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Leader selection and control design for topology estimation of dynamical networks

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Abstract— **We propose a framework for selecting leaders and employing active control to guarantee accurate topology estimation in finite time for dynamical networks. After determining the optimal or suboptimal solution of the minimum leader number which renders strongly structurally controllable, two topology estimation algorithms with active control design schemes are proposed. The first, employing the integral of the states and control input and building an equation of its topology matrix, based on the dynamics of the original network, gives a unique solution for a symmetric topological matrix and the subspace of an asymmetric topological matrix. The second, building upon an auxiliary network and comparing the difference with the original network, provides a guarantee for accurate topology estimation for both stable and unstable dynamical networks in finite time. Finally, a relevant simulation example verifies the performance of the proposed methods.**

I. Introduction

Networks of dynamical systems represent many physical systems, including power systems, biological systems, sensor systems, transport systems, etc. [1]. The topological network structures, displaying important properties of the networks and playing a crucial role in understanding the behaviors of the networks [2], are the basis for further performance improvement. In many practical systems, there is a lack of detailed topological structures, and hence additional efforts are needed to estimate them. For example, in the human immune system, inference of communication between different types of cells remains a challenging problem. There is significant work that designs algorithms for estimating the topology from network observations [3], prediction error-based methods [4], [5], constrained Lyapunov equation-based method [6], adaptive control-based method [7]–[9].

Data-driven topology estimation algorithms [6], [7], [10], [11] from pure observations indicate that the topology is uniquely identified when the trajectories of the network satisfy a rank condition indicating data richness. During the identification process, it is difficult to verify whether the identifiability condition is met and there is a lack of guarantee of when this condition holds. One effective way is to implement an active control input to ensure that the trajectories contain sufficient excitation and reduce the time of topology estimation. Previous relevant works [9], [12], [13] employ external control to track an auxiliary system, which guarantees that the trajectories of the network meet certain conditions, making it sufficient for topology estimation. However, all nodes in the network are required to implement the control input, which is not feasible for some networks and inefficient for a large network. One possible solution originating from leader-follower cooperation control [14] is to select a small group of agents to implement the cooperation task among the network where the cooperation aims to provide sufficiently rich information for accurate topology identification. In terms of partial excitation for topology estimation, some relevant works identify the edges among the network described by the pulse transfer function among nodes [15], [16]. However, they requires that the transfer functions between any two nodes and the network are strictly stable, which fails in identifying an unstable network. In fact, it is difficult to guarantee the stability of all the transfer functions before the topology identification. It thus becomes evident that, there is a lack of a systematic method for selecting the minimum leader and designing control input for general topology estimation and how to estimate the topology for unstable networks.

The motivation of this paper is to establish a framework for selecting the minimum number of leader nodes in a network and implementing active control to estimate edge weights in both stable and unstable networks. We propose a systematic method for estimating unknown edge weights by controlling the smallest possible leader set, integrating leader selection and controller design to effectively excite the network. Two weight estimation algorithms are proposed to accurately estimate the topology matrix in finite time. For an undirected graph, the first algorithm guarantees a unique solution for the unknown topology matrix after a finite time, while for a directed graph, it identifies the subspace to which the topology matrix belongs. Unlike previous works that assume the stability of the network, this paper considers scenarios where the network is unstable. To address both stable and unstable networks, a second parameter estimation algorithm is proposed. The designed control scheme only excites the selected leader nodes, and the proposed parameter estimation scheme ensures that edge weight estimation errors become zero in finite time.

The rest of the paper is structured as follows. In Section II, the problem formulation is introduced. Then we provide a solution for the minimum leader node set problem in Section III, and two weight estimation schemes for a stable network in Section IV. Section V proposes the controller design and the weight estimation scheme for an unstable network. A simulation example is given to verify the feasibility of the

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proposed algorithms in Section VI. Section VII concludes this paper.

