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On Topological Conditions to Maintain Leader-Follower Connectivity in Double-Integrator Multi-Agent Systems

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***Abstract*— In this work, a set of sufficient conditions that guarantee consensus towards a pre-specified target state in double-integrator leader-follower proximity-based networks are derived. Since only the leader agents are aware of the global objective and proximity-based communication between agents is considered, the follower agents must not lose contact to the leaders. We establish a connectivity analysis framework which is used to show that the initial network topology is maintained if the ratio of leaders-to-followers and the magnitude of the goal attraction force experienced by the leaders are below certain bounds. Various network topologies are examined, starting from a complete graph and extending to incomplete graphs. The theoretical results are illustrated by simulations.**

I. Introduction

Control of networked multiagent systems has earned more and more attention during the last decades. Great progress in wireless communication technology and robotics leading to continuously smaller, cheaper and more powerful mobile robots explain the growing interest in such networked systems. There are various real-world applications, such as exploration, surveillance and other cooperative tasks that can be tackled efficiently utilising a group of many simple agents.

The achievement of many group objectives naturally relies on a consistent "opinion" of all agents. For this reason, consensus algorithms are an extensively studied field in modern control theory; group objectives and applications that rely on consensus algorithms are, e.g., sensor networks [1]–[3], attitude alignment [4], [5] and formation control [6], just to name a few. Of particular interest in this work is flocking [7]–[10] and the rendezvous problem [11], [12]. Adequate modeling of many real world systems requires double-integrator dynamics; [7], [13], [14] propose a consensus protocol for double-integrator agents.

In order to control a networked system, so called leader or anchor agents can be applied. Those agents do not abide by the same control law as all other agents and may have access to additional information. The benefit of having only some leader agents is due to the fact that only those agents need to be equipped with specific sensors able to tell, e.g., absolute global positions. Many authors extended standard consensus protocols with leader agents, see for example [7], [11], [15], [16].

Technical limitations, such as limited operating distance of (wireless) communication systems or sensors, allow two agents to communicate only if they are close enough. This gives rise to the issue of connectivity maintenance. There are various results that achieve connectivity maintenance directly through the control law, e.g., [17]–[21]. This, however, requires all agents to be equipped with a particular control law explicitly incorporating the connectivity maintenance objective.

A novel approach to consensus and connectivity maintenance is presented in [11]. The authors provide indirect metrics on parameters and initial conditions for a single-integrator leader-follower proximity-based network, which inherently ensure connectivity maintenance and consensus of leaders and followers. In contrast to the previously mentioned approaches, the approach of [11] does not require a particular version of the consensus protocol such as infinite inter-agent potentials. Inspired by this approach, the paper in hand aims at finding corresponding conditions for proximity-based networks of double-integrator agents that inherently ensure consensus and maintenance of the initial connectivity of the state dependent interconnection graph. This work relies on the standard and well-studied consensus protocol introduced in [13]. Hence, the contribution of this work is *not* a further leader-follower consensus protocol with connectivity maintenance abilities, but an analysis framework to assess a given initial network topology with regard to connectivity maintenance and achievement of consensus. Based on this, several initial network topologies are analyzed and sufficient conditions for connectivity maintenance and consensus are synthesized. Besides making statements about connectivity maintenance and consensus for a given network configuration, these conditions might be used to initialize a group of leader and follower agents appropriately.

The remainder of this work is organized as follows: In Section II, a detailed system description is given. Section III elaborates sufficient conditions for various network topologies and their effectiveness is illustrated with simulations in Section IV. Finally, a conclusion is given in Section V.

II. System and Problem Statement

Consider N agents evolving in $\mathbb{R}^n$. All agents obey double-integrator dynamics

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= u_i\end{aligned} \qquad i \in \mathcal{N} = \{1, ..., N\}. \tag{1}$$

Since leader and follower agents are distinguished, every agent belongs either to the subset of leaders $\mathcal{N}^l$ or to the subset of followers $\mathcal{N}^f$, with the number of agents in each set $N_l = |\mathcal{N}^l|$ and $N_f = |\mathcal{N}^f|$. By the fact that $\mathcal{N}^l \cup \mathcal{N}^f = \mathcal{N}$

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and $\mathcal{N}^l \cap \mathcal{N}^f = \emptyset$, it holds $N = N_l + N_f$. Due to technical limitations of the sensors, every agent has a limited sensing range of $\Delta > 0$ in terms of the Euclidean norm. Thus, every agent can only access the states of its neighbors within this range given by the neighbor set $\mathcal{N}_i = \{j \in \mathcal{N} \mid \|x_i - x_j\| \leq \Delta\}$. The resulting interaction topology of all agents can be modeled as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices (or nodes) $\mathcal{V} = \{1, ..., N\}$ representing agents and edges $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid \|x_i - x_j\| \leq \Delta\}$ representing active communication links between agents. A graph $\mathcal{G}$ is called *complete* if any two vertices are neighbors. A *path* from vertex i to vertex j is a sequence of distinct neighboring vertices starting from i and ending at j. The graph $\mathcal{G}$ is said to be *connected* if there exists a path between any two vertices $i, j \in \mathcal{V}$. In the present work, we consider undirected graphs, i.e., $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$. A thorough presentation of algebraic graph theory can be found in [22].

