Second Order Consensus for Leader-follower Multi-agent Systems with Prescribed Performance ⋆

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Abstract: The problem of distributed control for second order leader-follower multi-agent systems with prescribed performance guarantees is investigated in this paper. Leader-follower is meant in the sense that a group of agents with external inputs are selected as leaders in order to drive the group of followers in a way that the entire system can achieve consensus within certain prescribed performance transient bounds. Under the assumption of tree graphs, we propose a distributed control law based on a backstepping approach for the group of leaders to steer the entire system achieving consensus within the prescribed performance bounds. Finally, a simulation example is given to verify the results.

Keywords: Leader-follower control, consensus, multi-agent systems, prescribed performance control.

1. INTRODUCTION

The study of consensus is a key topic in multi-agent systems due to its wide applications in robotics, cooperative control (Fax and Murray, 2003) and formation control (Balch and Arkin, 1998). We say that consensus is achieved when a group of agents converge to a common value. The first order consensus protocol was introduced in (Olfati-Saber and Murray, 2004), while second order consensus was investigated in (Ren and Atkins, 2007).

In this paper, consensus in a leader-follower framework is considered, that is, one or more agents are selected as *leaders* with external inputs in addition to the second order consensus protocol. The remaining agents are *followers* only obeying the second order consensus protocol. Within the leader-follower framework, an important branch that researchers usually deal with is the controllability of the leader-follower network. Network controllability was first investigated in (Tanner, 2004) by deriving conditions on the network topology, which ensure that the network can be controlled by a particular member which acts as a leader. Some other research in the leader-follower framework targets leader selection problems (Yazicioğlu and Egerstedt, 2013; Patterson and Bamieh, 2010; Franchi and Giordano, 2018). These involve the problem of how to choose the leaders among the agents enabling the leader-follower system to satisfy requirements like controllability, optimal performance or formation maintenance.

Prescribed performance control (PPC) was originally proposed in (Bechlioulis and Rovithakis, 2008), with the aim to prescribe the evolution of the system output or the tracking error within some predefined region. An agreement protocol that can additionally achieve prescribed performance for a combined error of positions and velocities is designed in (Macellari et al., 2017) for multi-agent systems with double integrator dynamics. In (Verginis et al., 2018), the authors apply PPC to platoon of vehicles with unknown second order nonlinear dynamics using a backstepping approach.

In this work, we aim at designing control strategies for the leaders such that the leader-follower multi-agent system achieves consensus within certain performance bounds. First order consensus control for leader-follower multi-agent systems under prescribed performance guarantees is presented in (Chen and Dimarogonas, 2019). Compared with existing work on PPC for multi-agent systems, we apply a PPC law only to the leaders, while most of the related work applies PPC to all the agents to achieve consensus. The benefit of this work is to lower the cost of the control effort since the followers will follow the leaders by obeying second order consensus protocols without any additional control and knowledge of the prescribed team bounds. Unlike other work that uses PPC in a leader-follower framework (Katsoukis and Rovithakis, 2016; Verginis et al., 2018), in which the multi-agent system only has one leader and the leader is treated as a reference for the followers, we focus on a more general framework in the sense that we can have more than one leader and the leaders are designed to steer the entire system achieving consensus within the prescribed performance bounds. This is difficult due to the combination of uncertain topologies, leader amount and positions. In addition, we focus on distributed control in the sense that each leader can only communicate with its neighbours. Within this general leader-follower framework, the main contribution is that, under the assumption of tree graphs, we propose a distributed control law based on a

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backstepping approach for the group of leaders to steer the entire system to consensus within certain prescribed performance transient bounds for the whole team.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and then formulate the problem, while Section 3 presents the main results, which are further illustrated by a simulation example in Section 4. Section 5 includes concluding remarks and future work.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Graph Theory

An undirected graph (Mesbahi and Egerstedt, 2010) is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the vertices set $\mathcal{V} = \{1, 2, \ldots, n\}$ and the edges set $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i\}$ indexed by $e_1, e_2, \ldots, e_m$. Here, $m = |\mathcal{E}|$ is the number of edges and $\mathcal{N}_i$ denotes the neighbourhood of agent i such that agent $j \in \mathcal{N}_i$ can communicate with i. The *adjacency matrix* $\mathbb{A}$ of $\mathcal{G}$ is the $n \times n$ symmetric matrix whose elements a_{ij} are given by $a_{ij} = 1$, if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The degree of vertex i is defined as $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$. Then the *degree matrix* is $\Delta = \text{diag}(d_1, d_2, \ldots, d_n)$. The *graph Laplacian* of $\mathcal{G}$ is $L = \Delta - \mathbb{A}$. A *path* is a sequence of edges connecting two distinct vertices. A graph is *connected* if there exists a path between any pair of vertices. By assigning an orientation to each edge of $\mathcal{G}$ we can define the *incidence matrix* $D = D(\mathcal{G}) = [d_{ij}] \in \mathbb{R}^{n \times m}$. The rows of D are indexed by the vertices and the columns are indexed by the edges with $d_{ij} = 1$ if the vertex i is the head of the edge (i, j), $d_{ij} = -1$ if the vertex i is the tail of the edge (i, j) and $d_{ij} = 0$ otherwise. Based on the incidence matrix, the graph Laplacian of $\mathcal{G}$ can be described as $L = DD^T$. In addition, $L_e = D^T D$ is the so-called *edge Laplacian* (Zelazo and Mesbahi, 2011).

2.2 System Description

We consider a multi-agent system with vertices $\mathcal{V} = \{1, 2, \ldots, n\}$. Without loss of generality, we suppose that the first n_f agents are followers while the last n_l agents are selected as leaders with respective vertices set $\mathcal{V}_F = \{1, \ldots, n_f\}$, $\mathcal{V}_L = \{n_f + 1, \ldots, n_f + n_l\}$ and $n = n_f + n_l$.