A. Notation and definitions

$X \succ 0$ denotes that X is a positive definite. For matrices X and Y, $X \succ Y$ means that $X - Y \succ 0$. Denote $\ker(A)$ and $\text{ran}(A)$ as the null space and range of $A \in R^{m\times n}$, $\ker(A) = \{x \in R^n | Ax = 0\}$ and $\text{ran}(A) = \{y \in R^m | y = Ax, x \in R^n\}$. $\otimes$ denotes the Kronecker product. For $A \in R^{m\times n}$, $\text{vec}(A) = [A_{11}, \dots, A_{m1}, A_{12}, \dots, A_{m2}, A_{1n}, \dots, A_{mn}]^\top$. $A^\top$ denotes the transpose of A.

Denote $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ a directed weighted graph, where $\mathcal{V} = \{1, 2, \dots, N\}$ is a node-set and $\mathcal{E} \subseteq \mathcal{V}^2$ is an edge set with M edges, characterizing the information exchange between agents. A directed edge $e_k := (i, j) \in \mathcal{E}$, indicates that agent j has access to information from node i, and let a diagonal matrix $W \in R^{M\times M}$, whose diagonal w_k entries represent the weights of the edges. Denote the directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ from $\mathcal{G}$, where $\mathcal{V}, \mathcal{E}$ are the same as in $\mathcal{G}$.

II. Problem formulation

The dynamics of a network is represented as

$$\dot{x} = Ax + Bu \tag{1}$$

where $x = [x_1, x_2, \cdots, x_N]^T \in R^N$ are the states of the network; N is the number of agents in this network; A could be a Laplacian matrix of the directed weighted graph $\mathcal{G}$ in [1], which can describe the connection of the agents for a weighted consensus problem; A could also be a general topological matrix here; $A_{ij} \in R$ is a constant value for $i, j = 1, 2, \cdots, N$; For $A_{ij} = 0$, it means that there is no connection among v_i and v_j, where $i, j = 1, 2, \cdots, N, i \neq j$; For $A_{ij} \neq 0$, there is an edge among node v_i and v_j, where $i, j = 1, 2, \cdots, N, i \neq j$; A_{ii} denotes that its own effect for each $i = 1, 2, \cdots, N$; The diagonal matrix $B \in R^{N\times N}$ with $B_{ii} \in \{0, 1\}$ for $i = 1, \cdots, N$, is a design parameter; $u \in R^N$ denotes the external control input. Here we call the node i as the leader if $B_{ii} \neq 0$.

The objective is to design B which determines the leader nodes to be controlled, and the external control u on these leader nodes which stabilizes and excites the network to estimate the unknown A.

In biological networks [17], [18], the existence of the connection among the networks could be known a priori. However, the connection strength of the edges is unclear. In social networks and traffic networks [19], the connections of the agents could also be obtained in advance but their connection strength is still unknown. Here we consider that the existence of edges of the network (1) is known a priori and we define it as pattern matrix $\bar{A}$

$$\bar{A}_{ij} = \begin{cases} 0, & \text{if } A_{ij} = 0, \\ 1, & \text{if } A_{ij} \neq 0. \end{cases} \tag{2}$$

However, the connection strength of the edges, which is abstracted as the weights of the edges, is unknown. The directed weighted graph $\mathcal{G}$ is linked to A while the corresponding directed graph $\mathcal{D}$ is linked to the pattern matrix $\bar{A}$.

Since there is no requirement for a stable A, the problem considered here is to select a group of leader nodes and design the controller to ensure sufficient excitation for topology estimation. We can classify this problem as two scenarios:

1) We first investigate the case where A is stable and design B and external control u to identify the topology;
2) The second scenario is for an unstable A, choosing the appropriate B and control input u guarantees that the topology algorithm ensures accurate edge weight estimation in a finite time before the states become unbounded.

III. Leader selection for weight estimation

Assuming a prior topology pattern $\bar{A}$ is given, this section involves how we implement a greedy algorithm to select a group of leaders who will implement the control and the constraints we consider to ensure sufficient excitation to estimate A.