Definition 1 (Goal attraction force): Let $d_i \in \mathbb{R}^n$ be the deviation of agent $i \in \mathcal{N}^l$ from a desired target state. Define the *goal attraction force* $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(d_i) = \begin{cases} -\nabla_{d_i} F(\|d_i\|) & \|d_i\| > 0 \\ 0 & \|d_i\| = 0 \end{cases} \tag{2}$$

and denote with $f_{\max}$ the maximum magnitude it can take, i.e., $\|f(d_i)\| \leq f_{\max}$. The scalar valued function $F : [0, \infty) \to [0, \infty)$ called *goal attraction potential* is a differentiable class $\mathcal{K}$ function of the deviation $\|d_i\|$. Continuity of $f(d_i)$ is guaranteed by imposing $\lim_{\|d_i\| \to 0} \frac{dF}{d\|d_i\|} \frac{1}{\|d_i\|} < \infty$ to ensure $\lim_{\|d_i\| \to 0} \nabla_{d_i} F(\|d_i\|) = 0$.
An example for such a goal attraction potential is given in Section IV. In the considered application, all agents should finally converge to the same *predefined* target state induced by the leader agents, thus either move at a desired velocity v_d or remain at a desired position x_d. Depending on this group objective, two different control laws are considered, namely the *relative* and the *absolute damping protocol*, which were both introduced in [13]. In a target velocity seeking group, all agents obey the *relative damping protocol*; to account for the fact that the target state is only available to leader agents, we have for all follower agents $i \in \mathcal{N}^f$

$$u_i = -\sum_{j \in \mathcal{N}_i} \big((x_i - x_j) + \gamma(v_i - v_j)\big) \tag{3a}$$

with $\gamma > 0$ and for all leader agents $i \in \mathcal{N}^l$

$$u_i = -\sum_{j \in \mathcal{N}_i} \big((x_i - x_j) + \gamma(v_i - v_j)\big) + f(v_i - v_d). \tag{3b}$$

In the case of a target position seeking group, the *absolute damping protocol* is used, which replaces the velocity alignment term in (3) by an absolute damping term

$$u_i = -\gamma v_i - \sum_{j \in \mathcal{N}_i} (x_i - x_j) \qquad \forall i \in \mathcal{N}^f \tag{4a}$$

$$u_i = -\gamma v_i - \sum_{j \in \mathcal{N}_i} (x_i - x_j) + f(x_i - x_d) \quad \forall i \in \mathcal{N}^l. \tag{4b}$$

For the overall system, the state variables can be written in stacked form as $x = [x_1^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^{Nn}$ and $v = [v_1^\top, \ldots, v_N^\top]^\top \in \mathbb{R}^{Nn}$. This form of stacking is also used for any other variable shared by all agents. To keep notation simple, all calculations are carried out for the one-dimensional case. Extensions to an arbitrary dimension n are (due to linearity) straightforward by application of the Kronecker product.

The protocols (3) and (4) are well-known to achieve consensus for leaderless networks if the underlying communication graph is connected at all times [10], [13], [14]. In the considered case including leader agents, consensus should be achieved towards the leader induced target position or target velocity. These extensions are rather straightforward, whereas the proofs of the following theorems are omitted in this paper due to the space restrictions. *Target velocity consensus* denotes the situation in which all agents move, concentrated to a single point, at target velocity v_d, i.e., $x_i = x_j, v_i = v_d$ for all $i, j \in \mathcal{N}$.

Theorem 1: A network of leader-follower agents of dynamics (1) obeying the relative damping protocol (3) asymptotically achieves target velocity consensus if the communication graph is connected at all times.
Target position consensus denotes the situation in which all agents remain at the target position x_d, i.e. $x_i = x_d$, $v_i = 0$ for all $i \in \mathcal{N}$.

Theorem 2: A network of leader-follower agents of dynamics (1) obeying the absolute damping protocol (4) asymptotically achieves target position consensus if the communication graph is connected at all times.

III. Sufficient Conditions for Consensus in Proximity-Based Networks

In this section we establish sufficient conditions for consensus and connectivity maintenance for various initial topologies of double-integrator leader-follower proximity-based networks. First, we provide a topology-based framework to analyze connectivity of any two initially connected agents. Based on this, we derive later sufficient conditions for consensus and connectivity maintenance for various specific initial network topologies.

