Feedback Control Strategies for Multi-Agent Systems under a Fragment of Signal Temporal Logic Tasks ⋆

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Abstract

Multi-agent systems under temporal logic tasks have great potential due to their ability to deal with complex tasks. The control of these systems, however, poses many challenges and the majority of existing approaches result in large computational burdens. We instead propose computationally-efficient and robust feedback control strategies for a class of systems that are, in a sense, feedback equivalent to single integrator systems, but where the dynamics are partially unknown for the control design. A bottom-up scenario is considered in which each agent is subject to a local task from a limited signal temporal logic fragment. Notably, the satisfaction of a local task may also depend on the behavior of other agents. We provide local continuous-time feedback control laws that, under some sufficient conditions, guarantee satisfaction of the local tasks. Otherwise, a local detection & repair scheme is proposed in combination with the previously derived feedback control laws to deal with infeasibilities, such as when local tasks are conflicting. The efficacy of the proposed method is demonstrated in simulations.

Key words: Multi-agent systems; formal methods-based control; signal temporal logic; robust control; autonomous systems; hybrid systems.

1 Introduction

Control of multi-agent systems is a promising research area where scholars have addressed multi-agent navigation (Dimarogonas et al., 2006), consensus (Ren and Beard, 2005), formation control (Tanner et al., 2003), and connectivity maintenance problems (Zavlanos and Pappas, 2008), see Mesbahi and Egerstedt (2010) for an overview. More recently, ideas from model checking (Baier and Katoen, 2008) have been used where the task, which is imposed on the system, is a complex temporal logic formula. Control of single-agent systems under linear temporal logic (LTL) tasks has been considered in Belta et al. (2007); Fainekos et al. (2009); Kress-Gazit et al. (2009) while the multi-agent case has been addressed in Filippidis et al. (2012); Guo and Dimarogonas (2013, 2017); Kloetzer and Belta (2010); Tumova and Dimarogonas (2016). The aforementioned works rely on an abstraction of the workspace and the agent dynamics. This abstraction process and the subsequent plan synthesis are limited, especially for multi-agent systems, due to their high computational complexity. For single-agent systems, probabilistically optimal and complete sampling-based methods have been proposed in Vasile and Belta (2013) and Karaman and Frazzoli (2012) to avoid these computational burdens by incrementally building and model-checking a transition system. For the multi-agent setup, two paradigms exist: one where all agents are subject to a global task (top-down) and one where each agent is subject to a local task (bottom-up). These local tasks can be obtained in two ways. Either a global task is decomposed into local ones as in Schillinger et al. (2016), which in some sense is a mix of top-down and bottom-up, or each agent is assigned a local task regardless of what other agents are assigned. Especially in the latter case it may happen that the satisfaction of a local task also depends on the behavior of other agents. By behavior of an agent we mean the corresponding agent trajectory. A challenge is hence that local tasks may be in conflict, i.e., satisfiability of each local task does not imply satisfiability of the conjunction of all local tasks. The authors in Guo and Dimarogonas (2013) find least violating solutions in these conflicting situations. Opposed to the aforemen-

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tioned works using LTL, signal temporal logic (STL) (Maler and Nickovic, 2004) is based on continuous-time signals and suited to impose continuous-time tasks. STL entails space robustness (Donzé and Maler, 2010), a form of the robust semantics (Fainekos and Pappas, 2009), which states how robustly a signal satisfies a task. These robust semantics are mainly used for control under STL tasks, which is, however, a difficult problem due to the nonlinear, nonconvex, noncausal, and nonsmooth semantics. Control of discrete-time single-agent systems under STL tasks has been considered in Raman et al. (2014) and Lindemann and Dimarogonas (2019); Raman et al. (2014) obtain a computationally expensive mixed integer linear program for which robust extensions have been presented in Raman et al. (2015) and Sadraddini and Belta (2015), while an extension to multi-agent systems has been reported in Liu et al. (2017). The authors in Pant et al. (2018) use smooth approximations of the robust semantics within a nonconvex optimization problem for discrete-time systems, while providing conservative continuous-time guarantees for the corresponding continuous-time system.

We consider continuous-time multi-agent systems in a bottom-up fashion. Each local task stems from a limited STL fragment and may depend on the behavior of other agents, while agents may also be dynamically coupled. The agent dynamics are described by nonlinear control-affine systems where the corresponding driftless system is feedback equivalent to a single integrator system, but where the drift term is unknown. The limited STL fragment allows to encode concave temporal tasks such as *eventually within* 5 *sec reach a region and stay there for the next* 10 *sec while staying close to other agents*. Specifications such as *eventually within* 5 *sec reach a region while always avoiding another region* induce a mix of convex and concave temporal tasks and are not permitted here. These assumptions are necessary to achieve finite time stability results under arbitrarily short deadlines and to obtain closed-form and continuous feedback control laws. We derive a robust continuous-time feedback control law that, under some sufficient conditions, guarantees satisfaction of all local tasks. When these conditions do not hold, we propose a local detection & repair scheme, expressed as a hybrid control system (Goebel et al., 2009), that detects critical events to repair them in a three-stage procedure. We discretize neither the environment nor the agent dynamics in space or time. To the best of our knowledge, this is the first approach not making use of discretizations in space or time and directly providing continuous-time satisfaction guarantees. Our main contribution is a robust, abstraction-free, and computationally-efficient control strategy for coupled multi-agent systems that finds least violating solutions for conflicting local specifications. This paper is an extension of Lindemann and Dimarogonas (2018).

Section 2 presents preliminaries and the problem definition. Section 3 states our proposed problem solution, while simulations in Section 4 demonstrate the efficacy of our method. Conclusions are given in Section 5.