When selecting the leaders in the network (1), the critical problem is how we can provide sufficient excitation for all the agents for further topology estimation. One possible solution here is to choose the leaders that can maintain the controllability of (A, B) of this network (1). Without knowing the exact value of A, we employ the controllability of $(\bar{A}, B)$ to select the leaders. There are two kinds of structural controllability: weakly structural controllability and strongly structural controllability, as per [20]. Weak structural controllability denotes that there exists a group (A_0, B_0) that is controllable, where A_0 and B_0 share the same pattern matrix $\bar{A}$ and $\bar{B}$. Strong structural controllability means that (A_0, B_0) is controllable for any A_0 and B_0 that share the same pattern matrix $\bar{A}$ and $\bar{B}$. Since $\bar{A}$ represents all the networks with pattern structure $\bar{A}$, we aim to find a solution B that can guarantee strongly structural controllability of (A, B) based on the pattern matrix $\bar{A}$.

Now the problem bows down to how to select an input matrix B which renders $(\bar{A}, B)$ strongly structurally controllable. We can formulate it as the following optimization problem:

$$\begin{aligned} &\min_{B\in\{0,1\}^{n\times n}} \|B\|_0 \\ &\text{s.t.} \quad (\bar{A}, B) \text{ is strongly structurally controllable.} \end{aligned} \tag{3}$$

where $\|B\|_0$ denotes the semi-norm of B, meaning its number of nonzero entries.

Finding a minimum number of leaders keeping $(\bar{A}, B)$ strongly structural controllability, is equal to finding a maximum uniquely restricted matching based on $\bar{A}$, which is a NP-complete problem [21]. For small graph, we can implement exhaustive algorithms to obtain the maximum uniquely restricted matching. While for bigger graphs, we have to resort to a greedy algorithm [20] or Markov Chain Monte Carlo (MCMC) [22] to find a feasible suboptimal leader set.

We will use B_* to denote the optimal solution or the suboptimal solution of Problem (3). Denote the cardinality of set $\mathcal{S}$ as M_1. Denote B_{*i} as the i-th column of B_*. Delete the zero columns of B_* and form a new matrix $B^* \in R^{N\times M_1}$ for further control design. We will use B^* and $u \in R^{M_1}$ for the network (1) as

$$\dot{x} = Ax + B^*u \tag{4}$$

with the same x and A as in (1).

IV. Topology estimation for a asymptotically stable A

If A is asymptotically stable, then we can use partial excitation to provide a guarantee for topology estimation. In this section, we will design two topology estimation algorithms to estimate A by implementing input excitation based on B^*.

Firstly, we give the definitions of the persistently exciting (PE) condition and sufficiently rich signals for sinusoid signals [23].

Definition 1 (PE condition): A function $\phi : R_{\geq 0} \to R^{n\times m}$ is said to be PE if for any $t \geq 0$, there exists $\gamma > 0$ and $T > t$ such that

$$\int_t^{t+T} \phi(\tau)\phi(\tau)^\top d\tau \succ \gamma I. \tag{5}$$

Definition 2 (Sufficiently rich of order n [23]): A function $\phi : R_{\geq 0} \to R^{n\times m}$ is said to be sufficiently rich of order n if it consists of at least of $n/2$ distinct frequencies.

A. Weight estimation algorithm for a symmetric A

Condition 1 (Controller): Design a controller $u(t) = (u_1(t), \ldots, u_{M_1}(t))^\top \in R^{M_1}$ where M_1 is the number of the selected leader nodes: $u_i(t)$ is designed to be bounded, stationary, whose definition (refering to Definition 5.2.2 of [23]) and sufficiently rich of order n for $i = 1, \ldots, M_1$.

Let $Y(t) := x(t)x(t)^\top$. Construct two matrix functions $P(t)$ and $H(t)$ as

$$\dot{P} = x(t)x(t)^\top, \tag{6}$$

$$\dot{H} = B^*u(t)x(t)^\top + x(t)u(t)^\top B^{*\top}. \tag{7}$$

with $P(0) = x(0)x(0)^\top$ and $H(0) = B^*u(0)x(0)^\top + x(0)u(0)^\top B^{*\top}$. Denote $Q(t) = Y(t) - Y(0) - H(t)$. Define the topology estimation time t_e

$$t_e := \inf\{t > 0 | P(t) \succ \gamma I\}. \tag{8}$$

with $\gamma > 0$.