A. Connectivity Analysis Framework

Consider relative agent states $\epsilon_{ij} := [x_{ij}, v_{ij}]^\top$ with $x_{ij} := x_i - x_j$ and $v_{ij} := v_i - v_j$. Two initially connected agents in a proximity-based network stay connected if their relative states remain in the set $\Omega = \{(x_{ij}, v_{ij}) \mid \|x_{ij}\| \leq \Delta\}$ which is invariant under the relative agent dynamics

$$\begin{aligned} \dot{x}_{ij} &= v_{ij} \\ \dot{v}_{ij} &= u_i - u_j \end{aligned} \qquad (i,j) \in \mathcal{E}. \tag{5}$$

Definition of such a set Ω is elaborated in the following and sufficient conditions to render it invariant under dynamics (5) are presented. To this end, assume that the relative agent dynamics (5) can be expressed in the form

$$\dot{\epsilon}_{ij} = \begin{pmatrix} \dot{x}_{ij} \\ \dot{v}_{ij} \end{pmatrix} = A\epsilon_{ij} + \underbrace{\begin{pmatrix} 0 \\ \alpha_{ij} \end{pmatrix}}_{\bar{\alpha}_{ij}} \tag{6}$$

with A being a Hurwitz matrix and $\bar{\alpha}_{ij}$ considered as disturbance which will become more clear later. The subsequent

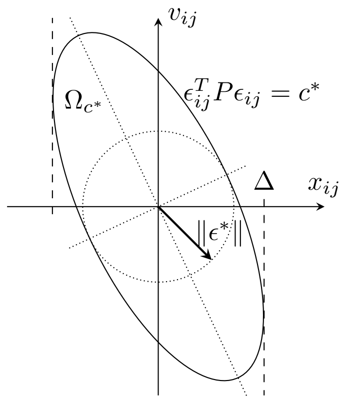


Fig. 1: Exemplary invariant set Ω_{c^*} in the ϵ_{ij}-plane.

procedure uses standard invariance arguments. It is noted, however, that this procedure basically confirms and exploits *input-to-state stability (ISS)* [23, Theorem 4.19] for the system of relative agent dynamics (6) with respect to the disturbance $\bar{\alpha}_{ij}$.

With A being Hurwitz, the Lyapunov equation $A^\top P + PA = -Q$ yields for any positive definite matrix Q a Lyapunov function candidate for the system, namely $V_{ij}(\epsilon_{ij}) = \epsilon_{ij}^\top P \epsilon_{ij}$ (with $P = \left[\begin{smallmatrix} P_1 & P_2 \\ P_2 & P_3 \end{smallmatrix}\right]$ symmetric and positive definite). Derivation along the system trajectories yields

$$\begin{aligned} \dot{V}_{ij} &= -\epsilon_{ij}^\top Q \epsilon_{ij} + 2\epsilon_{ij}^\top P \bar{\alpha}_{ij} \\ &\leq -\lambda_{\min}(Q)\|\epsilon_{ij}\| \left(\|\epsilon_{ij}\| - 2\|\alpha_{ij}\| \frac{\sqrt{P_2^2 + P_3^2}}{\lambda_{\min}(Q)} \right) \\ &\leq 0 \quad \text{for} \quad \|\epsilon_{ij}\| \geq 2\|\alpha_{ij}\| \frac{\sqrt{P_2^2 + P_3^2}}{\lambda_{\min}(Q)}. \end{aligned} \tag{7}$$

Choosing $c = c^*$ such that

$$\max_{\epsilon_{ij}^\top P \epsilon_{ij} = c^*} \|x_{ij}\| \leq \Delta \tag{8}$$

results in $c^* = \left(P_1 - \frac{P_2^2}{P_3}\right) \Delta^2$ and completes the definition of the set $\Omega_{c^*} = \{\epsilon_{ij} \mid \epsilon_{ij}^\top P \epsilon_{ij} \leq c^*\}$. According to (7), the set Ω_{c^*} is invariant under system dynamics (6) if it holds $\|\epsilon_{ij}\| \geq 2\|\alpha_{ij}\| \frac{\sqrt{P_2^2+P_3^2}}{\lambda_{\min}(Q)}$ for all $\epsilon_{ij} \in \partial\Omega_{c^*}$. This puts a constraint on the magnitude of the disturbance $\|\alpha_{ij}\|$, namely

$$\|\alpha_{ij}\| \leq \frac{\lambda_{\min}(Q)}{2\sqrt{P_2^2 + P_3^2}} \|\epsilon^*\| = \frac{\lambda_{\min}(Q)}{2\sqrt{P_2^2 + P_3^2}} \sqrt{\frac{c^*}{\lambda_{\max}(P)}} \tag{9}$$

with $\epsilon^* \in \partial\Omega_{c^*}$ such that $||\epsilon^*|| \leq ||\epsilon_{ij}||$ for all $\epsilon_{ij} \in \partial\Omega_{c^*}$. An exemplary set Ω_{c^*} is shown in Figure 1.