2 Preliminaries

The set of integers, natural, non-negative, and positive real numbers are $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$, respectively. True and false are $\top$ and $\perp$ and $\mathbf{0}_n$ is the n-dimensional zero vector. For the column vectors $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ and instead of $\begin{bmatrix}\boldsymbol{\zeta}_1^T & \boldsymbol{\zeta}_2^T\end{bmatrix}^T$, we let $\begin{bmatrix}\boldsymbol{\zeta}_1 & \boldsymbol{\zeta}_2\end{bmatrix}$ be a column vector, while partial derivatives are assumed to be row vectors.

2.1 Signal Temporal Logic (STL)

Signal temporal logic (STL) (Maler and Nickovic, 2004) consists of predicates μ that are obtained by evaluating a continuously differentiable predicate function $h : \mathbb{R}^d \to \mathbb{R}$ as $\mu := \top$ if $h(\boldsymbol{\zeta}') \geq 0$ and $\mu := \perp$ if $h(\boldsymbol{\zeta}') < 0$ for $\boldsymbol{\zeta}' \in \mathbb{R}^d$. The STL syntax is then defined as

$$\varphi \ ::= \ \top \mid \mu \mid \neg\varphi \mid \varphi' \wedge \varphi'' \mid F_{[a,b]}\varphi \mid G_{[a,b]}\varphi,$$

where φ' and φ'' are STL formulas and where $\neg$, $\wedge$, $F_{[a,b]}$, and $G_{[a,b]}$ are the negation, conjunction, eventually and always operators, respectively, with $a, b \in \mathbb{R}_{\geq 0}$. For a continuous-time signal $\boldsymbol{\zeta} : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, the satisfaction relation $(\boldsymbol{\zeta}, t) \models \varphi$ indicates if $\boldsymbol{\zeta}$ satisfies φ at time t. These STL semantics are formally defined in Maler and Nickovic (2004). Robust semantics have been introduced in Fainekos and Pappas (2009) as a robustness measure for temporal logics. Space robustness (Donzé and Maler, 2010) are robust semantics for STL and defined as

$$\begin{aligned}
\bar{\rho}^{\mu}(\boldsymbol{\zeta}, t) &:= h(\boldsymbol{\zeta}(t)) \\
\bar{\rho}^{\neg\varphi}(\boldsymbol{\zeta}, t) &:= -\bar{\rho}^{\varphi}(\boldsymbol{\zeta}, t) \\
\bar{\rho}^{\varphi'\wedge\varphi''}(\boldsymbol{\zeta}, t) &:= \min\left(\bar{\rho}^{\varphi'}(\boldsymbol{\zeta}, t), \bar{\rho}^{\varphi''}(\boldsymbol{\zeta}, t)\right) \\
\bar{\rho}^{F_{[a,b]}\varphi}(\boldsymbol{\zeta}, t) &:= \max_{t_1\in[t+a,t+b]} \bar{\rho}^{\varphi}(\boldsymbol{\zeta}, t_1) \\
\bar{\rho}^{G_{[a,b]}\varphi}(\boldsymbol{\zeta}, t) &:= \min_{t_1\in[t+a,t+b]} \bar{\rho}^{\varphi}(\boldsymbol{\zeta}, t_1).
\end{aligned}$$

The function $\bar{\rho}^{\varphi}$ determines how robustly $\boldsymbol{\zeta}$ satisfies φ at time t and it holds that $(\boldsymbol{\zeta}, t) \models \varphi$ if $\bar{\rho}^{\varphi}(\boldsymbol{\zeta}, t) > 0$. We assume a fragment of the STL introduced above. Let

$$\psi \ ::= \ \top \mid \mu \mid \neg\mu \mid \psi_{(1)} \wedge \psi_{(2)} \tag{1a}$$

$$\phi \ ::= \ F_{[a,b]}\psi \mid G_{[a,b]}\psi \mid F_{[\underline{a},\underline{b}]}G_{[\bar{a},\bar{b}]}\psi \tag{1b}$$

where ψ in (1b) and $\psi_{(1)}, \psi_{(2)}$ in (1a) are formulas of class ψ given in (1a) and where $a, \underline{a}, \bar{a}, \underline{b} \in \mathbb{R}_{\geq 0}$ and $b, \bar{b} \in \mathbb{R}_{\geq 0} \cup \infty$ with $a \leq b$, $\underline{a} \leq \underline{b}$, and $\bar{a} \leq \bar{b}$. We refer to ψ as *non-temporal (boolean) formulas* and to ϕ as *temporal formulas*. Let $\mathcal{S}$ be the set of all possible signals with time domain $\mathbb{R}_{\geq 0}$. Note that $\bar{\rho}^{\psi}(\boldsymbol{\zeta}, t)$ and $\bar{\rho}^{\phi}(\boldsymbol{\zeta}, t)$ map from $\mathcal{S} \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}$. For *non-temporal formulas* ψ, we can equivalently use $\bar{\rho}^{\psi}(\boldsymbol{\zeta}(t))$ by a change of notation where now $\bar{\rho}^{\psi}:\mathbb{R}^n\to\mathbb{R}$. The notation of $\bar{\rho}^{\psi}(\boldsymbol{\zeta}(t))$ is used to highlight that t is only contained in $\bar{\rho}^{\psi}$ through the composition of $\bar{\rho}^{\psi}$ with $\boldsymbol{\zeta}$. The non-smooth robust semantics of $\psi_{(1)}\wedge\psi_{(2)}$ are replaced by a smooth approximation, denoted by $\rho^{\phi}(\boldsymbol{\zeta},t)$. For the formulas in (1), we define