If the directed graph $\mathcal{G}$ we aim to estimate is constrained to be undirected, then the topological matrix A will be symmetric. We can find the unique solution as follows.

Theorem 1: Assume that A is asymptotically stable and symmetric for the network (1). Design a controller $u(t)$ that satisfies Condition 1, then A is uniquely determined as

$$\text{vec}(A) = (P(t_e) \otimes I + I \otimes P(t_e))^{-1}\text{vec}(Q(t_e)) \tag{9}$$

where $I \in R^{N\times N}$ is the identity matrix. □

Proof: We first prove that $P(t_e)$ is a positive definite matrix. From the discussion in Section III, we know that $(\bar{A}, B^*)$ is strongly structurally controllable. Therefore, as A has the same structure with $\bar{A}$, (A, B^*) is controllable. Then from the proof of Theorem 5.2.3 in [23], we know that x is PE if (A, B^*) is controllable, A is asymptotically stable and each u_i is stationary and sufficiently rich of order n for $i = 1, \ldots, M_1$. Then there exists such T that

$$\int_0^T x(\tau)x(\tau)^\top d\tau \succ \gamma I. \tag{10}$$

with the same γ as in (8) and the topology estimation time $T \geq t_e$. Then $P(t_e)$ is positive definite and its rank is N.

The time derivative of $Y(t)$ is

$$\dot{Y}(t) = \dot{x}(t)x(t)^\top + x(t)\dot{x}(t)^\top.$$

Along the trajectory of the network (1), it is

$$\begin{aligned}\dot{Y}(t) &= (Ax(t) + B^*u(t))x(t)^\top + x(t)(Ax(t) + B^*u(t))^\top \\ &= Ax(t)x(t)^\top + x(t)x(t)^\top A^\top \\ &\quad + B^*u(t)x(t)^\top + x(t)u(t)^\top B^{*\top}.\end{aligned}$$

By integrating $\dot{Y}$ from 0 to t, we obtain $Y(t) - Y(0)$ as

$$\begin{aligned}Y(t) - Y(0) &= \textstyle\int_0^t (Ax(\tau)x(\tau)^\top + x(\tau)x(\tau)^\top A^\top)d\tau \\ &\quad + \textstyle\int_0^t (B^*u(\tau)x(\tau)^\top + x(\tau)u(\tau)^\top B^{*\top})d\tau.\end{aligned}$$

From $P(t)$ and $H(t)$ described in (6) and (7), we have

$$Y(t) - Y(0) = AP(t) + P(t)A^\top + H(t). \tag{11}$$

With $Q(t) = Y(t) - Y(0) - H(t)$, we get

$$Q(t) = AP(t) + P(t)A^\top. \tag{12}$$

Let $t = t_e$. Since A is symmetric, $A = A^\top$. Then we have

$$Q(t_e) = AP(t_e) + P(t_e)A. \tag{13}$$

As $P(t_e)$ is a positive definite matrix, according to Theorem 4.4.6 in [24], the equation (13) has a unique solution A.

Making (13) vectorized, then we obtain

$$\text{vec}(Q(t_e)) = \text{vec}(AP(t_e)) + \text{vec}(P(t_e)A). \tag{14}$$

Using the property of $\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B)$ where A, B, C have appropriate dimensions, equation (14) follows

$$\text{vec}(Q(t_e)) = ((P(t_e)^\top \otimes I)\text{vec}(A) + (I \otimes (P(t_e)^\top)\text{vec}(A).$$

Since $P(t_e) \succ 0$, then $((P(t_e)^\top \otimes I) \succ 0$ and $(I \otimes (P(t_e)^\top) \succ 0$ from Corollary 4.2.13 [24]. Hence $((P(t_e)^\top \otimes I + I \otimes (P(t_e)^\top) \succ 0$. The solution of A is thus determined as in (9). ■

Remark 1: The topology reconstruction problem in [25] cannot guarantee whether the rank condition holds during the topology reconstruction and it also means finding the unknown topological matrix probably takes quite a long time until the rank condition holds. The proposed method implementing the input to provide sufficient excitation guarantees accurate topology estimation in finite time.