Theorem 3: Two agents whose dynamics can be written as (6) stay connected at all times if their initial conditions are chosen from the set $\Omega_{c^*} = \{\epsilon_{ij}^\top P \epsilon_{ij} \leq c^*\}$, with P being the unique solution of the Lyapunov equation for any positive definite matrix Q, c^* such that (8) holds and $\|\alpha_{ij}\|$ being bounded by (9).

Remark: If we are dealing with several (completely connected) groups of agents at the same time, we add subindices to refer to the respective variables $A_a, \alpha_{a_{ij}}$, P_a, Q_a, c_a^*, ϵ_a^*, $\Omega_{c^*}^a$ determined for agent group $\mathcal{N}^a$.

B. Consensus and Connectivity Maintenance for Specific Initial Topologies

This section uses the framework established before to examine double-integrator leader-follower proximity-based networks of different initial topologies. Sufficient conditions depending on the initial topology are stated, which ensure connectivity maintenance and consensus to the leader induced target state. For each considered topology in III-B.1–III-B.3 we examine every pair of initially connected agents with the presented connectivity analysis framework and require them to stay connected at all times in order to guarantee connectivity maintenance and consensus. The influence of the goal attraction force on the leaders and interconnections not part of the initial topology are treated as disturbance. This lets us derive sufficient conditions which guarantee connectivity maintenance and consensus of the respective leader-follower proximity-based network for each initial topology. It is noted that it might be quite restrictive to maintain the initial topology exactly, particularly to also require pairs of leaders to remain connected at all times.

Intuitively, there are two parameters that need to be constrained in order to maintain the initial topology in leader-follower proximity-based networks, namely

1) the magnitude of the goal attraction force pulling the leader agents towards their common reference and
2) initial conditions of all agents.

In this context, the term *proper complete connectivity* is introduced.

Definition 2: A group of agents $\mathcal{N}^a$ that are initially completely connected are said to be *properly completely connected* if the initial relative state of any two agents belongs to the invariant set $\Omega_{c^*}^a$, i.e., $\epsilon_{ij}(t_0) = [x_{ij}(t_0), v_{ij}(t_0)]^\top \in \Omega_{c^*}^a$ for all $i, j \in \mathcal{N}^a$ with $\Omega_{c^*}^a = \{\epsilon_{ij} \mid \epsilon_{ij}^\top P_a \epsilon_{ij} \leq c_a^*\}$.

1) Complete Graph: Consider a network of leader and follower agents aiming for velocity consensus, thus the relative damping protocol (3) is applied. In a complete graph topology, relative agent dynamics are

$$\dot{\epsilon}_{ij} = \underbrace{\begin{pmatrix} 0 & 1 \\ -N & -N\gamma \end{pmatrix}}_{A} \epsilon_{ij} + \underbrace{\begin{pmatrix} 0 \\ \bar{f}_i(v_i - v_d) - \bar{f}_j(v_j - v_d) \end{pmatrix}}_{\alpha_{ij}} \tag{10}$$

with $\bar{f}_i := 0$ for $i \in \mathcal{N}^f$ and $\bar{f}_i := f$ for $i \in \mathcal{N}^l$. These dynamics are of the same form as (6), with matrix A being Hurwitz. Thus, Theorem 3 provides conditions for connectivity maintenance between all pairs of agents. Three different kinds of interconnections have to be examined:

Follower-follower connections: In this case $\alpha_{ij} = 0$ and thus condition (9) is fulfilled for any two follower agents. Correct choice of their initial conditions from Ω_{c^*} (thus ensuring proper complete connectivity) suffices to keep connectivity of any two followers in the complete graph by virtue of Theorem 3.

Leader-follower connections: For the interconnection of a leader and a follower agent we have $\|\alpha_{ij}\| \leq f_{\max}$ which turns by (9) into an additional condition on $f_{\max}$, namely $f_{\max} \leq \frac{\lambda_{\min}(Q)}{2\sqrt{P_2^2+P_3^2}} \|\epsilon^*\|$.

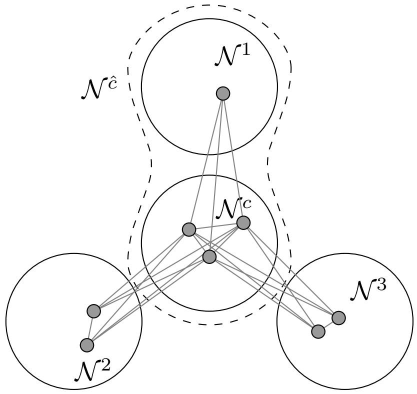


Fig. 2: Exemplary topology of an incomplete interconnection graph of followers.

