)-(13) satisfies the conditions of Theorem 1 for $r = 1$.

Proof: In order to verify that the conditions of Theorem 1 are satisfied by the vector field in (12)-(13), we exploit (14) and the fact that for each $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_N) = \mathbf{H}(\boldsymbol{\nu}) \in \mathcal{B}(1)^N$ there exists a (unique) $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_N) \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$ such that $\mathbf{x} = \mathbf{H}(\boldsymbol{\nu})$. We proceed with the verification of Condition 1) of Theorem 1 and pick $\mathbf{x} \in \mathcal{B}(1)^N$, where $\mathbf{x} = \mathbf{H}(\boldsymbol{\nu})$ for certain $\boldsymbol{\nu} \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$. Notice, that since $\boldsymbol{\nu} \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$, it holds by definition that $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i > \cos(\alpha^\star) \geq 0 \forall i \in \mathcal{N}$. Therefore, since Condition 1)a) depends exclusively on the sign of (14), we can ignore the effect of the positive term $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i$.

In order to show Condition 1)a), pick any $p \in \mathcal{P}$ and notice that, due to (5) and continuity of $\tilde{w}_{ij}(\cdot, \cdot)$, it holds that $\tilde{w}_{ij}(\boldsymbol{\nu}_i, \boldsymbol{\nu}_j) \geq 0$ for any $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})$ and $i, j \in \mathcal{N}$. Recall that $\mathcal{H}(\mathbf{x}) = \{i \in \mathcal{N} : i = \arg\max_{j \in \mathcal{N}} \|\mathbf{x}_j\|\}$. Thus, for $i \in \mathcal{H}(\mathbf{x})$, $\|\mathbf{x}_i\| \geq \|\mathbf{x}_j\|$ for all $j \in \mathcal{N} \backslash \{i\}$, and it follows from Proposition 6 that $\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i \leq \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_j$ for all $j \in \mathcal{N} \backslash \{i\}$. From the latter and the result of Proposition 7, we get that $\bar{\boldsymbol{\nu}}^T \Pi(\boldsymbol{\nu}_i) \boldsymbol{\nu}_j \geq 0$ for all $j \in \mathcal{N}(p)$. Thus, we conclude from (14) that for any $p \in \mathcal{P}$ and $\mathbf{x} \in \mathcal{B}(1)^N$ it holds $\mathbf{x}_i^T \mathbf{f}_{i,p}(\mathbf{x}) \leq \mathbf{v}_i^{\max} \leq 0$, which implies that Condition 1)a) is satisfied.

For the verification of Condition 1)b), we additionally assume that $\mathbf{x} \notin \mathcal{C}$, where $\mathcal{C} = \{\mathbf{x} \in (\mathbb{R}^n)^N : \mathbf{x}_1 = \cdots = \mathbf{x}_N\}$. We will show that for each $p \in \mathcal{P}$ there exists $k \in \mathcal{H}(\mathbf{x})$ such that $\mathbf{x}_k^T \mathbf{f}_{k,p}(\mathbf{x}) < 0$. Indeed, suppose on the contrary that there exists $p \in \mathcal{P}$ such that

$$\mathbf{x}_i^T \mathbf{f}_{i,p}(\mathbf{x}) = 0 \forall i \in \mathcal{H}(\mathbf{x}), \tag{15}$$

and recall that the agents' network is not synchronized, since $\mathbf{x} \notin \mathcal{C}$. Consider then an $l \in \mathcal{H}(\mathbf{x})$, for which $\mathbf{x}_l^T \mathbf{f}_{l,p}(\mathbf{x}) = 0$ according to assumption (15). Notice also, that due to (5), Propositions 6, 7 and (14), it can be shown (as in the proof of Condition 1)a) above) that $\mathbf{x}_l^T \mathbf{f}_{l,p}(\mathbf{x}) = 0$ is satisfied only if all neighbors of agent l are synchronized with agent l, i.e., only if $\boldsymbol{\nu}_j = \boldsymbol{\nu}_l \Leftrightarrow \mathbf{x}_j = \mathbf{x}_l$ for all $j \in \mathcal{N}_l(p)$. This implies that all $j \in \mathcal{N}_l(p)$ are contained in $\mathcal{H}(\mathbf{x})$, i.e., $\mathcal{N}_l(p) \cup \{l\} \subseteq \mathcal{H}(\mathbf{x})$. As such, by assumption (15), $\mathbf{x}_j^T \mathbf{f}_{j,p}(\mathbf{x}) = 0$ for all $j \in \mathcal{N}_l(p)$, which means that the previous rationale is applicable for all $j \in \mathcal{N}_l(p)$, thus leading to the conclusion that all neighbors of all neighbors of agent l are necessarily synchronized with each other. Since the graph encoded by $p \in \mathcal{P}$ is connected, the previous rationale, applied $N-1$ times, leads to the conclusion that all agents are synchronized. Since $\mathbf{x} \notin \mathcal{C}$, a contradiction has been reached, and therefore, for each $p \in \mathcal{P}$, there exists a $k \in \mathcal{H}(\mathbf{x})$ for which $\mathbf{x}_k^T \mathbf{f}_{k,p}(\mathbf{x}) < 0$, and therefore condition 1)b) of Theorem 1 is satisfied.