$$\begin{aligned}
\rho^{\mu}(\boldsymbol{\zeta}(t)) &:= h(\boldsymbol{\zeta}(t))\\
\rho^{\neg\mu}(\boldsymbol{\zeta}(t)) &:= -\rho^{\mu}(\boldsymbol{\zeta}(t))\\
\rho^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t)) &:= -\frac{1}{\eta}\ln\Big(\sum_{j=1}^{2}\exp\big(-\eta\rho^{\psi_{(j)}}(\boldsymbol{\zeta}(t))\big)\Big)\\
\rho^{F_{[a,b]}\psi}(\boldsymbol{\zeta},t) &:= \max_{t_1\in[t+a,t+b]}\rho^{\psi}(\boldsymbol{\zeta}(t_1))\\
\rho^{G_{[a,b]}\psi}(\boldsymbol{\zeta},t) &:= \min_{t_1\in[t+a,t+b]}\rho^{\psi}(\boldsymbol{\zeta}(t_1))\\
\rho^{F_{[\underline{a},\underline{b}]}G_{[\bar{a},\bar{b}]}\psi}(\boldsymbol{\zeta},t) &:= \max_{t_1\in[t+\underline{a},t+\underline{b}]}\min_{t_2\in[t_1+\bar{a},t_1+\bar{b}]}\rho^{\psi}(\boldsymbol{\zeta}(t_2))
\end{aligned}$$

where $\eta>0$ is a design parameter that determines the accuracy by which $\rho^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))$ approximates $\bar{\rho}^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))$. In fact, it can be shown that $\rho^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))=\bar{\rho}^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))$ as $\eta\to\infty$. Regardless of η, it holds that $\rho^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))\leq\bar{\rho}^{\psi_{(1)}\wedge\psi_{(2)}}(\boldsymbol{\zeta}(t))$, which is stated in the next lemma where the proof can be derived from Boyd and Vandenberghe (2004, p.72).

Lemma 1 *Consider a conjunction of q* non-temporal formulas *$\psi_{(j)}$ as $\psi:=\wedge_{j=1}^{q}\psi_{(j)}$ where each $\psi_{(j)}$ does not contain any further conjunctions itself. Then,*

$$\rho^{\psi}(\boldsymbol{\zeta}(t))\leq\bar{\rho}^{\psi}(\boldsymbol{\zeta}(t))\leq\rho^{\psi}(\boldsymbol{\zeta}(t))+\ln(q)/\eta.$$

Remark 1 *The fragment in* (1) *can be extended to include sequential formulas. For single-agent systems, this extension is explained in Lindemann et al. (2017) by employing a hybrid control strategy. For the multi-agent setup, this extension can be done in the same way and is omitted. Note that the fragment in* (1) *does not allow to pose reach-avoid specifications, which require at least discontinuous feedback control laws (Liberzon, 2003, Ch. 4), while we focus on continuous feedback control laws.*

2.2 A Bottom-up Approach for Multi-Agent Systems

Consider M agents modeled by an undirected graph $\mathcal{G}:=(\mathcal{V},\mathcal{E})$ (Mesbahi and Egerstedt, 2010) where $\mathcal{V}:=\{v_1,\ldots,v_M\}$ while $\mathcal{E}\in\mathcal{V}\times\mathcal{V}$ indicates communication links. At time t, let $\boldsymbol{x}_i(t)\in\mathbb{R}^{n_i}$, $\boldsymbol{u}_i(t)\in\mathbb{R}^{m_i}$, and $\boldsymbol{w}_i(t)\in\mathcal{W}_i\subset\mathbb{R}^{n_i}$ be the state, input, and additive noise of agent v_i with $\mathcal{W}_i\subset\mathbb{R}^{n_i}$ being a bounded set and

$$\dot{\boldsymbol{x}}_i(t)=f_i(\boldsymbol{x}_i(t))+f_i^c(\boldsymbol{x}(t))+g_i(\boldsymbol{x}_i(t))\boldsymbol{u}_i(t)+\boldsymbol{w}_i(t) \tag{2}$$

where $\boldsymbol{x}(t):=\begin{bmatrix}\boldsymbol{x}_1(t) & \ldots & \boldsymbol{x}_M(t)\end{bmatrix}\in\mathbb{R}^n$ with $n:=n_1+\ldots+n_M$. Also define $\boldsymbol{x}_i^{\text{ext}}(t):=\begin{bmatrix}\boldsymbol{x}_{j_1}(t) & \ldots & \boldsymbol{x}_{j_{M-1}}(t)\end{bmatrix}$ such that $v_{j_1},\ldots,v_{j_{M-1}}\in\mathcal{V}\setminus\{v_i\}$. We also define the *behavior* of agent v_i to be agent v_i's trajectory, i.e., the solution $\boldsymbol{x}_i:\mathbb{R}_{\geq 0}\to\mathbb{R}^{n_i}$ to (2). The functions f_i and f_i^c are unknown apart from a regularity assumption.

Assumption 1 *The functions $f_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i}$, $f_i^c:\mathbb{R}^n\to\mathbb{R}^{n_i}$, and $g_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i\times m_i}$ are locally Lipschitz continuous, and $g_i(\boldsymbol{x}_i)g_i(\boldsymbol{x}_i)^T$ is positive definite for all $\boldsymbol{x}_i\in\mathbb{R}^{n_i}$; $\boldsymbol{w}_i:\mathbb{R}_{\geq 0}\to\mathbb{R}^{n_i}$ is piecewise continuous.*