Leader-leader connections: Analogously, for two leader agents, we get $\|\alpha_{ij}\| \leq 2f_{\max}$ and thus the condition $f_{\max} \leq \frac{\lambda_{\min}(Q)}{4\sqrt{P_2^2+P_3^2}}\|\epsilon^*\|$.

These conditions maintain all initial links (and thus keep the network graph complete at all times). From Theorem 1 it is known that target velocity consensus is achieved.

Theorem 4: Consider a group of double-integrator leader-follower agents obeying the relative damping protocol (3). Connectivity maintenance and target velocity consensus is achieved if the magnitude of the goal attraction force is bounded by $f_{\max} \leq \frac{\lambda_{\min}(Q)}{4\sqrt{P_2^2+P_3^2}}\|\epsilon^*\|$ and the initial graph is properly completely connected.

Remark: The bound $f_{\max}$ on the goal attraction force depends *neither on the state nor on the target*. Thus, choosing an adequately bounded function according to Definition 1 enables a priori definition of a suitable goal attraction force.

Remark: Since the result relies solely on an upper bound of the goal attraction force $f_{\max}$, the theorem also applies to target position consensus with a slight change of matrix A, according to the absolute damping protocol (4).

2) Incomplete graph of followers: Subsequently, we treat the case of position consensus. First, a network of only follower agents in a specific initial interconnection topology is investigated. This particular topology is illustrated in Figure 2 and consists of $K+1$ groups of agents $\mathcal{N}^1, \mathcal{N}^2, ..., \mathcal{N}^K, \mathcal{N}^c$. All of these groups are disjoint and their union is the entire group of agents $\mathcal{N}^f = \mathcal{N}$. Additionally, for $k = 1, ..., K$, each union $\mathcal{N}^c \cup \mathcal{N}^k$ is assumed to be properly completely connected.

Theorem 5: Let the sets of follower agents $\mathcal{N}^1, \mathcal{N}^2, ..., \mathcal{N}^K, \mathcal{N}^c$ be $K+1$ disjoint sets and their union be $\mathcal{N}^f = \mathcal{N}$. All agents obey dynamics (1) with the absolute damping protocol (4a). If

1) each union $\mathcal{N}^c \cup \mathcal{N}^k$ is properly completely connected for all $k = 1, \ldots, K$ and
2) $N \leq \dfrac{\lambda_{\min}(Q_{\hat{c}})}{2\Delta\sqrt{P_{\hat{c}_2}^2 + P_{\hat{c}_3}^2}}\|\epsilon_{\hat{c}}^*\| + |\mathcal{N}^{\hat{c}}|$

with $\mathcal{N}^{\hat{c}} := \mathcal{N}^c \cup \operatorname{argmin}_{\mathcal{N}^k}\{|\mathcal{N}^k|\}$, then consensus and connectivity maintenance is achieved.

Proof: The unions of $\mathcal{N}^c$ and any other agent group $\mathcal{N}^k$ (called group pairs) are all properly completely connected by assumption. This assumption ensures connectivity maintenance within these complete graphs *separately*. During evolution of the system, additional links between these separate complete graphs are created which are considered as disturbance. It has to be shown that this disturbance does not break the initial topology. Examine the group pair $\mathcal{N}^{\hat{c}}$ containing the least number of agents. Intuitively, for this group pair, the bound on the disturbance α_{ij} in order to maintain connectivity is the lowest. The relative dynamics of two agents in this group, i.e., for $i, j \in \mathcal{N}^{\hat{c}}$, can be formulated as

$$\begin{aligned} \dot{x}_{ij} &= v_{ij} \\ \dot{v}_{ij} &= -\gamma v_{ij} - \sum_{k\in\mathcal{N}_i}(x_i - x_k) + \sum_{k\in\mathcal{N}_j}(x_j - x_k) \qquad (11) \\ &= -|\mathcal{N}^{\hat{c}}|x_{ij} - \gamma v_{ij} \underbrace{-\sum_{k\in\mathcal{N}_i\setminus\mathcal{N}^{\hat{c}}}(x_i - x_k) + \sum_{k\in\mathcal{N}_j\setminus\mathcal{N}^{\hat{c}}}(x_j - x_k)}_{\alpha_{\hat{c}_{ij}}}. \end{aligned}$$