Finally, since $\mathbf{x}_1 = \cdots = \mathbf{x}_N \Leftrightarrow \boldsymbol{\nu}_1 = \cdots = \boldsymbol{\nu}_N$, it follows from Proposition 7 that $\mathbf{x}^T \mathbf{f}_p(\mathbf{x}) \overset{(14)}{=} \mathbf{0}$ for all $p \in \mathcal{P}$. Thus, the second condition of Theorem 1 is also satisfied. ■

Theorem 9: Consider the solution $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \mapsto (\mathcal{S}^n)^N$ of (3) with $\boldsymbol{\nu}(0) \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$ for some $\bar{\boldsymbol{\nu}} \in \mathcal{S}^n$. Then, for a network graph connected at all times, *i)* $\boldsymbol{\nu}(t) \in \bar{\mathcal{C}}(\alpha, \bar{\boldsymbol{\nu}})^N$ for all $t \geq 0$, where $\alpha = \arccos(\max_{i \in \mathcal{N}} \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i(0)) \in [0, \alpha^\star)$; *ii)* $\boldsymbol{\nu}(\cdot)$ synchronizes asymptotically and $\lim_{t \to \infty} \bar{\boldsymbol{\nu}} \boldsymbol{\nu}_i(t)$ exists for all $i \in \mathcal{N}$; and *iii)* all unit vectors converge to a constant unit vector, i.e. $\exists \boldsymbol{\nu}^\star \in \mathcal{S}^n : \lim_{t \to \infty} \boldsymbol{\nu}(t) = \boldsymbol{\nu}^\star$.

Proof: Consider a solution $\boldsymbol{\nu}(\cdot)$ of (3), and $\mathbf{x}(\cdot) = \mathbf{H}_{\bar{\nu}}(\boldsymbol{\nu}(\cdot))$. Since $\alpha^\star$ is either $\frac{\pi}{2}$ or $\frac{\pi}{4}$, then $\boldsymbol{\nu}(0)$ is within the domain of $\mathbf{H}_{\bar{\nu}}(\cdot)$ and moreover $\bar{\mathcal{B}}(r_0)^N \subset \mathcal{B}(1)^N$ where $r_0 \overset{(11)}{=} \max_{i \in \mathcal{N}} \|Q_{\bar{\nu}}^T \boldsymbol{\nu}_i(0)\| \overset{\text{Prop. 6}}{=} \sqrt{1 - \max_{i \in \mathcal{N}} (\bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i(0))^2} < 1$ (where the latter inequality follows from the fact that $\boldsymbol{\nu}(0) \in \mathcal{C}(\alpha^\star, \bar{\boldsymbol{\nu}})^N$). From Proposition 8, the dynamics (12) satisfy Theorem's 1 conditions and therefore the set $\bar{\mathcal{B}}(r_0)^N$ is positively invariant for trajectories of $\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t))$. This in turn implies that the set $\mathbf{H}_{\bar{\nu}}^{-1}(\bar{\mathcal{B}}(r_0)^N) = \bar{\mathcal{C}}(\alpha, \bar{\boldsymbol{\nu}})^N$, where $\alpha = \arccos(\max_{i \in \mathcal{N}} \bar{\boldsymbol{\nu}}^T \boldsymbol{\nu}_i(0)) \in [0, \alpha^\star)$, is positively invariant for trajectories of $\dot{\boldsymbol{\nu}}(t) = \tilde{\mathbf{f}}_{\sigma(t)}(\boldsymbol{\nu}(t))$; i.e, all unit vectors are forever contained in the closed α-cone they start on. This suffices to conclude *i)* in the Theorem.

Let us now focus on part *ii)* of the Theorem. From Proposition 8, the dynamics (12) satisfy Theorem's 1 conditions. It follows from Theorem 1 that $\lim_{t \to \infty} (\mathbf{x}_i(t) - \mathbf{x}_j(t)) = 0$ for all $i, j \in \mathcal{N}$, which implies that $\lim_{t \to \infty} (\boldsymbol{\nu}_i(t) - \boldsymbol{\nu}_j(t)) = 0$, for all $i, j \in \mathcal{N}$ (see Proposition 6). Moreover, it follows that the Lyapunov function in Theorem 1 converges to a constant, i.e., $\lim_{t \to \infty} V(\mathbf{x}(t)) = \lim_{t \to \infty} \max_{i \in \mathcal{N}} \frac{1}{2} \|\mathbf{x}_i(t)\|^2 = \lim_{t \to \infty} \frac{1}{2} \|\mathbf{x}_1(t)\|^2 = V^\infty$, for some constant $0 \leq V^\infty \leq V(0) < \frac{1}{2}$. From Proposition 6, it follows that $\lim_{t \to \infty} \bar{\boldsymbol{\nu}} \boldsymbol{\nu}_i(t) = \lim_{t \to \infty} \sqrt{1 - \|\mathbf{x}_i(t)\|^2} = \sqrt{1 - 2V^\infty}$ for all $i \in \mathcal{N}$. The proof of part *iii)* of the Theorem is omitted here for brevity, and is found in [31]. ■