Remark 2 *We emphasize that f_i and f_i^c are unknown so that* (2) *is* not *feedback equivalent to $\dot{\boldsymbol{x}}_i(t)=\boldsymbol{u}_i(t)+\boldsymbol{w}_i(t)$. Note that $g_i(\boldsymbol{x}_i)g_i(\boldsymbol{x}_i)^T$ is positive definite if and only if $g_i(\boldsymbol{x}_i)$ has full row rank. This assumption captures, for instance, the dynamics of omnidirectional robots as in Section 4; f_i^c describes* given dynamical couplings *between agents, e.g., a physical connection when two or more agents collaboratively carry an object. Using $\boldsymbol{u}_i=g_i(\boldsymbol{x}_i)^T(g_i(\boldsymbol{x}_i)g_i(\boldsymbol{x}_i)^T)^{-1}f_i^u(\boldsymbol{x})+\boldsymbol{v}_i$, the dynamics $\dot{\boldsymbol{x}}_i=f_i(\boldsymbol{x}_i)+f_i^c(\boldsymbol{x})+f_i^u(\boldsymbol{x})+g_i(\boldsymbol{x}_i)\boldsymbol{v}_i+\boldsymbol{w}_i$ resemble* (2) *if $f_i^u(\boldsymbol{x})$ is locally Lipschitz continuous; $f_i^u(\boldsymbol{x})$ can be used for other objectives such as collision avoidance, consensus, formation control, or connectivity maintenance.*

Each agent $v_i\in\mathcal{V}$ is now subject to a local STL formula ϕ_i of class ϕ given in (1b), i.e., ϕ_i is of the form

$$\phi_i ::= F_{[a_i,b_i]}\psi_i \mid G_{[a_i,b_i]}\psi_i \mid F_{[\underline{a}_i,\underline{b}_i]}G_{[\bar{a}_i,\bar{b}_i]}\psi_i. \tag{3}$$

where ψ_i is a *non-temporal formula* of the form (1a). Satisfaction of ϕ_i depends on the behavior of v_i and may also depend on the behavior of agents in $\mathcal{V}\setminus\{v_i\}$. If the satisfaction of ϕ_i depends on the behavior of $v_j\in\mathcal{V}$, we say that ϕ_i depends on v_j. Equivalently, we then say that agent v_j is participating in ϕ_i; ϕ_i consequently depends on a set of agents denoted by $\mathcal{V}_i:=\{v_{j_1},\ldots,v_{j_{P_i}}\}\subseteq\mathcal{V}$ where P_i indicates the total number of agents that participate in ϕ_i. We call ϕ_i a *non-collaborative formula* if $P_i=1$. Otherwise, i.e., if $P_i>1$, we call ϕ_i a *collaborative formula*. Let $p_i:=n_{j_1}+\ldots+n_{j_{P_i}}$ and define

$$\bar{\boldsymbol{x}}_i(t):=\begin{bmatrix}\boldsymbol{x}_{j_1}(t) & \ldots & \boldsymbol{x}_{j_{P_i}}(t)\end{bmatrix}\in\mathbb{R}^{p_i}.$$

Assumption 2 *Each formula ψ_i in* (3) *is: 1) s.t. $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$ is concave and 2) well-posed in the sense that $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)>0$ implies $\|\bar{\boldsymbol{x}}_i\|\leq\bar{C}<\infty$ for some $\bar{C}\geq 0$.*

Remark 3 *Recall the syntax of class ψ formulas in* (1a) *and consider $\psi_i:=\psi_{(1)}\wedge\psi_{(2)}$. Part 1) of Assumption 2 is satisfied if $\rho^{\psi_{(1)}}(\bar{\boldsymbol{x}}_i)$ and $\rho^{\psi_{(2)}}(\bar{\boldsymbol{x}}_i)$ are concave. This assumption is needed since the controller presented in this paper uses the gradient of $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$. Local extrema may hence lead the system to get stuck. Part 2) of Assumption 2 will ensure bounded solutions and is not restrictive since $\psi_i^{Ass.2}:=(\|\bar{\boldsymbol{x}}_i\|<\bar{C})$ can be combined with ψ_i for a sufficiently large $\bar{C}$ so that $\psi_i\wedge\psi_i^{Ass.2}$ is well-posed.*

The signal $\bar{\boldsymbol{x}}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ locally satisfies ϕ_i if $(\bar{\boldsymbol{x}}_i, 0) \models \phi_i$ and ϕ_i is locally satisfiable if $\exists \bar{\boldsymbol{x}}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ such that $\bar{\boldsymbol{x}}_i$ locally satisfies ϕ_i. The signal $\boldsymbol{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ globally satisfies $\{\phi_1, \ldots, \phi_M\}$ if $\bar{\boldsymbol{x}}_i$, which is naturally contained in $\boldsymbol{x}$, locally satisfies ϕ_i for all agents $v_i \in \mathcal{V}$. The set of formulas $\{\phi_1, \ldots, \phi_M\}$ is globally satisfiable if $\exists \boldsymbol{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\boldsymbol{x}$ globally satisfies $\{\phi_1, \ldots, \phi_M\}$. Consider also the undirected graph $\mathcal{G}_d := (\mathcal{V}, \mathcal{E}_d)$ where there is an edge $(v_i, v_j) \in \mathcal{E}_d \subseteq \mathcal{V} \times \mathcal{V}$ if ϕ_i depends on v_j; $\Xi \subseteq \mathcal{V}$ is a *maximal dependency cluster* if $\forall v_i, v_j \in \Xi$ there is a path from v_i to v_j in $\mathcal{G}_d$ and $\nexists v_i \in \Xi, v_k \in \mathcal{V} \setminus \Xi$ such that there is a path from v_i to v_k. Here, a path is a sequence $v_i, v_{k_1}, \ldots, v_{k_P}, v_j$ such that $(v_i, v_{k_1}), \ldots, (v_{k_P}, v_j) \in \mathcal{E}_d$. A multi-agent system under $\{\phi_1, \ldots, \phi_M\}$ hence induces $L \leq M$ maximal dependency clusters $\bar{\Xi} := \{\Xi_1, \ldots, \Xi_L\}$.