Let $x_i \in \mathbb{R}$ be the position of agent i, where we only consider the one dimensional case, without loss of generality. Specifically, the results can be extended to higher dimensions with appropriate use of the Kronecker product. The state evolution of each follower $i \in \mathcal{V}_F$ is governed by the second order consensus protocol:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \sum_{j \in \mathcal{N}_i} \left((x_j - x_i) + (v_j - v_i) \right), \end{aligned} \tag{1}$$

while the state evolution of each leader $i \in \mathcal{V}_L$ is governed by the second order consensus protocol with an external input $u_i \in \mathbb{R}$:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \sum_{j \in \mathcal{N}_i} \left((x_j - x_i) + (v_j - v_i) \right) + u_i, \end{aligned} \tag{2}$$

Let $x = [x_1, \ldots, x_{n_f}, \ldots, x_n]^T, v = [v_1, \ldots, v_{n_f}, \ldots, v_n]^T \in \mathbb{R}^n$ be the respective stack vector of absolute positions and velocities and $u = [u_{n_f+1}, \ldots, u_{n_f+n_l}]^T \in \mathbb{R}^{n_l}$ be the control input vector including the external inputs of leader agents in (2). Denote $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_m]^T$, $\bar{v} = [\bar{v}_1, \ldots, \bar{v}_m]^T \in \mathbb{R}^m$ as the respective stack vector of relative positions and relative velocities between the pair of communication agents $(i, j) = k \in \mathcal{E}$, where $\bar{x}_k \triangleq x_{ij} = x_i - x_j, \bar{v}_k \triangleq v_{ij} = v_i - v_j, k = 1, 2, \ldots, m$. It can be easily verified that $Lx = D\bar{x}$ and $\bar{x} = D^T x$. In addition, if $\bar{x} = 0$, we have that $Lx = 0$. Similarly, it holds that $Lv = D\bar{v}$ and $\bar{v} = D^T v$. By stacking (1) and (2), the dynamics of the leader-follower multi-agent system is rewritten as:

$$\Sigma : \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -L & -L \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0_{n \times n_l} \\ B \end{bmatrix} u, \tag{3}$$

where L is the graph Laplacian and $B = \begin{bmatrix} 0_{n_f \times n_l} \\ I_{n_l} \end{bmatrix}$.

2.3 Prescribed Performance Control

The aim of PPC is to prescribe the evolution of the system output or the tracking error within some predefined region described as follows:

$$-M_{\bar{x}_i} \rho_{\bar{x}_i}(t) < \bar{x}_i(t) < \rho_{\bar{x}_i}(t) \qquad \text{if } \bar{x}_i(0) > 0, \tag{4}$$

$$-\rho_{\bar{x}_i}(t) < \bar{x}_i(t) < M_{\bar{x}_i} \rho_{\bar{x}_i}(t) \qquad \text{if } \bar{x}_i(0) < 0, \tag{5}$$

where $\rho_{\bar{x}_i}(t) : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}, i = 1, 2, \ldots, m$ are positive, smooth and strictly decreasing performance functions that introduce the predefined bounds for the target system outputs or the tracking errors. One example choice is $\rho_{\bar{x}_i}(t) = (\rho_{\bar{x}_{i0}} - \rho_{\bar{x}_{i\infty}}) e^{-l_{\bar{x}_i} t} + \rho_{\bar{x}_{i\infty}}$ with $\rho_{\bar{x}_{i0}}, \rho_{\bar{x}_{i\infty}}$ and $l_{\bar{x}_i}$ positive parameters and $\rho_{\bar{x}_{i\infty}} = \lim_{t \to \infty} \rho_{\bar{x}_i}(t)$ represents the maximum allowable tracking error at steady state; $M_{\bar{x}_i}$ represents the maximum allowed overshoot.

Normalizing $\bar{x}_i(t)$ with respect to the performance function $\rho_{\bar{x}_i}(t)$, we define the modulated error as $\hat{\bar{x}}_i(t)$ and the corresponding prescribed performance region $\mathcal{D}_{\bar{x}_i}$ as:

$$\hat{\bar{x}}_i(t) = \frac{\bar{x}_i(t)}{\rho_{\bar{x}_i}(t)} \tag{6}$$

$$\mathcal{D}_{\bar{x}_i} \triangleq \{\hat{\bar{x}}_i : \hat{\bar{x}}_i \in (-M_{\bar{x}_i}, 1)\} \qquad \text{if } \bar{x}_i(0) > 0 \tag{7}$$

$$\mathcal{D}_{\bar{x}_i} \triangleq \{\hat{\bar{x}}_i : \hat{\bar{x}}_i \in (-1, M_{\bar{x}_i})\} \qquad \text{if } \bar{x}_i(0) < 0 \tag{8}$$

Then the modulated error is transformed through a transformed function $T_{\bar{x}_i}$ that defines the smooth and strictly increasing mapping $T_{\bar{x}_i} : \mathcal{D}_{\bar{x}_i} \to \mathbb{R}$ and $T_{\bar{x}_i}(0) = 0$. One example choice is

$$T_{\bar{x}_i}(\hat{\bar{x}}_i) = \ln \left(-M_{\bar{x}_i} \frac{\hat{\bar{x}}_i + 1}{\hat{\bar{x}}_i - M_{\bar{x}_i}} \right). \tag{9}$$

The transformed error is then defined as

$$\varepsilon_{\bar{x}_i}(\hat{\bar{x}}_i) = T_{\bar{x}_i}(\hat{\bar{x}}_i) \tag{10}$$