B. Topology estimation for an asymmetric A

This part discusses that the case of an asymmetric A, which is associated with the directed graph. Then from (13), there exist multiple solutions for A. Since A is asymmetric, we cannot use $A = A^\top$ now. Firstly let $t = t_e$ and render (12) vectorized as

$$\mathrm{vec}(Q(t_e)) = \mathrm{vec}(AP(t_e)) + \mathrm{vec}(P(t_e)A^\top). \tag{15}$$

Theorem 2: Assume that A is asymptotically stable and design a controller $u(t)$ that satisfies the same conditions as in Theorem 1. Then the equation (15) has infinitely many solutions and is given by $\mathrm{vec}(A) = \Big(\mathrm{vec}\hat{A} + \ker(I+K)\Big)$, where $\mathrm{vec}\hat{A}$ satisfies

$$(I+K)\mathrm{vec}(\hat{A}) = (P(t_e)^\top \otimes I)^{-1}\mathrm{vec}(Q(t_e))$$

and K is a permutation matrix which satisfies $\mathrm{vec}(A^\top) = K\mathrm{vec}(A)$. □

Proof: Using the property $\mathrm{vec}(ABC) = (C^\top \otimes A)\mathrm{vec}(B)$ where A, B, C have appropriate dimensions, the first right item of equation (15) becomes $(P(t_e)^\top \otimes I)\mathrm{vec}(A)$. Similarly, we get $\mathrm{vec}(P(t_e)A^\top) = K\mathrm{vec}(AP(t_e))$. From $\mathrm{vec}(P(t_e)A^\top) = K\mathrm{vec}(AP(t_e))$, equation (15) can be rewritten as

$$\mathrm{vec}(Q(t_e)) = (P(t_e)^\top \otimes I)\mathrm{vec}(A) + K(P(t_e)^\top \otimes I)\mathrm{vec}(A).$$

Combining all the items, we obtain

$$\mathrm{vec}(Q(t_e)) = (P(t_e)^\top \otimes I)(I+K)\mathrm{vec}(A). \tag{16}$$

Using the proof of Theorem 1, $P(t_e)^\top \otimes I$ is positive definite. Moving it to the left of the equation (16), we have

$$(P(t_e)^\top \otimes I)^{-1}\mathrm{vec}(Q(t_e)) = (I+K)\mathrm{vec}(A). \tag{17}$$

For the permutation matrix K, A_{ij} in $\mathrm{vec}(A)$ exchanges with A_{ji} in $\mathrm{vec}(A^\top)$ for $i, j = 1, 2, \ldots, N$ and $i \neq j$. The corresponding permutation graph, whose definition refers to [26], will form a two-length cycle, from Lemma 2 in [26], and some of the eigenvalues of K are -1, and thus $(I+K)$ has equivalent zero eigenvalues. With $(P(t_e)^\top \otimes I)^{-1}\mathrm{vec}(Q(t_e)) \in \mathrm{ran}(I+K)$, we know that (17) has infinitely many solutions. (17) is equivalent to (15). This also means that (15) has infinitely many solutions in A.

From Theorem 2, $\mathrm{vec}(\hat{A})$ is a special solution of (17) and the solution of A can thus be expressed as $\mathrm{vec}(A) = \mathrm{vec}(\hat{A}) + \ker(I+K)$. ■

C. A novel algorithm for uniquely determining the topology for an asymmetric A

From Theorem 2, the proposed method has infinitely many solutions if A is asymmetric. In this part, we will design a new algorithm to find the unique solution of the topology matrix by building an auxiliary network.