Assumption 3 *For each Ξ_l with $l \in \{1, \ldots, L\}$ it holds that $(v_i, v_j) \in \mathcal{E}$ for each $v_i, v_j \in \Xi_l$.*

In Section 3, each agent $v_i \in \mathcal{V}$ is equipped with logical variables so that v_i is associated with a hybrid state $\boldsymbol{z}_i \in \mathcal{Z}_i \subseteq \mathbb{R}^{n_{\boldsymbol{z}}}$ where $\mathcal{Z}_i$ is a hybrid domain of dimension $n_{\boldsymbol{z}}$. Hybrid systems with internal and external inputs $\boldsymbol{u}_i^{\text{int}} \in \mathcal{U}_i^{\text{int}}$ and $\boldsymbol{u}_i^{\text{ext}} \in \mathcal{U}_i^{\text{ext}}$, respectively, have been presented in Sanfelice (2016) where $\mathcal{U}_i^{\text{int}}$ and $\mathcal{U}_i^{\text{ext}}$ are input domains. Let $\mathfrak{H}_i := \mathcal{Z}_i \times \mathcal{U}_i^{\text{int}} \times \mathcal{U}_i^{\text{ext}}$. The value of the state $\boldsymbol{z}_i$ after a jump, i.e., after an instantaneous change in $\boldsymbol{z}_i$, is denoted by $\hat{\boldsymbol{z}}_i$. A hybrid system is now a tuple $\mathcal{H}_i := (C_i, F_i, D_i, G_i)$ where $C_i \subseteq \mathfrak{H}_i$, $D_i \subseteq \mathfrak{H}_i$, $F_i : \mathfrak{H}_i \rightrightarrows \mathbb{R}^{n_{\boldsymbol{z}}}$, and $G_i : \mathfrak{H}_i \rightrightarrows \mathbb{R}^{n_{\boldsymbol{z}}}$ are the flow and jump set and the set-valued flow and jump map, respectively. The continuous and discrete dynamics are governed by

$$\begin{cases} \dot{\boldsymbol{z}}_i \in F_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) & \text{for} \quad (\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in C_i \\ \hat{\boldsymbol{z}}_i \in G_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) & \text{for} \quad (\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in D_i. \end{cases} \tag{4}$$

Solutions to (4) are parametrized by (t, j); t represents continuous time indicating flow according to $F_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ while j is a counter of the jumps that occur according to $G_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$. A detailed review can be found in Goebel et al. (2009, 2012).

2.3 Problem Statement

Let the supremum of $\bar{\rho}^{\psi_i}(\bar{\boldsymbol{x}}_i)$ and $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$ be

$$\bar{\rho}_i^{\text{opt}} := \sup_{\bar{\boldsymbol{x}}_i \in \mathbb{R}^{p_i}} \bar{\rho}^{\psi_i}(\bar{\boldsymbol{x}}_i) \text{ and } \rho_i^{\text{opt}} := \sup_{\bar{\boldsymbol{x}}_i \in \mathbb{R}^{p_i}} \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$$

where ρ_i^{opt} is in particular straighforward to compute since $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$ is continuously differentiable and concave.

Assumption 4 *ρ_i^{opt} is such that $\rho_i^{opt} > 0$.*

Assumption 4 implies that ψ_i and hence ϕ_i are locally satisfiable. Note that there always exists an η such that $\rho_i^{\text{opt}} > 0$ if $\bar{\rho}_i^{\text{opt}} > 0$ since $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i) = \bar{\rho}^{\psi_i}(\bar{\boldsymbol{x}}_i)$ as $\eta \to \infty$. We then find η by solving a convex feasibility problem by selecting $\eta > 0$ such that $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i) > 0$ for some $\bar{\boldsymbol{x}}_i \in \mathbb{R}^{p_i}$. If this feasibility problem, however, is not feasible, we set $\eta := 1$. It then holds that $\rho_i^{\text{opt}} \leq 0$ and ϕ_i is not satisfiable if $\rho_i^{\text{opt}} + \ln(q) < 0$ with q as in Lemma 1. Let $\rho_i^{\text{gap}} > 0$ be a parameter indicating how much ϕ_i is violated. If $\bar{\boldsymbol{x}}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ is such that $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0)$ for r_i with $\rho_i^{\text{opt}} - \rho_i^{\text{gap}} \leq r_i < \rho_i^{\text{opt}}$, we say that $\bar{\boldsymbol{x}}_i$ is a *least violating solution* with a given gap of ρ_i^{gap}. The goal is to derive $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)$ such that $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$ where $r_i \in \mathbb{R}$ is a robustness measure, while $\rho_i^{\max} \in \mathbb{R}$ with $r_i < \rho_i^{\max}$ is a robustness delimiter.

Problem 1 *Let η be selected for each ψ_i as instructed above. Given the parameters $\rho_i^{gap} > 0$ and $\delta_i > 0$, derive for each cluster $\Xi_l \in \bar{\Xi}$ a control strategy as follows:*
Case A*) Under the assumption that $\phi_i = \phi_j$ for each $v_i, v_j \in \Xi_l$, design $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)$ such that $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$ for each $v_i \in \Xi_l$ and where r_i is chosen such that $0 < r_i < \rho_i^{max}$ if $\rho_i^{opt} > 0$ and $r_i \geq \rho_i^{opt} - \rho_i^{gap}$ if $\rho_i^{opt} \leq 0$.*
Case B*) Otherwise, i.e., $\exists v_i, v_j \in \Xi_l$ such that $\phi_i \neq \phi_j$, assume nevertheless that each agent $v_i \in \Xi_l$ initially applies the derived control law $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)$ for Case A. Design a local online detection & repair scheme for each $v_i \in \Xi_l$ such that $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$ where either $r_i > 0$ or r_i is maximized up to a precision of $\delta_i > 0$.*

By a *maximization up to a precision of* δ_i we mean that r_i is successively reduced by δ_i whenever it turns out that $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)$ can not achieve $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$.