It can be verified that if the transformed error $\varepsilon_{\bar{x}_i}(\hat{\bar{x}}_i)$ is bounded, then the modulated error $\hat{\bar{x}}_i$ is constrained within the regions (7), (8). This also implies the error $\bar{x}_i$ evolves within the predefined performance bounds (4), (5). Differentiating (10) with respect to time, we derive

$$\dot{\varepsilon}_{\bar{x}_i}(\hat{\bar{x}}_i) = \mathcal{J}_{T_{\bar{x}_i}}(\hat{\bar{x}}_i, t)[\dot{\bar{x}}_i + \alpha_{\bar{x}_i}(t)\bar{x}_i] \tag{11}$$

where

$$\mathcal{J}_{T_{\bar{x}_i}}(\hat{\bar{x}}_i, t) \triangleq \frac{\partial T_{\bar{x}_i}(\hat{\bar{x}}_i)}{\partial \hat{\bar{x}}_i} \frac{1}{\rho_{\bar{x}_i}(t)} > 0 \tag{12}$$

$$\alpha_{\bar{x}_i}(t) \triangleq -\frac{\dot{\rho}_{\bar{x}_i}(t)}{\rho_{\bar{x}_i}(t)} > 0 \tag{13}$$

are the normalized Jacobian of the transformation function $T_{\bar{x}_i}$ and the normalized derivative of the performance function, respectively.

2.4 Problem Statement

In this work, we are interested in how to design a control strategy for the leader-follower multi-agent system (3) such that it can achieve consensus within the prescribed performance bounds. The control strategy is only applied to the leaders and these drive the followers to guarantee that multi-agent system (3) meets the requirements. Formally,

Problem 1. Let the leader-follower multi-agent system Σ be defined by (3) with the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the prescribed performance functions $\rho_{\bar{x}_i}, i = 1, 2, \ldots, m$. Derive a control strategy such that the controlled leader-follower multi-agent system achieves consensus while satisfying (4),(5).

3. MAIN RESULTS

In this section, we design the control for the leader-follower multi-agent system (3) such that the system can achieve consensus within the prescribed performance functions

$$\rho_{\bar{x}_i}(t) = (\rho_{\bar{x}_{i0}} - \rho_{\bar{x}_{i\infty}})e^{-l_{\bar{x}_i} t} + \rho_{\bar{x}_{i\infty}}. \tag{14}$$

Here the performance functions are chosen as (14) without loss of generality and we assume that communicating agents share information about their performance and transformation functions. Hence, the communication between neighbouring agents is bidirectional and the graph $\mathcal{G}$ is assumed undirected.

Consensus is achieved in the sense that the stack vector $\bar{x}$ of relative positions converges to zero as $t \to \infty$. We then rewrite the dynamics of the leader-follower multi-agent system (3) into the edge space in order to characterise the dynamics of the relative positions. We first rewrite (3) into the dynamics corresponding to followers and leaders, respectively. The corresponding incidence matrix is decomposed as $D = \begin{bmatrix} D_F^T \ D_L^T \end{bmatrix}^T$ (Mesbahi and Egerstedt, 2010). Multiplying with D^T on both sides of (3), we obtain the dynamics on the edge space as

$$\Sigma_e : \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} 0_m & I_m \\ -L_e & -L_e \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix} + \begin{bmatrix} 0_{m \times n_l} \\ D_L^T \end{bmatrix} u. \tag{15}$$

It is known that the edge Laplacian L_e is positive definite if the graph is a tree (Dimarogonas and Johansson, 2010). We thus here assume the following

Assumption 1. The leader-follower multi-agent system (3) described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a connected tree.

We consider tree graphs as a starting point since we exploit the positive definiteness of L_e in the analysis, and motivated by the fact that they require less communication load (less edges) for their implementation. Note however that further results for a general graph could be built based on the results of tree graphs, for example, through appropriate graph decompositions (Zelazo and Mesbahi, 2011). For the leader-follower multi-agent system (15), we first design the reference velocity $v_d \in \mathbb{R}^n$ and the corresponding reference relative velocity $\bar{v}_d \in \mathbb{R}^m$ as:

$$v_d = -D\mathcal{J}_{T_{\hat{\bar{x}}}} G_{\bar{x}} \varepsilon_{\hat{\bar{x}}}; \qquad \bar{v}_d = -L_e \mathcal{J}_{T_{\hat{\bar{x}}}} G_{\bar{x}} \varepsilon_{\hat{\bar{x}}}, \tag{16}$$

where $\hat{\bar{x}} \in \mathbb{R}^m$ is the stack vector of transformed errors $\hat{\bar{x}}_i$, $G_{\bar{x}} \in \mathbb{R}^{m \times m}$ is a positive definite diagonal gain matrix with entries the positive constant parameters $g_{\bar{x}_i}$. $\mathcal{J}_{T_{\hat{\bar{x}}}} \triangleq \mathcal{J}_T(\hat{\bar{x}}, t) \in \mathbb{R}^{m \times m}$ is a time varying diagonal matrix with diagonal entries $\mathcal{J}_{T_{\bar{x}_i}}(\hat{\bar{x}}_i, t)$, and $\varepsilon_{\hat{\bar{x}}} \triangleq \varepsilon(\hat{\bar{x}}) \in \mathbb{R}^m$ is a stack vector with entries $\varepsilon_{\bar{x}_i}(\hat{\bar{x}}_i)$.

We then define the relative velocity error vector as $\bar{e} = [\bar{e}_1, \ldots, \bar{e}_m]^T = \bar{v} - \bar{v}_d \in \mathbb{R}^m$. The corresponding prescribed performance functions $\rho_{\bar{e}_i}(t), i = 1, 2, \ldots, m$ related to the relative velocity errors are defined as

$$\rho_{\bar{e}_i}(t) = (\rho_{\bar{e}_{i0}} - \rho_{\bar{e}_{i\infty}})e^{-l_{\bar{e}_i} t} + \rho_{\bar{e}_{i\infty}}, \tag{17}$$

and (17) is designed in a way such that the initial condition of $\bar{e}_i$ is within the performance bounds, i.e., $|\bar{e}_i(0)| < \rho_{\bar{e}_i}(0) = \rho_{\bar{e}_{i0}}, i = 1, 2, \ldots, m$. The related prescribed performance region is described as

$$-\rho_{\bar{e}_i}(t) < \bar{e}_i(t) < \rho_{\bar{e}_i}(t), i = 1, 2, \ldots, m. \tag{18}$$