Note that these dynamics are again of form (6) with the influence of additional agents from $(\mathcal{N}\setminus\mathcal{N}^{\hat{c}})$ considered as the disturbance $\alpha_{\hat{c}_{ij}}$

$$\dot{\epsilon}_{ij} = \underbrace{\begin{pmatrix} 0 & 1 \\ -|\mathcal{N}^{\hat{c}}| & -\gamma \end{pmatrix}}_{A_{\hat{c}}} \epsilon_{ij} + \begin{pmatrix} 0 \\ \alpha_{\hat{c}_{ij}} \end{pmatrix}. \qquad (12)$$

Thus, Theorem 3 can be applied to ensure connectivity maintenance of the considered group. This requires the disturbance $\alpha_{\hat{c}_{ij}}$ to be bounded as $\|\alpha_{\hat{c}_{ij}}\| \leq \frac{\lambda_{\min}(Q_{\hat{c}})}{2\sqrt{P_{\hat{c}_2}^2+P_{\hat{c}_3}^2}}\|\epsilon_{\hat{c}}^*\|$. An upper bound for $\|\alpha_{\hat{c}_{ij}}\|$ in (11) can be obtained by Lemma 6 given below this proof. By this lemma, $\|\alpha_{\hat{c}_{ij}}\|$ can be bounded as $\|\alpha_{\hat{c}_{ij}}\| \leq |\mathcal{N}\setminus\mathcal{N}^{\hat{c}}|\Delta = (N - |\mathcal{N}^{\hat{c}}|)\Delta$. Thus, the condition $N \leq \frac{\lambda_{\min}(Q_{\hat{c}})}{2\Delta\sqrt{P_{\hat{c}_2}^2+P_{\hat{c}_3}^2}}\|\epsilon_{\hat{c}}^*\| + |\mathcal{N}^{\hat{c}}|$ ensures preservation of complete connectivity of the smallest group pair $\mathcal{N}^{\hat{c}}$. What remains to discuss is that this condition also guarantees connectivity maintenance of all other group pairs. This holds true by the same arguments as before and by the fact that $|\mathcal{N}^c \cup \mathcal{N}^k| \geq |\mathcal{N}^{\hat{c}}|$ for all $k = 1, ..., K$. Thus, the initial topology is maintained, which concludes the proof of consensus and connectivity maintenance. ■

Remark: Condition 2) does not limit the number of agents, but only requires the total number of agents and the number of agents belonging to $\mathcal{N}^{\hat{c}}$ to be in an appropriate ratio.

Lemma 6: Consider two neighboring agents i and j and the set of agents $\mathcal{N}^a$. Let $\mathcal{N}_k^a$ denote the subset of agents from $\mathcal{N}^a$ that are neighboring agent k, i.e., $\mathcal{N}_k^a = \mathcal{N}^a \cap \mathcal{N}_k$. Then the following estimation holds

$$\Big\| \sum_{k\in\mathcal{N}_i^a}(x_i - x_k) - \sum_{k\in\mathcal{N}_j^a}(x_j - x_k)\Big\| \leq |\mathcal{N}^a|\Delta.$$

Proof: Define the set of common neighbors $\mathcal{N}_c^a = \mathcal{N}_i^a \cap \mathcal{N}_j^a$, thus all those agents from $\mathcal{N}^a$ which are simultaneously neighboring agent i and j. The given term can then

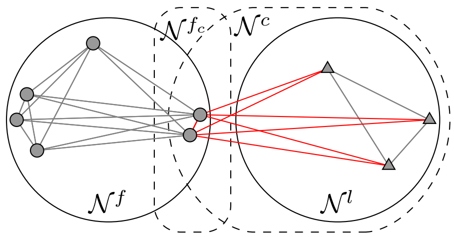


Fig. 3: Exemplary topology of an incomplete leader-follower interconnection graph.

be rewritten as

$$\Big\| \sum_{k\in\mathcal{N}_c^a}(x_i - x_j) + \sum_{k\in\mathcal{N}_i^a\setminus\mathcal{N}_c^a}(x_i - x_k) - \sum_{k\in\mathcal{N}_j^a\setminus\mathcal{N}_c^a}(x_j - x_k)\Big\|$$
$$\le |\mathcal{N}_c^a|\Delta + (|\mathcal{N}_i^a| - |\mathcal{N}_c^a|)\Delta + (|\mathcal{N}_j^a| - |\mathcal{N}_c^a|)\Delta \le |\mathcal{N}^a|\Delta.$$

This calculation uses the fact that the distance between neighboring agents is always smaller than Δ as well as the fact that the number of individual neighbors of agents i and j, $(|\mathcal{N}_i^a| - |\mathcal{N}_c^a| + |\mathcal{N}_j^a| - |\mathcal{N}_c^a|)$, plus the number of common neighbors $|\mathcal{N}_c^a|$ is limited by the number of agents $|\mathcal{N}^a|$. ■

3) Incomplete leader-follower graph: In this scenario, an initial topology of a leader-follower proximity-based network is studied, in which the group of followers is properly completely connected. Consider a subset $\mathcal{N}^{f_c} \subseteq \mathcal{N}^f$ so that $\mathcal{N}^{f_c} \cup \mathcal{N}^l =: \mathcal{N}^c$ forms a proper complete graph as well. Hence, this initial topology consists of two complete (sub)graphs corresponding to $\mathcal{N}^f$ and $\mathcal{N}^c$ as illustrated in Figure 3. In order to maintain this topology, all links within these two groups have to be maintained. Again, target position consensus under the absolute damping protocol (4) is examined.