3 Proposed Problem Solution

Following a funnel-based control strategy (Bechlioulis and Rovithakis, 2014), we define a performance function

$$\gamma_i(t) := (\gamma_i^0 - \gamma_i^\infty) \exp(-l_i t) + \gamma_i^\infty$$

where $\gamma_i^0, \gamma_i^\infty \in \mathbb{R}_{>0}$ with $\gamma_i^0 \geq \gamma_i^\infty$ and $l_i \in \mathbb{R}_{\geq 0}$. We achieve $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$ by prescribing a temporal behavior to $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t))$ through γ_i and $\rho_i^{\max}$ as

$$-\gamma_i(t) + \rho_i^{\max} < \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t)) < \rho_i^{\max}. \tag{5}$$

If γ_i and $\rho_i^{\max}$ are chosen properly (as shown later) and if (5) holds for all $t \geq 0$, then $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$. Let $e_i(\bar{\boldsymbol{x}}_i) := \rho^{\psi_i}(\bar{\boldsymbol{x}}_i) - \rho_i^{\max}$, $\xi_i(\bar{\boldsymbol{x}}_i, t) := \frac{e_i(\bar{\boldsymbol{x}}_i)}{\gamma_i(t)}$, and

$$\epsilon_i(\bar{\boldsymbol{x}}_i, t) := S\big(\xi_i(\bar{\boldsymbol{x}}_i, t)\big),$$

where $S(\xi_i) := \ln\left(-\frac{\xi_i + 1}{\xi_i}\right)$, then (5) is equivalent to $-\gamma_i(t) < e_i(t) < 0$ and hence to $-1 < \xi_i(t) < 0$ where $e_i(t) := e_i(\bar{\boldsymbol{x}}_i(t), t)$ and $\xi_i(t) := \xi_i(\bar{\boldsymbol{x}}_i(t), t)$. Applying

$S(\cdot)$ to $-1 < \xi_i(t) < 0$ gives $-\infty < \epsilon_i(t) < \infty$ with $\epsilon_i(t) := \epsilon_i(\bar{\boldsymbol{x}}_i(t), t)$. If $\epsilon_i(t)$ is bounded for all $t \geq 0$, i.e., $\xi_i(t) \in \Omega_\xi := (-1, 0)$, then (5) holds for all $t \geq 0$. The connection between $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t))$ and $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0)$ is made by γ_i and $\rho_i^{\max}$, which need to be designed as instructed in Lindemann et al. (2017). If Assumption 4 holds, select

$$t_i^* \in \begin{cases} [a_i, b_i] & \text{if } \phi_i = F_{[a_i,b_i]}\psi_i \\ \{a_i\} & \text{if } \phi_i = G_{[a_i,b_i]}\psi_i \\ [\underline{a}_i + \bar{a}_i, \underline{b}_i + \bar{a}_i] & \text{if } \phi_i = F_{[\underline{a}_i,\underline{b}_i]}G_{[\bar{a}_i,\bar{b}_i]}\psi_i, \end{cases} \quad (6)$$

$$\rho_i^{\max} \in \Big(\max\big(0, \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(0))\big), \rho_i^{\text{opt}} \Big) \quad (7)$$

$$r_i \in (0, \rho_i^{\max}) \quad (8)$$

$$\gamma_i^0 \in \begin{cases} \big(\rho_i^{\max} - \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(0)), \infty\big) & \text{if } t_i^* > 0 \\ \big(\rho_i^{\max} - \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(0)), \rho_i^{\max} - r_i\big] & \text{else} \end{cases} \quad (9)$$

$$\gamma_i^\infty \in \big(0, \min\big(\gamma_i^0, \rho_i^{\max} - r_i\big)\big] \quad (10)$$

$$l_i = \begin{cases} 0 & \text{if } -\gamma_i^0 + \rho_i^{\max} \geq r_i \\ -\ln\big(\frac{r_i + \gamma_i^\infty - \rho_i^{\max}}{-(\gamma_i^0 - \gamma_i^\infty)}\big)/t_i^* & \text{else} \end{cases} \quad (11)$$

where it needs to hold that $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(0)) > r_i$ if $t_i^* = 0$. It now holds that $0 < r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$ if (5) holds for all $t \geq 0$. The intuition here is that by the choice of γ_i it is ensured that $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t)) \geq r_i$ for all $t \geq t_i^*$. By the choice of t_i^* it consequently holds that $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \geq r_i$. For the case that Assumption 4 does not hold and to guarantee a least violating solution with a given gap of ρ_i^{gap}, $\rho_i^{\max}$ and r_i are instead of (7) and (8) chosen as

$$\rho_i^{\max} \in \Big(\max\big(\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(0)), \rho_i^{\text{opt}} - \rho_i^{\text{gap}}\big), \rho_i^{\text{opt}} \Big) \quad (12)$$

$$r_i \in [\rho_i^{\text{opt}} - \rho_i^{\text{gap}}, \rho_i^{\max}) \quad (13)$$

Remark 4 *The choices of t_i^*, ρ_i^{max}, r_i, and γ_i allow some freedom. In practice and to avoid input saturations, however, it is advisable to avoid a steep performance function γ_i. Furthemore, a broader funnel (5) is recommended as $t \to \infty$, i.e., γ_i^∞ and $\rho_i^{\max}$ should be selected rather large.*

3.1 Global and Local Task Satisfaction Guarantees

We first present global task satisfaction guarantees.