Similar to (6), $\bar{e}_i(t)$ is normalized as

$$\hat{\bar{e}}_i(t) = \frac{\bar{e}_i(t)}{\rho_{\bar{e}_i}(t)}. \tag{19}$$

Then the normalized error is transformed through a transformed function $T_{\bar{e}_i}$ such that $T_{\bar{e}_i}(0) = 0$, with one example choice being

$$T_{\bar{e}_i}(\hat{\bar{e}}_i) = \ln\left(\frac{1 + \hat{\bar{e}}_i}{1 - \hat{\bar{e}}_i}\right). \tag{20}$$

Therefore, the transformed error $\varepsilon_{\bar{e}_i}$ is defined as

$$\varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i) = T_{\bar{e}_i}(\hat{\bar{e}}_i). \tag{21}$$

Similar to (11), differentiating (21) with respect to time, we derive

$$\dot{\varepsilon}_{\bar{e}_i}(\hat{\bar{e}}_i) = \mathcal{J}_{T_{\bar{e}_i}}(\hat{\bar{e}}_i, t)[\dot{\bar{e}}_i + \alpha_{\bar{e}_i}(t)\bar{e}_i] \tag{22}$$

where

$$\mathcal{J}_{T_{\bar{e}_i}}(\hat{\bar{e}}_i, t) \triangleq \frac{\partial T_{\bar{e}_i}(\hat{\bar{e}}_i)}{\partial \hat{\bar{e}}_i} \frac{1}{\rho_{\bar{e}_i}(t)} > 0 \tag{23}$$

$$\alpha_{\bar{e}_i}(t) \triangleq -\frac{\dot{\rho}_{\bar{e}_i}(t)}{\rho_{\bar{e}_i}(t)} > 0 \tag{24}$$

are the normalized Jacobian of the transformation function $T_{\bar{e}_i}$ and the normalized derivative of the performance function, respectively. Using $T_{\bar{e}_i}(0) = 0$, we can derive that

$$\hat{\bar{e}}_i \frac{\partial \varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i)}{\partial \hat{\bar{e}}_i} \varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i) \geq \mu_{\bar{e}_i} \varepsilon^2_{\bar{e}_i}(\hat{\bar{e}}_i) \tag{25}$$

for some positive constant $\mu_{\bar{e}_i}$ (Karayiannidis and Doulgeri, 2012). (25) is useful for the forthcoming stability analysis.

For the leader-follower multi-agent system (15), the proposed controller applied to the leader agents is the composition of the term based on prescribed performance of the relative velocity errors of the neighbouring agents:

$$u_j = -\sum_{i \in \Phi_j} g_{\bar{e}_i} \mathcal{J}_{T_{\bar{e}_i}}(\hat{\bar{e}}_i, t) \varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i), \qquad j \in \mathcal{V}_L, \tag{26}$$

where $\Phi_j = \{i | (j, k) = i, k \in \mathcal{N}_j\}$, i.e., the set of all the edges that include agent $j \in \mathcal{V}_L$ as a node. Then the stack input vector is

$$u = -D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}}, \tag{27}$$

where $\hat{\bar{e}} \in \mathbb{R}^m$ is the stack vector of transformed errors $\hat{\bar{e}}_i$, $G_{\bar{e}} \in \mathbb{R}^{m \times m}$ is a positive definite diagonal gain matrix with entries as the positive constant parameters $g_{\bar{e}_i}$, $\mathcal{J}_{T_{\hat{\bar{e}}}} \triangleq \mathcal{J}_T(\hat{\bar{e}}, t) \in \mathbb{R}^{m \times m}$ is a time varying diagonal matrix with diagonal entries $\mathcal{J}_{T_{\bar{e}_i}}(\hat{\bar{e}}_i, t)$, and $\varepsilon_{\hat{\bar{e}}} \triangleq \varepsilon(\hat{\bar{e}}) \in \mathbb{R}^m$ is a stack vector with entries $\varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i)$.

Next, we derive the following result and will use Lyapunov-like methods to prove that the prescribed performance can be guaranteed for both relative positions and relative velocity errors. In addition, consensus can be achieved.

Theorem 1. Consider the leader-follower multi-agent system Σ under Assumption 1 with dynamics (3), and the predefined performance functions $\rho_{\bar{x}_i}$ and $\rho_{\bar{e}_i}$ as in (14) and (17), respectively. The transformation functions are chosen satisfying $T_{\bar{x}_i}(0) = 0, T_{\bar{e}_i}(0) = 0$, and assume that the initial conditions $\bar{x}_i(0)$ and $\bar{e}_i(0)$ are within the performance bounds (4),(5) and (18), respectively. If the following condition holds:

$$\bar{\gamma} \geq l = \max_{i=1,\ldots,m}(l_{\bar{e}_i}), \tag{28}$$

where l is the largest decay rate of $\rho_{\bar{e}_i}(t)$ and $\bar{\gamma}$ is the maximum value of γ that ensures:

$$\Gamma = \begin{bmatrix} D_L^T D_L & \frac{1}{2}\big(L_e - \gamma(I_m - D_L^T D_L)\big) \\ \frac{1}{2}\big(L_e - \gamma(I_m - D_L^T D_L)\big) & \gamma L_e \end{bmatrix} \geq 0, \tag{29}$$

then, the relative position $\bar{x}$ under the control law (27) goes to an arbitrary small ball around zero while satisfying (4),(5). In addition, the relative velocity errors satisfy (18).