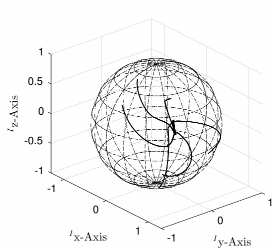
VI. Simulations

In this section, we present simulations that illustrate some of the results in the previous sections for a network of six agents, i.e., $\mathcal{N} = \{1, \cdots, 6\}$. The neighbor sets for all agents change in time: for agent 1, $\mathcal{N}_1(\cdot)$ alternates between $\{2\}$ and $\{2, 4\}$; for agent 2, $\mathcal{N}_2(\cdot)$ alternates between $\{3\}$ and $\{3, 6\}$; for agent 3, $\mathcal{N}_3(\cdot)$ alternates between $\{4\}$ and $\{4, 5\}$; for agent 4, $\mathcal{N}_4(\cdot)$ alternates between $\{5\}$ and $\{5, 1\}$; for agent 5, $\mathcal{N}_5(\cdot)$ alternates between $\{6\}$ and $\{6, 3\}$; for agent 6, $\mathcal{N}_6(\cdot)$ alternates between $\{1\}$ and $\{1, 2\}$. For these time-varying neighbor sets, the network graph is connected at all times. The switching instants, for each agent $i \in \mathcal{N}$, are those from the sequences $\mathcal{T}^i = \{\frac{1}{i} + ki\}_{k \in \mathbb{N}}$. Regarding the weight functions, for agents whose $i \in \mathcal{N}$ is even, $g_{ij}(\theta) = j$, and for agents whose $i \in \mathcal{N}$ is odd, $g_{ij}(\theta) = j(2 - \cos(\theta))$.

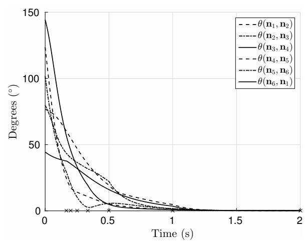
In Figs. 3(a)-3(b), six unit vectors are randomly initialized in an open $\frac{\pi}{2}$-cone around $[1\,0\,0]^T$. In Figs. 3(a) and 3(b), the trajectories of the unit vectors on the unit sphere is shown, and a visual inspection indicates convergence to a synchronized network. In Fig. 3(b), the angular distance, i.e., $\theta(\cdot, \cdot)$ as in Definition 3, between some agents is presented, and it indicates convergence to a synchronized network.

In Figs. 3(c)-3(d), six rotation matrices were randomly initialized such that $\theta(\mathbf{I}, \mathcal{R}_i) \leq \frac{\pi}{2}$ for all $i \in \mathcal{N}$. In Figs. 3(c), the trajectories of the rotation matrices are shown on a sphere of π radius[1], and a visual inspection indicates convergence to a synchronized network. In Figs. 3(d), the angular distance, i.e., $\theta(\cdot, \cdot)$ as in Definition 2, between

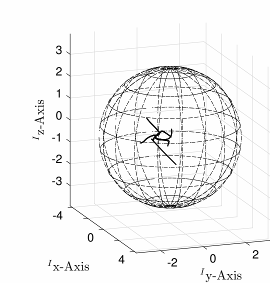
[1]Given $\mathcal{R} \in \mathcal{SO}(3)$, we plot $\frac{\theta(\mathbf{I}, \mathcal{R}) \mathcal{S}^{-1}(\mathcal{R} - \mathcal{R}^T)}{2 \sin(\theta(\mathbf{I}, \mathcal{R}))} \in \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| \leq \pi\}$.



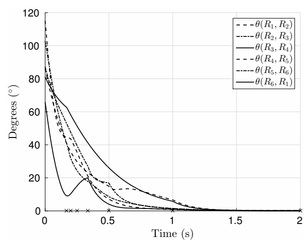
(a) Trajectories of unit vectors in unit sphere



(b) Angular distance between some pairs of unit vectors



(c) Trajectories of rotation matrices in sphere of π radius



(d) Angular distance between some pairs of rotation matrices

Fig. 3. Simulations (a cross on the time axis for each switching instant).

some agents is presented, and it indicates convergence to a synchronized network.

Figures showing the evolution of $V(\mathbf{x}) = \max_{i \in \mathcal{N}} \frac{1}{2}\|\mathbf{x}_i\|^2$ – used in Theorem 9 – are found in [31]. In particular, $V(\mathbf{x}(\cdot))$ is not smooth, but it is almost always decreasing.

VII. Conclusions

In this paper, we study attitude synchronization in $\mathcal{S}^2$ and in $\mathcal{SO}(3)$, for a network of agents under connected switching topologies. We propose switching control laws for each of the agent's angular velocity, which are decentralized and do not require a common orientation frame among agents. Our main contribution lies in transforming those two problems into a common framework, where all agents dynamics are transformed into unit vectors' dynamics on a sphere of appropriate dimension. Convergence to a synchronized network is guaranteed for a wide range of initial conditions. Directions for future work include extending all results to agents controlled at the torque level, rather than the angular velocity level.

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