Theorem 1 *Let Assumptions 1-4 hold. Assume for each $\Xi_l \in \bar{\Xi}$ that for all $v_i, v_j \in \Xi_l$: 1) $\phi_i = \phi_j$ and 2) $t_i^* = t_j^*$, $\rho_i^{max} = \rho_j^{max}$, $r_i = r_j$, and $\gamma_i = \gamma_j$ are chosen as in (6)-(11). If each agent $v_i \in \mathcal{V}$ applies*

$$\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t) := -\epsilon_i(\bar{\boldsymbol{x}}_i, t) g_i(\boldsymbol{x}_i)^T \frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i}, \quad (14)$$

then it holds that $0 < r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$ for all agents $v_i \in \mathcal{V}$, i.e., $\{\phi_1, \ldots, \phi_M\}$ are globally satisfied. All closed-loop signals are continuous and bounded.

PROOF. In Step A, we apply Sontag (2013, Thm. 54) and show that for each $v_i \in \mathcal{V}$ there exists a maximal solution $\xi_i(t)$ such that $\xi_i(t) := \xi_i(\bar{\boldsymbol{x}}_i(t), t) \in \Omega_\xi := (-1, 0)$ for all $t \in \mathcal{J} := [0, \tau_{\max}) \subseteq \mathbb{R}_{\geq 0}$ where $\tau_{\max} > 0$, which is the same as requiring that (5) holds for all $t \in \mathcal{J}$. Step B consists of using Sontag (2013, Prop. C.3.6) to show that $\tau_{\max} = \infty$. Note beforehand that

$$\frac{d\epsilon_i}{dt} = \frac{\partial \epsilon_i}{\partial \xi_i}\frac{d\xi_i}{dt} = \frac{-1}{\gamma_i \xi_i (1 + \xi_i)}\Big(\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - \xi_i \dot{\gamma}_i\Big) \quad (15)$$

since $\frac{\partial \epsilon_i}{\partial \xi_i} = \frac{-1}{\xi_i(1+\xi_i)}$ and $\frac{d\xi_i}{dt} = \frac{1}{\gamma_i}\big(\frac{de_i}{dt} - \xi_i \dot{\gamma}_i\big)$. It also holds that $\frac{de_i}{dt} = \frac{\partial e(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ with $\frac{\partial e_i(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}} = \frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}}$.

Step A: First, define $\boldsymbol{\xi} := \begin{bmatrix}\xi_1 & \ldots & \xi_M\end{bmatrix}$ and $\boldsymbol{y} := \begin{bmatrix}\boldsymbol{x} & \boldsymbol{\xi}\end{bmatrix}$. By inserting (14) into (2), we get $\dot{\boldsymbol{x}}_i = H_{\boldsymbol{x}_i}(\boldsymbol{x}, \xi_i)$ with

$$H_{\boldsymbol{x}_i}(\boldsymbol{x}, \xi_i) := f_i(\boldsymbol{x}_i) + f_i^{\text{c}}(\boldsymbol{x}) - \ln\Big(-\frac{\xi_i + 1}{\xi_i}\Big) g_i(\boldsymbol{x}_i) g_i(\boldsymbol{x}_i)^T \frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i} + \boldsymbol{w}_i.$$

Let $H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) := \begin{bmatrix}H_{\boldsymbol{x}_1}(\boldsymbol{x}, \xi_1) & \ldots & H_{\boldsymbol{x}_M}(\boldsymbol{x}, \xi_M)\end{bmatrix}$ so that $\dot{\boldsymbol{x}} = H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi})$. We also obtain $\frac{d\xi_i}{dt} = H_{\xi_i}(\boldsymbol{x}, \xi_i, t)$ with

$$H_{\xi_i}(\boldsymbol{x}, \xi_i, t) := \frac{1}{\gamma_i(t)}\Big(\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}} H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) - \xi_i \dot{\gamma}_i(t)\Big).$$