Proof. The proof is based on three steps. We first show that there exists a maximal solution for both $\hat{\bar{x}}$ and $\hat{\bar{e}}$. Equivalently, that $\hat{\bar{x}}_i(t)$ and $\hat{\bar{e}}_i(t)$ remain in $\mathcal{D}_{\bar{x}_i}$ and $\mathcal{D}_{\bar{e}_i} = (-1, 1)$, respectively within the maximal time solution interval $[0, \tau_{\max})$, where, $\mathcal{D}_{\bar{x}_i}$ is defined in (7), (8). Next, we prove that the proposed control strategy restricts $\hat{\bar{x}}_i(t)$ and $\hat{\bar{e}}_i(t)$ in compact subsets of $\mathcal{D}_{\bar{x}_i}$ and $\mathcal{D}_{\bar{e}_i}$ for $t \in [0, \tau_{\max})$, which by contradiction results in $\tau_{\max} = \infty$ in the last step and the proof is completed. In the sequel, we show the proof in detail step by step. We first define the target open set $\mathcal{D} = \mathcal{D}_{\bar{x}} \times \mathcal{D}_{\bar{e}}$ such that:

$$\begin{aligned} \mathcal{D}_{\bar{x}} &= \mathcal{D}_{\bar{x}_1} \times \mathcal{D}_{\bar{x}_2} \times \cdots \times \mathcal{D}_{\bar{x}_m}, \\ \mathcal{D}_{\bar{e}} &= \mathcal{D}_{\bar{e}_1} \times \mathcal{D}_{\bar{e}_2} \times \cdots \times \mathcal{D}_{\bar{e}_m}. \end{aligned} \tag{30}$$

Step 1. Since the initial conditions $\bar{x}_i(0), \bar{e}_i(0)$ are chosen within the performance bounds, we can verify that the initial normalized relative positions $\hat{\bar{x}}(0)$ and the initial normalized relative velocity errors $\hat{\bar{e}}(0)$ are within the open sets $\mathcal{D}_{\bar{x}}$ and $\mathcal{D}_{\bar{e}}$, respectively. We can conclude that $z(0) \in \mathcal{D}$, where $z(t) = [\hat{\bar{x}}(t), \hat{\bar{e}}(t)]^T$. By calculating the derivative of $\hat{\bar{x}}(t)$ and $\hat{\bar{e}}(t)$, we can verify that $\dot{z}$ is continuous and also locally Lipschitz on z. Hence, according to Theorem 54 of (Sontag, 2013), there exists a maximal solution $z(t)$ in a time interval $[0, \tau_{\max})$ such that $z(t) \in \mathcal{D}, \forall t \in [0, \tau_{\max})$.

Step 2. Based on Step 1, we know that $\bar{x}_i$ and $\bar{e}_i$ satisfy (4),(5) and (18), respectively for all $t \in [0, \tau_{\max})$. We first consider the Lyapunov-like function $V_{\bar{x}} = \frac{1}{2}\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \varepsilon_{\hat{\bar{x}}}$ related to the relative positions. Differentiating $V_{\bar{x}}$ with respect to time and using the stacked vector version of equation (11), we obtain

$$\dot{V}_{\bar{x}} = \varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \dot{\varepsilon}_{\hat{\bar{x}}} = \varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} (\dot{\bar{x}} + \alpha_{\bar{x}}(t)\bar{x}), \tag{31}$$

where $\alpha_{\bar{x}}(t)$ is the diagonal matrix with diagonal entries $\alpha_{\bar{x}_i}(t)$. According to (13) and (14), we know that

$$\alpha_{\bar{x}_i}(t) \triangleq -\frac{\dot{\rho}_{\bar{x}_i}(t)}{\rho_{\bar{x}_i}(t)} = l_{\bar{x}_i} \frac{\rho_{\bar{x}_i}(t) - \rho_{\bar{x}_{i\infty}}}{\rho_{\bar{x}_i}(t)} < l_{\bar{x}_i}, \forall t \tag{32}$$

Since $\dot{\bar{x}} = \bar{v} = \bar{v}_d + \bar{e}$ where $\bar{v}_d$ is given in (16), we obtain $\dot{\bar{x}} = -L_e \mathcal{J}_{T_{\hat{\bar{x}}}} G_{\bar{x}} \varepsilon_{\hat{\bar{x}}} + \bar{e}$, and then by replacing $\dot{\bar{x}}$ in (31), we further derive that

$$\begin{aligned} \dot{V}_{\bar{x}} =& \varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} (-L_e \mathcal{J}_{T_{\hat{\bar{x}}}} G_{\bar{x}} \varepsilon_{\hat{\bar{x}}} + \bar{e} + \alpha_{\bar{x}}(t)\bar{x}) \\ =& -\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} L_e \mathcal{J}_{T_{\hat{\bar{x}}}} G_{\bar{x}} \varepsilon_{\hat{\bar{x}}} + \varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} \bar{e} \\ &+ \varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} \alpha_{\bar{x}}(t)\bar{x} \\ \leq& -\lambda_{\min}(L_e)\|\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}}\|^2 + \|\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}}\| \bar{M}_{\bar{x}}, \end{aligned} \tag{33}$$

where $\bar{M}_{\bar{x}}$ is a positive constant satisfying

$$\|\bar{e} + \alpha_{\bar{x}}(t)\bar{x}\| \leq \bar{M}_{\bar{x}}. \tag{34}$$

(34) holds for a bounded $\bar{M}_{\bar{x}}$ due to the boundedness of $\alpha_{\bar{x}_i}(t)$ as shown in (32) and the boundedness of $\bar{x}_i, \bar{e}_i, i = 1, \cdots, m$, which is shown in the beginning of Step 2. Then, we can conclude that $\dot{V}_{\bar{x}} < 0$ when $\|\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}}\| > \frac{\bar{M}_{\bar{x}}}{\lambda_{\min}(L_e)}$. This condition is guaranteed when $\|\varepsilon_{\hat{\bar{x}}}\| > \frac{\bar{M}_{\bar{x}}}{\underline{\beta}\lambda_{\min}(L_e)}$ due to the fact that

$$\|\varepsilon_{\hat{\bar{x}}}^T G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}}\| \geq \underline{\beta}\|\varepsilon_{\hat{\bar{x}}}\|, \tag{35}$$

where $\underline{\beta}$ is selected satisfying $G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}} \geq \underline{\beta} I_m$ and $G_{\bar{x}} \mathcal{J}_{T_{\hat{\bar{x}}}}$ is a diagonal positive definite matrix. It can be concluded that $\|\varepsilon_{\hat{\bar{x}}}\|$ is upper bounded by

$$\|\varepsilon_{\hat{\bar{x}}}\| \leq \bar{\varepsilon}_1 = \max\left\{\|\varepsilon_{\hat{\bar{x}}}(0)\|, \frac{\bar{M}_{\bar{x}}}{\underline{\beta}\lambda_{\min}(L_e)}\right\}, \tag{36}$$