Choose $A_m \in R^{N\times N}$ as a reference matrix for A, where there is no requirement for A_m. Similar to the original network (1), we establish an auxiliary network as

$$\dot{\hat{x}} = A_m x + B^* u + \tau\tilde{x}, \tag{18}$$

with $\hat{x}(0) = 0$ and $\tau > 0$, and the prediction error is $\tilde{x} = x - \hat{x}$. A filter $w \in R^N$ for x is designed as

$$\dot{w} = x - \tau w, \tag{19}$$

with $w(0) = 0$.

The following result states that the output keeps PE after a stable filter if the input is PE.

Lemma 1: If x is PE, then the output w of this filter also keeps PE. □

Proof: From the dynamics of (19), we can obtain the transfer function from x to w as

$$H(s) = \frac{W(s)}{X(s)} = \frac{1}{s+\tau}. \tag{20}$$

$H(s)$ has a pole which has a negative real part. And $H(s)$ is a minimum phase system, proper rational transfer function, and x and $\dot{x}$ are bounded. Thus from Lemma 4.8.3 in [23], $w(t)$ also satisfies PE condition. ■

Denote the error between the reference matrix A_m and A as $\tilde{A} = A_m - A$. Based on the original network (1) and the auxiliary network (18), the prediction error $\tilde{x}$ is

$$\dot{\tilde{x}} = -\tilde{A}x - \tau\tilde{x}, \tag{21}$$

with $\tilde{x}(0) = x(0)$. Denote $\zeta := \tilde{x} + \tilde{A}w$ and its time derivative yields

$$\dot{\zeta} = \dot{\tilde{x}} + \tilde{A}\dot{w}. \tag{22}$$

From (21) and (19), the time derivative $\dot{\zeta}$ becomes

$$\dot{\zeta} = -\tau(\tilde{x} + \tilde{A}w) = -\tau\zeta \tag{23}$$

with $\zeta(0) = \tilde{x}(0)$. The second equation is derived from $\zeta = \tilde{x} - \tilde{A}w$.

Construct two matrix functions $Y(t)$ and $Z(t)$ as

$$\begin{aligned} \dot{Y} &= ww^\top, \\ \dot{Z} &= (A_m w + \tilde{x} - \zeta)w^\top \end{aligned} \tag{24}$$

with $Y(0) = Z(0) = 0$. With $Y(0) = 0$, integrate the derivative $\dot{Y}$ from 0 to t to obtain

$$Y(t) = \int_0^{t_e} w(\tau)w(\tau)^\top d\tau. \tag{25}$$

Define the topology estimation time t_e

$$t_e := \inf\{t > 0 | Y(t) \succ \gamma I\}. \tag{26}$$

with $\gamma > 0$.

The weight estimation $\hat{A}(t)$ is then estimated as

$$\hat{A}(t) = Z(t)Y^{-1}(t), \tag{27}$$

for all $t \geq t_e$.

We provide the result of accurately estimating the topology matrix.

Theorem 3: Assume an asymptotically stable A for the network (1) and design a controller u that satisfies the same conditions as in Theorem 1. Then, if we implement the weight estimation algorithm (27), the weight estimation error $\tilde{A}' = \hat{A} - A$ will be zero after t_e. □

Proof: Integrating $\dot{Z}$ from 0 to t, we obtain

$$\begin{aligned} Z(t) &= \textstyle\int_0^t (A_m w + \tilde{x} - \zeta) w^\top d\tau \\ &= \textstyle\int_0^t (A_m w - \tilde{A} w) w^\top d\tau. \end{aligned} \tag{28}$$

The first equation uses $Z(0) = 0$. The second equation is obtained from $\zeta = \tilde{x} - \tilde{A}w$. Using $\tilde{A} = A_m - A$, we get

$$Z(t) = \int_0^t (Aw) w^\top d\tau = A \int_0^t w w^\top d\tau. \tag{29}$$

The second equation holds as A is a constant matrix. From the proof of Theorem 1, x is PE. From Lemma 1, w keeps PE. This means that there exists a T such that $Y(T) \succ \gamma I$, where γ is the same as in (26) and $T \geq t_e$. Then

$$A = Z(t) Y^{-1}(t), \quad t > t_e \tag{30}$$

Then from (27), the estimation error $\tilde{A}'$ is zero after $t > t_e$. ■

V. Topology Estimation Algorithm for a More General A

Theorem 3 estimates the unknown topological matrix A for a stable A. However, it may not work in the scenario of an unstable A. This section discusses the design of the control input u for the topology estimation for a more general topological matrix A, which may be either stable or unstable.