Follower-follower connection: Consider the interconnection between two arbitrary follower agents $i, j \in \mathcal{N}^f$. Their relative dynamics are

$$\begin{aligned}\dot{x}_{ij} &= v_{ij}\\ \dot{v}_{ij} &= -N_f x_{ij} - \gamma v_{ij} \overbrace{- \sum_{k\in\mathcal{N}_i^l}(x_i - x_k) + \sum_{k\in\mathcal{N}_j^l}(x_j - x_k)}^{\alpha_{f_{ij}}}.\end{aligned} \tag{13}$$

The dynamics are again of form (6) with the influence of leader agents considered as disturbance

$$\dot{\epsilon}_{ij} = \underbrace{\begin{pmatrix} 0 & 1 \\ -N_f & -\gamma \end{pmatrix}}_{A_f}\epsilon_{ij} + \begin{pmatrix} 0 \\ \alpha_{f_{ij}} \end{pmatrix}. \tag{14}$$

According to Lemma 6, $\|\alpha_{f_{ij}}\|$ can be bounded as $\|\alpha_{f_{ij}}\| \le N_l\Delta$. Consequently, condition (9) from Theorem 3 puts a limit on the ratio of leaders-to-followers, since the right-hand side of the equation scales with the number of followers

$$N_l \le \frac{\lambda_{\min}(Q_f)}{2\Delta\sqrt{P_{f_2}^2 + P_{f_3}^2}}\|\epsilon_f^*\|. \tag{15}$$

Connection within $\mathcal{N}^c$: Consider the interconnection between two agents within the group $\mathcal{N}^c$, i.e., $i, j \in \mathcal{N}^c$. Their relative dynamics are

$$\dot{\epsilon}_{ij} = \underbrace{\begin{pmatrix} 0 & 1 \\ -N_c & -\gamma \end{pmatrix}}_{A_c}\epsilon_{ij} + \begin{pmatrix} 0 \\ \alpha_{c_{ij}} \end{pmatrix} \tag{16}$$

where

$$\begin{aligned}\alpha_{c_{ij}} = &-\sum_{k\in\mathcal{N}_i\setminus\mathcal{N}^c}(x_i - x_k) + \sum_{k\in\mathcal{N}_j\setminus\mathcal{N}^c}(x_j - x_k)\\ &+ \bar{f}_i(x_i - x_d) - \bar{f}_j(x_j - x_d)\end{aligned} \tag{17}$$

with $\bar{f}_i := 0$ for $i \in \mathcal{N}^f$ and $\bar{f}_i := f$ for $i \in \mathcal{N}^l$. An upper bound for $\|\alpha_{c_{ij}}\|$ is again obtained from Lemma 6. This provides the bound $\|\alpha_{c_{ij}}\| \le (N_f - N_{f_c})\Delta + 2f_{\max}$ which sets an upper bound on $f_{\max}$

$$f_{\max} \le \frac{\lambda_{\min}(Q_c)}{4\sqrt{P_{c_2}^2 + P_{c_3}^2}}\|\epsilon_c^*\| - \frac{\Delta(N_f - N_{f_c})}{2}. \tag{18}$$

Theorem 7: Consider a group of double-integrator leader-follower agents such that the follower graph is properly completely connected. Let $\mathcal{N}^{f_c} \cup \mathcal{N}^l = \mathcal{N}^c$ with $\mathcal{N}^{f_c} \subseteq \mathcal{N}^f$ form a proper complete graph as well and let conditions (15) and (18) hold. Then consensus to the leader-induced target position and connectivity maintenance is achieved.