$\forall t \in [0, \tau_{\max})$. Due to the boundedness of $\|\varepsilon_{\hat{\bar{x}}}\|$ in $t \in [0, \tau_{\max})$, we can restrict $\hat{\bar{x}}_i$ in a compact subset of $\mathcal{D}_{\bar{x}_i}$ as

$$\hat{\bar{x}}_i(t) \in [\underline{\delta}_{\bar{x}_i}, \bar{\delta}_{\bar{x}_i}] \triangleq [-T_{\bar{x}_i}^{-1}(\bar{\varepsilon}_1), T_{\bar{x}_i}^{-1}(\bar{\varepsilon}_1)] \subset \mathcal{D}_{\bar{x}_i}, \tag{37}$$

where $T_{\bar{x}_i}^{-1}$ is the inverse function of the transformed function $T_{\bar{x}_i}$. $T_{\bar{x}_i}^{-1}$ always exists because $T_{\bar{x}_i}$ is a smooth and strictly increasing function. Therefore, the reference relative velocity vector $\bar{v}_d$ as designed in (16) and its derivative $\dot{\bar{v}}_d$ are both bounded in $t \in [0, \tau_{\max})$. Moreover, since $\bar{v} = \bar{v}_d + \bar{e}$, we can also conclude that $\bar{v}(t)$ is bounded for all $t \in [0, \tau_{\max})$ due to the boundedness of $\bar{v}_d$ and $\bar{e}$.

Next, for the velocity part, we consider the Lyapunov-like function $V_{\bar{e}} = \frac{1}{2}\varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} + \frac{\gamma}{2}\bar{e}^T\bar{e}$. Differentiating $V_{\bar{e}}$ with respect to time and using the stacked vector version of equation (22), we obtain

$$\begin{aligned} \dot{V}_{\bar{e}} &= \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \dot{\varepsilon}_{\hat{\bar{e}}} + \gamma \bar{e}^T \dot{\bar{e}} \\ &= \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\dot{\bar{e}} + \alpha_{\bar{e}}(t)\bar{e}) + \gamma \bar{e}^T \dot{\bar{e}}. \end{aligned} \tag{38}$$

Then, based on $\dot{\bar{v}} = -L_e\bar{x} - L_e\bar{v} + D_L^T u$ that is shown in the edge dynamics (15) and $\bar{v} = \bar{e} + \bar{v}_d$, and further substituting the control strategy (27), we derive that

$$\begin{aligned} \dot{\bar{e}} &= \dot{\bar{v}} - \dot{\bar{v}}_d \\ &= -L_e\bar{e} - L_e\bar{v}_d - L_e\bar{x} - D_L^T D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} - \dot{\bar{v}}_d. \end{aligned} \tag{39}$$

Thus, replacing the above expression of $\dot{\bar{e}}$ in (38) and denoting $\Omega = -L_e\bar{v}_d - L_e\bar{x} - \dot{\bar{v}}_d$, we further obtain

$$\begin{aligned} \dot{V}_{\bar{e}} =& \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (-L_e\bar{e} - D_L^T D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} + \alpha_{\bar{e}}(t)\bar{e} + \Omega) \\ &+ \gamma \bar{e}^T(-L_e\bar{e} - D_L^T D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} + \Omega) \\ =& -\varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} D_L^T D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} L_e \bar{e} \\ &+ \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \alpha_{\bar{e}}(t)\bar{e} - \gamma \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} D_L^T D_L \bar{e} - \gamma \bar{e}^T L_e \bar{e} \\ &+ \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma \bar{e}^T \Omega, \end{aligned} \tag{40}$$

Adding and subtracting $\gamma \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \bar{e}$ on the right hand side of (40), we obtain

$$
\begin{aligned}
\dot{V}_{\bar{e}} =& - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\gamma I_m - \alpha_{\bar{e}}(t))\bar{e} - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} D_L^T D_L \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} \\
& - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} L_e \bar{e} - \gamma \bar{e}^T L_e \bar{e} + \gamma \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (I_m - D_L^T D_L)\bar{e} \\
& + \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma \bar{e}^T \Omega \\
=& - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\gamma I_m - \alpha_{\bar{e}}(t))\bar{e} + \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma \bar{e}^T \Omega \\
& - y^T \begin{bmatrix} D_L^T D_L & \frac{1}{2}\big(L_e - \gamma(I_m - D_L^T D_L)\big) \\ \frac{1}{2}\big(L_e - \gamma(I_m - D_L^T D_L)\big) & \gamma L_e \end{bmatrix} y \\
=& - \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\gamma I_m - \alpha_{\bar{e}}(t))\bar{e} - y^T \Gamma y \\
& + \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma \bar{e}^T \Omega
\end{aligned} \tag{41}
$$

with

$$
y = \begin{bmatrix} \mathcal{J}_{T_{\hat{\bar{e}}}} G_{\bar{e}} \varepsilon_{\hat{\bar{e}}} \\ \bar{e} \end{bmatrix}, \tag{42}
$$

and where the block matrix Γ is defined in(29). We have that $G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}}$ is a diagonal positive definite matrix. Using (32), we can verify that $(\gamma I_m - \alpha_{\bar{e}}(t))$ is a diagonal positive definite matrix if $\gamma \geq l = \max(l_{\bar{e}_i}) > \bar{\alpha} = \sup \alpha_{\bar{e}_i}(t)$. Since $T_{\bar{e}_i}$ is smooth, strictly increasing and $T_{\bar{e}_i}(0) = 0$, we have $\varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i)\hat{\bar{e}}_i \geq 0$. Then, by setting $\gamma := \theta + \bar{\alpha}$, with θ being a positive constant, we get:

$$
- \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\gamma I_m - \alpha_{\bar{e}}(t))\bar{e} \leq -\theta \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \bar{e} \tag{43}
$$