Firstly, we introduce the definition of the PE conditions of order l, piecewise uniform continuouity and minimum interdiscontinuity distance.

Definition 3 (Excitation of order l): A function $\phi : R_{\geq 0} \to R^n$ is said to be PE of order l if ϕ is $(l-1)$ times differentiable and $\phi_l = [\phi^{0\top} \ldots \phi^{(l-1)\top}]^\top$ satisfies PE condition, where ϕ^i is the i-th times derivative of ϕ and $i = 1, \ldots, (l-1)$.

Definition 4 (Piecewise uniform continuouity [27]): A function $f : X \to Y$ is said to be piecewise uniformly continuous for $X \subset R$ and $Y \subset R^m$ if a) f is piecewise continuous, i.e. there exists a countable set $D \subset X$ such that f is continuous on $X \setminus D$ and $\Omega \cap D$ is finite for every bounded set $\Omega \subset R$; b) f is uniformly continuous on X.

Definition 5 (Minimum interdiscontinuity distance): The minimum interdiscontinuity distance of a function $f : X \to Y$ is $\kappa(f) = \inf \|d_1 - d_2\|$ for any $d_1, d_2 \in D$ where $D \subset X$ and D is the set of discontinuity points of f.

Define r as the minimum relative degree for the transfer function $H(s) = (sI - A)^{-1}B$. $u^{(n-r)}$ denotes the $(n-r)$th time derivative of u for $n - r \geq 1$.

Lemma 2 (Corollary 1 [27]): If (A, B) is controllable, and the input u is bounded, $u^{(n-r)}$ is piecewise uniformly continuous with minimum interdiscontinuity distance $\kappa > 0$ and PE of order $n - r + 1$, then the state trajectory x of the linear system (1) is PE. □

Theorem 4: Assume there exists a feasible solution B for the optimization problem (3). Choose a controller u that satisfies the same conditions as in Lemma 2. Then implementing the weight estimation algorithm (27) guarantees that the weight estimation error $\tilde{A}' = \hat{A} - A$ will be zero after t_e, where t_e is the same as in (26). □

Proof: For the network (1) with a general A, the control input satisfies the conditions of Lemma 2 and (A, B^*) is controllable. Then according to Lemma 2, the trajectory of state x satisfies PE condition (5). From Lemma 1, $w(t)$ will maintain this property if x satisfies PE condition (5). Then, we know that $Y(t)$ defined in (24) will become positive definite after a finite time. Similarly to the proof in Theorem 3, the weight estimation error $\tilde{A}$ will be zero for all $t \geq t_e$ using the weight estimation algorithm (27). ■

Remark 2: For an unstable A, the proposed method is able to estimate the unknown topological matrix A by implementing the control input on the selected leader node.

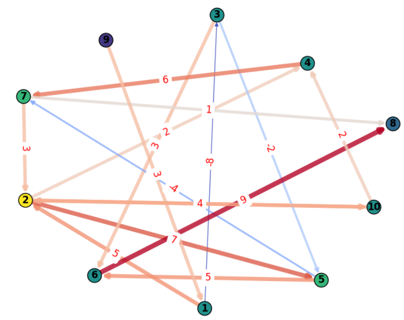

Fig. 1: Directed weighted graph

VI. Simulation

We considered 10 agents whose communication network is represented by a weighted directed graph depicted in Fig. 1. The number of the edges in Fig. 1 denotes their edges' weights. A and $\bar{A}$ is obtained from Fig. 1. Assume that we have access to $\bar{A}$. After implementing the minimum leader selection using the exhaustive algorithm, choose the leader set as $S = \{3, 6, 8\}$ using the exhaustive algorithm and $B^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

The parameter of the (19) is $\tau = 4$. The control input $u(t)$ is $\begin{bmatrix} \cos(\pi/2t - \pi/8) - \sin(\pi t + 2\pi/3) + \cos(2\pi t) + \sin(\pi t) \\ \sin(2\pi t) - \sin(3\pi t - \pi/3) + \cos(4\pi t - \pi/4) - \cos(\pi t) \\ \cos(1.5\pi t) + \sin(\pi t - \pi/4) + \sin(1.3\pi t - \pi/4) + \end{bmatrix}$.