IV. Simulations

In this section, the theoretical results are illustrated in simulations for agents evolving in the plane. Snapshots of follower and leader agents (depicted by dots and triangles, respectively) and their velocity vectors are shown at several timestamps. The following goal attraction potential is chosen

$$\begin{aligned}F(\|d_i\|) &= \int \frac{2\eta}{\pi}\arctan(\mu\|d_i\|)\, d\|d_i\| \\ &= \frac{2\eta}{\pi}\left[\|d_i\|\arctan(\mu\|d_i\|) - \frac{1}{2\mu}\log(\mu^2\|d_i\|^2 + 1)\right].\end{aligned} \tag{19}$$

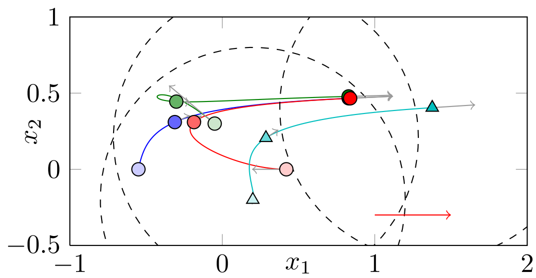
The norm of $\nabla_{d_i}F(\|d_i\|)$ is known to be bounded by the same value as $\frac{2\eta}{\pi}\arctan(\|d_i\|) \le \eta$ for all $\|d_i\|$. This facilitates limitation of the magnitude of the goal attraction force $\|f(d_i)\| \le f_{\max} = \eta$ by setting this parameter. The parameter μ is set to $\mu = 10$. Unless otherwise stated, the damping factor γ is set to $\gamma = 1$ and the choice $Q = I$ is made. The sensing range is set to $\Delta = 1$.

A. Complete graph case

First, the influence of the magnitude of the goal attraction function is investigated in the complete graph case. Consider 3 follower and 1 leader agents in a target velocity seeking scenario. All initial relative agent states are in the invariant set Ω_{c^*}, i.e., $V_{ij}(\epsilon_{ij_0}) = \epsilon_{ij_0}^\top P\epsilon_{ij_0} \le c^*$ for all $i, j \in \mathcal{N}$. Besides correct choice of initial conditions, Theorem 4 puts a bound on $f_{\max}$ which is calculated to $f_{\max} \le 2.369$. The resulting system trajectories are shown in Figure 4. For $f_{\max} = 2.3$ the initial topology is maintained and velocity consensus is finally achieved. Setting $f_{\max} = 4.5$, however, leads to loss of the complete graph topology and finally even detaches the leader from the followers.

B. Incomplete graph case

The topology investigated in this example consists of a proper complete follower subgraph and the proper complete subgraph $\mathcal{N}^c = \mathcal{N}^l \cup \mathcal{N}^{f_c}$. Sufficient conditions to maintain this specific topology and to achieve convergence of all agents to the common target position are given by



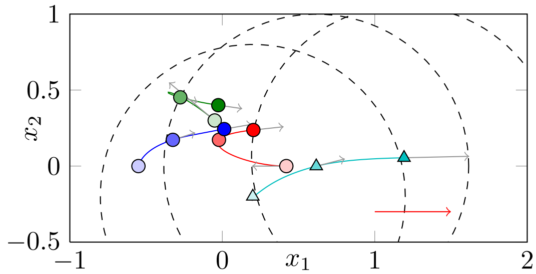


Fig. 4: Evolution of $N = 4$ agents with suitable initial conditions in a proper complete graph structure.

Theorem 7. Figure 5 shows a simulation of such a scenario with $N = 10$ agents. A total number of $N_f = 7$ follower agents and a damping factor of $\gamma = 9$ permits $N_l = 3$ leader agents and requires $N_{f_c} = 4$. The bound on the magnitude of the goal attraction force is $f_{\max} \leq 1.012$. All initial relative agent states are chosen according to the requirements. Connectivity maintenance and consensus to the leader induced target position is achieved.

V. Conclusion

In this work we presented sufficient conditions which guarantee consensus and connectivity maintenance in double-integrator leader-follower proximity-based networks based on the initial topology. The established sufficient conditions serve as an analysis tool to predict whether a given group of agents stays connected and reaches consensus to a target which is only known to a subset of the agents. On the other hand, these sufficient conditions also advice how to initialize double-integrator leader-follower proximity-based networks in order to guarantee connectivity maintenance and consensus. Bounds on the magnitude of the goal attraction force and on the ratio of leaders-to-followers turned out to be the crucial parameters.

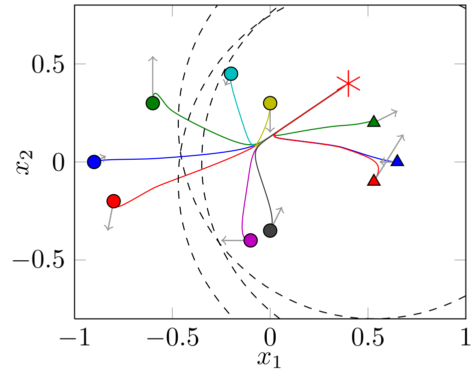


Fig. 5: Evolution of $N = 10$ agents in an initially connected but not complete graph structure.

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