Then, according to (19), (23), we further obtain

$$
- \theta \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \bar{e} = -\theta \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \frac{\partial \varepsilon_{\hat{\bar{e}}}}{\partial \hat{\bar{e}}} \hat{\bar{e}} \leq 0. \tag{44}
$$

(44) holds because the transformed function $T_{\bar{e}_i}$ is smooth and strictly increasing and $\varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i)\hat{\bar{e}}_i \geq 0$. Then, based on condition (28), and choosing $\gamma = \bar{\gamma}$, we obtain $-\varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} (\gamma I_m - \alpha_{\bar{e}}(t))\bar{e} \leq 0$ and $\Gamma \geq 0$. Then $\dot{V}_{\bar{e}}$ is further upper bounded as

$$
\dot{V}_{\bar{e}} \leq -\theta \varepsilon_{\hat{\bar{e}}}^T G_{\bar{e}} \frac{\partial \varepsilon_{\hat{\bar{e}}}}{\partial \hat{\bar{e}}} \hat{\bar{e}} + \varepsilon_{\hat{\bar{e}}}^T (G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma Q \Omega), \tag{45}
$$

where Q is a time-varying diagonal positive definite matrix such that $\bar{e} = Q \varepsilon_{\hat{\bar{e}}}$. This matrix Q always exists with the diagonal entries $q_i = \rho_{\bar{e}_i} \hat{\bar{e}}_i / \varepsilon_{\bar{e}_i}(\hat{\bar{e}}_i) > 0$. Next, according to inequality (25), we further derive that

$$
\dot{V}_{\bar{e}} \leq -\theta \mu \|\varepsilon_{\hat{\bar{e}}}\|^2 + \|\varepsilon_{\hat{\bar{e}}}\| \bar{M}_{\bar{e}}, \tag{46}
$$

where $\mu = \min(\mu_{\bar{e}_i}), i = 1, \cdots, m$ and $\mu_{\bar{e}_i}$ is defined in (25). $\bar{M}_{\bar{e}}$ is a positive constant satisfying

$$
\|G_{\bar{e}} \mathcal{J}_{T_{\hat{\bar{e}}}} \Omega + \gamma Q \Omega\| \leq \bar{M}_{\bar{e}}. \tag{47}
$$

(47) holds with a bounded $\bar{M}_{\bar{e}}$ due to the boundedness of $\bar{x}, \bar{v}_d, \dot{\bar{v}}_d$. Similarly, it can be concluded that $\dot{V}_{\bar{e}} < 0$ when $\|\varepsilon_{\hat{\bar{e}}}\| > \frac{\bar{M}_{\bar{e}}}{\theta \mu}$ and further $\|\varepsilon_{\hat{\bar{e}}}\|$ is upper bounded by

$$
\|\varepsilon_{\hat{\bar{e}}}\| \leq \bar{\varepsilon}_2 = \max \left\{ \|\varepsilon_{\hat{\bar{e}}}(0)\|, \frac{\bar{M}_{\bar{e}}}{\theta \mu} \right\}, \tag{48}
$$

$\forall t \in [0, \tau_{\max})$. Due to the boundedness of $\|\varepsilon_{\hat{\bar{e}}}\|$ in $t \in [0, \tau_{\max})$, we can restrict $\hat{\bar{e}}_i$ in a compact subset of $\mathcal{D}_{\bar{e}_i}$ as

$$
\hat{\bar{e}}_i(t) \in [\underline{\delta}_{\bar{e}_i}, \bar{\delta}_{\bar{e}_i}] \triangleq [-T_{\bar{e}_i}^{-1}(\bar{\varepsilon}_2), T_{\bar{e}_i}^{-1}(\bar{\varepsilon}_2)] \subset \mathcal{D}_{\bar{e}_i}, \tag{49}
$$

where $T_{\bar{e}_i}^{-1}$ is the inverse function of $T_{\bar{e}_i}$.

Step 3. Finally, we need to prove that $\tau_{\max}$ can be extended to ∞. According to (37) and (49), we know that $z(t) \in \mathcal{D}' = \mathcal{D}'_{\bar{x}} \times \mathcal{D}'_{\bar{e}}, \forall t \in [0, \tau_{\max})$, where $\mathcal{D}'_{\bar{x}} = [\underline{\delta}_{\bar{x}_1}, \bar{\delta}_{\bar{x}_1}] \times \cdots \times [\underline{\delta}_{\bar{x}_m}, \bar{\delta}_{\bar{x}_m}]$ and $\mathcal{D}'_{\bar{e}} = [\underline{\delta}_{\bar{e}_1}, \bar{\delta}_{\bar{e}_1}] \times \cdots \times [\underline{\delta}_{\bar{e}_m}, \bar{\delta}_{\bar{e}_m}]$. Hence, $\mathcal{D}' \subset \mathcal{D}$ is a nonempty and compact subset of $\mathcal{D}$ and it can be concluded that $z(t) \in \mathcal{D}', \forall t \in [0, \tau_{\max})$. Let us now assume that $\tau_{\max} < \infty$. According to Proposition C.3.6 of (Sontag, 2013), there exists a $t' \in [0, \tau_{\max})$ such that $z(t') \notin \mathcal{D}'$, which leads to a contradiction. Hence, we conclude that $\tau_{\max}$ is extended to ∞, that is $z(t) \in \mathcal{D}' \subset \mathcal{D}, \forall t \geq 0$. Therefore $\varepsilon_{\hat{\bar{x}}}, \varepsilon_{\hat{\bar{e}}}$ are bounded for all $t \geq 0$ and the boundedness of the transformed errors $\varepsilon_{\hat{\bar{x}}}, \varepsilon_{\hat{\bar{e}}}$ implies that the relative position $\bar{x}(t)$ and the relative velocity error $\bar{e}(t)$ evolve while satisfying (4),(5) and (18), respectively for all $t \geq 0$. Finally, the convergence result is discussed next. By choosing small enough $\rho_{\bar{x}_{i\infty}}$, we can conclude that consensus is achieved in the sense that $\bar{x}_i \in (-\epsilon, \epsilon)$ as $t \to \infty$, where ϵ is close to 0 and satisfies $\epsilon < \rho_{\bar{x}_{i\infty}}, i = 1, \ldots, m$. □