Set A_m of the auxiliary network (18) as $\begin{bmatrix} -2 & -3 & -3 & -4 & -5 & -3 & -4 & -3 & & \\ & -2 & 1 & & & 2 & & & 1 & 1 \\ & & -1 & -1 & 1 & 1 & & 1 & & 1 \\ & & & -1 & 1 & 1 & 2 & 2 & & 1 \\ & & & & -1 & & 1 & 1 & 2 & \\ & & & & & -2 & -2 & -1 & & \\ & & & & & & -2 & -2 & 1 & 1 \\ 1 & 2 & & & & & & -2 & & \\ & 1 & 2 & & & -2 & & & & \\ & -2 & & -2 & & & & 1 & & 1 \end{bmatrix}$. We set the initial value as $x(0) = [0.7; 1.5; 1.4; 0.1; 0.6; 1.1; 0.2; 2; 1; 3]$. The Laplacian matrix A has a positive eigenvalue 3, which makes the network (1) unstable. We obtained the topology estimation time $t_e = 1.02s$ when setting $\gamma = 1e - 15$ in (8) and $\|\tilde{A}\| = 4.1e - 03$, which verifies Theorem 4. When setting different γ, we get different t_e. The smaller γ we set, the

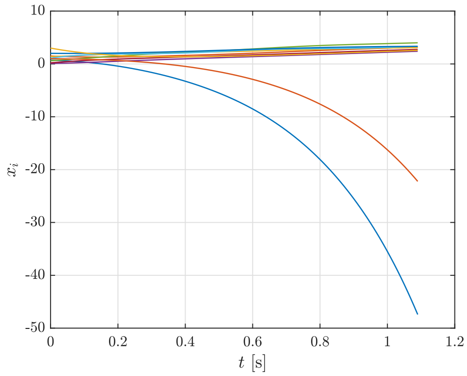

Fig. 2: The evolution of state x_i

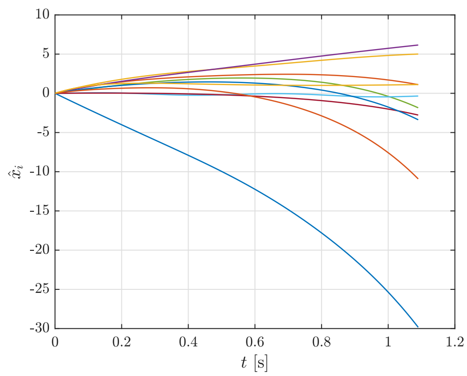

Fig. 3: The evolution of $\hat{x}_i$ in the auxiliary network

smaller t_e it is. The trajectory of state x is displayed in Fig.2 and the evolution of $\hat{x}$ of the auxiliary network is shown in Fig.3 during $t \in [0, 1.1]s$. From them, one can witness that the state x and $\hat{x}$ of the auxiliary network are still bounded before the topology estimation is achieved.

VII. Conclusion

This paper considers leader selection and design for accurate topology estimation and proposes a framework that combines leader selection and input design to reduce the number of nodes required for excitation while ensuring reliable topology estimation. Two topology estimation algorithms are presented: one determines the unique solution of the topological matrix for a symmetric graph and identifies the subspace where the topological matrix resides for a general graph, while the other guarantees that the topological matrix estimation errors converge to zero in finite time. Future work will focus on incorporating the prior pattern matrix into topological matrix estimation and exploring its impact on the subspace where the topological matrix resides.

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