Remark 1. The consensus achieved in Theorem 1 is practical convergence since ϵ can be arbitrarily small but not exactly zero. This kind of practical convergence is also shown in (Bechlioulis and Rovithakis, 2014). It is reasonable in practical design and it cannot be guaranteed that the errors converge to 0 since the Lyapunov-like function is bounded by its level set rather than negative definite.

4. SIMULATION

In this section, a simulation example is presented. The communication graph is shown as Fig. 1, where the leaders and followers are represented by grey and white nodes, respectively. We choose, without loss of generality, the same $\rho_{\bar{x}_i}$ and $\rho_{\bar{e}_i}$ for all edges:

$$
\rho_{\bar{x}_i}(t) = 4.9e^{-0.4t} + 0.1; \rho_{\bar{e}_i}(t) = 2e^{-0.4t} + 0.1. \tag{50}
$$

For all $(i, j) \in \mathcal{E}$, we choose $M_{\bar{x}_i} = 1$ and

$$
T_{\bar{x}_i}(\hat{\bar{x}}_i) = \ln\left(\frac{1 + \hat{\bar{x}}_i}{1 - \hat{\bar{x}}_i}\right); T_{\bar{e}_i}(\hat{\bar{e}}_i) = \ln\left(\frac{1 + \hat{\bar{e}}_i}{1 - \hat{\bar{e}}_i}\right).
$$

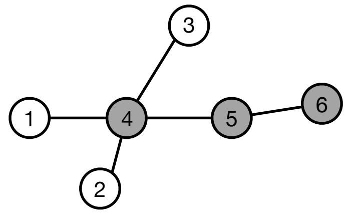

Fig. 1. Communication graph with a tree topology.

The relative positions are initialised as $[3\ 2\ 2.5\ 1\ 0.5]^T$ and the relative velocity error is initialised as $[1\ 1\ 1\ 1\ 1]^T$. According to Theorem 1, the matrix inequality is feasible with $\bar{\gamma} = 1$, l is chosen to be 0.4 and thus satisfies the constraint $l \leq \bar{\gamma} = 1$. The simulation result when applying the PPC control law (27) with gain matrices $G_{\bar{x}}, G_{\bar{e}}$ whose diagonal entries are all equal to 1 is shown in Fig. 2. As a comparison, the simulation result without the PPC control law (27) is shown in Fig. 3. In both figures, the black lines indicate the prescribed performance functions. We can see that the trajectories intersect the performance bound without extra control, which can be improved by applying the PPC law (27) such that the controlled system achieves consensus while satisfying (4),(5) and (18).

5. CONCLUSIONS

In this paper, the consensus problem of second order leader-follower multi-agent systems with prescribed performance guarantees has been investigated. Under the assumption of tree graphs, a distributed prescribed performance control law has been proposed for the group

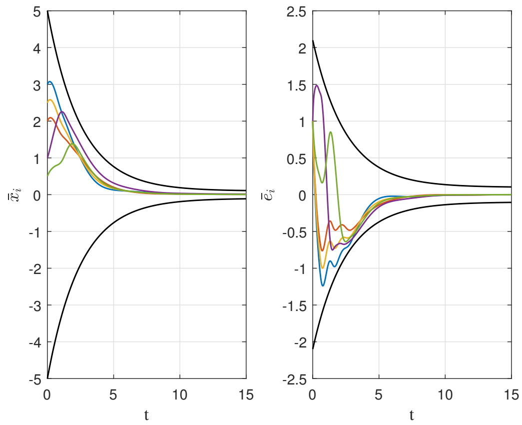

Fig. 2. The left and right figures show the trajectories of relative positions and relative velocity errors *with* PPC control (27), respectively.

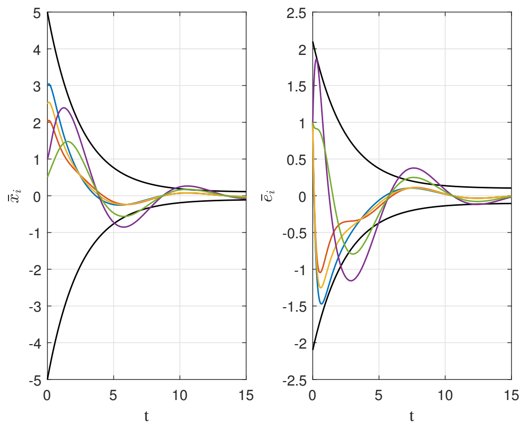

Fig. 3. The left and right figures show the trajectories of relative positions and relative velocity errors *without* PPC, respectively.

of leaders to drive the followers ensuring that the entire system can achieve consensus within the prescribed performance bounds. We have proved that when the decay rate of the performance functions of the relative velocity errors is within a sufficient bound, the relative velocity errors can also evolve within certain performance bounds.

Future research directions include considering more general graphs that include circles, applying other transient approaches to this leader-follower framework and also investigating leader selection problems.

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