Let now $H_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{\xi}, t) := \begin{bmatrix}H_{\xi_1}(\boldsymbol{x}, \xi_1, t) & \ldots & H_{\xi_M}(\boldsymbol{x}, \xi_M, t)\end{bmatrix}$ so that $\dot{\boldsymbol{\xi}} = H_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{\xi}, t)$, which results in $\dot{\boldsymbol{y}} = H(\boldsymbol{y}, t)$ with $H(\boldsymbol{y}, t) := \begin{bmatrix}H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) & H_{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{\xi}, t)\end{bmatrix}$. Note that $\boldsymbol{x}(0)$ is such that $\xi_i(\bar{\boldsymbol{x}}_i(0), 0) \in \Omega_\xi := (-1, 0)$ holds for all agents $v_i \in \Xi_l$ due to the choice of γ_i^0. Next define $\Omega_i(t) := \{\bar{\boldsymbol{x}}_i \in \mathbb{R}^{p_i} \,|\, -1 < \xi_i(\bar{\boldsymbol{x}}_i, t) < 0\}$ and note that $\Omega_i(t_2) \subseteq \Omega_i(t_1)$ for $t_1 < t_2$ since γ_i is non-increasing in t. Note also that $\bar{\boldsymbol{x}}_i(0) \in \Omega_i(0)$ and that $\Omega_i(t)$ is bounded due to Assumption 2 and since γ_i is bounded. Due to Aubin and Frankowska (2009, Prop. 1.4.4) the inverse image of an open set under a continuous function is open. By defining $\xi_{i,0}(\bar{\boldsymbol{x}}_i) := \xi_i(\bar{\boldsymbol{x}}_i, 0)$, it holds that ${\xi_{i,0}}^{-1}(\Omega_\xi) = \Omega_i(0)$ is open. Next, select $v_{i_l} \in \Xi_l$ for each $l \in \{1, \ldots, L\}$ and define $\Omega_{\boldsymbol{x}} := \Omega_{i_1}(0) \times \ldots \times \Omega_{i_L}(0) \subset \mathbb{R}^n$, $\Omega_{\boldsymbol{\xi}} := \Omega_\xi \times \ldots \times \Omega_\xi \subset \mathbb{R}^M$, and $\Omega_{\boldsymbol{y}} := \Omega_{\boldsymbol{x}} \times \Omega_{\boldsymbol{\xi}} \subset \mathbb{R}^{n+M}$, which are open, non-empty, and bounded sets so that $\boldsymbol{y}(0) = \begin{bmatrix}\boldsymbol{x}(0) & \boldsymbol{\xi}(0)\end{bmatrix} \in \Omega_{\boldsymbol{y}}$. Let us check the existence conditions of solutions for the initial value problem $\dot{\boldsymbol{y}} = H(\boldsymbol{y}, t)$ with $\boldsymbol{y}(0) \in \Omega_{\boldsymbol{y}}$ and $H(\boldsymbol{y}, t) : \Omega_{\boldsymbol{y}} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+M}$: 1) $H(\boldsymbol{y}, t)$ is locally Lipschitz continuous on $\boldsymbol{y}$ since $f_i(\boldsymbol{x}_i)$, $f_i^{\text{c}}(\boldsymbol{x})$, $g_i(\boldsymbol{x}_i)$, $\epsilon_i = \ln\big(-\frac{\xi_i+1}{\xi_i}\big)$, and $\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i}$ are locally Lipschitz continuous on $\boldsymbol{y}$ for each $t \in \mathbb{R}_{\geq 0}$. 2) $H(\boldsymbol{y}, t)$ is continuous on t for each fixed $\boldsymbol{y} \in \Omega_{\boldsymbol{y}}$ due to continuity of γ_i and $\dot{\gamma}_i$. Due to Sontag (2013, Thm. 54) there exists a maximal solution $\boldsymbol{y}(t) \in \Omega_{\boldsymbol{y}}$ for all $t \in \mathcal{J} := [0, \tau_{\max}) \subseteq \mathbb{R}_{\geq 0}$ and $\tau_{\max} > 0$ and hence $\boldsymbol{\xi}(t) \in \Omega_{\boldsymbol{\xi}}$ and $\boldsymbol{x}(t) \in \Omega_{\boldsymbol{x}}$ for all $t \in \mathcal{J}$.

Step B: We show that $\tau_{\max} = \infty$ by contradiction of

Sontag (2013, Prop. C.3.6). Therefore, assume $\tau_{\max} < \infty$. Note that $\xi_i(\bar{\boldsymbol{x}}_i, t) = \xi_j(\bar{\boldsymbol{x}}_j, t)$, $\epsilon_i(\bar{\boldsymbol{x}}_i, t) = \epsilon_j(\bar{\boldsymbol{x}}_j, t)$, and $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i) = \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)$ for all $v_i, v_j \in \Xi_l$ since $\bar{\boldsymbol{x}}_i = \bar{\boldsymbol{x}}_j$ and since $\rho_i^{\max} = \rho_j^{\max}$ and $\gamma_i = \gamma_j$. We first show that $\epsilon_i(t)$ is bounded for all $t \in \mathbb{R}_{\geq 0}$ and then it consequently follows that $\epsilon_j(t)$ is bounded for all $v_j \in \Xi_l \setminus \{v_i\}$. Since the clusters are maximal we can then deduce the same result for the other clusters. Consider the Lyapunov function candidate $V(\epsilon_i) := \frac{1}{2}\epsilon_i^2$. By using (15), it follows

$$\dot{V}(\epsilon_i) = \epsilon_i \frac{d\epsilon_i}{dt} = \epsilon_i \Big(- \frac{1}{\gamma_i \xi_i (1+\xi_i)} \big(\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} - \xi_i \dot{\gamma}_i \big) \Big) \leq \epsilon_i \alpha_i \frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}} H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) + |\epsilon_i| \alpha_i k_i \quad (16)$$

where $\alpha_i(t) := -\frac{1}{\gamma_i \xi_i (1+\xi_i)}$ and k_i is a positive constant such that $0 \leq |\xi_i \dot{\gamma}_i| \leq k_i < \infty$. This follows since $\xi_i(t) \in \Omega_\xi$ for all $t \in \mathcal{J}$ and $\dot{\gamma}_i$ is bounded by definition; $\alpha_i(t)$ satisfies $\alpha_i(t) \in [\frac{4}{\gamma_i^0}, \infty)$ for all $t \in \mathcal{J}$ since $\frac{4}{\gamma_i^0} \leq -\frac{1}{\gamma_i^0 \xi_i (1+\xi_i)} \leq -\frac{1}{\gamma_i \xi_i (1+\xi_i)} \leq -\frac{1}{\gamma_i^\infty \xi_i (1+\xi_i)} < \infty$ for $\xi_i \in \Omega_\xi$. We note that $\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}} H_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{v_j \in \Xi_l} \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j} H_{\boldsymbol{x}_j}(\boldsymbol{x}, \xi_j)$. Using (2) and (14), then

$$\epsilon_i \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j} H_{\boldsymbol{x}_j}(\boldsymbol{x}, \xi_j) = \epsilon_i \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j} \Big(f_j(\boldsymbol{x}_j) + f_j^c(\boldsymbol{x}) - \epsilon_j g_j(\boldsymbol{x}_j) g_j(\boldsymbol{x}_j)^T \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j} + \boldsymbol{w}_j \Big) \leq |\epsilon_i| M_j - |\epsilon_i|^2 \lambda_j J_j$$

where $\epsilon_i = \epsilon_j$ as remarked previously; $\lambda_j > 0$ is the positive minimum eigenvalue of $g_j(\boldsymbol{x}_j) g_j(\boldsymbol{x}_j)^T$ according to Assumption 1, and $\| \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j} \big(f_j(\boldsymbol{x}_j) + f_j^c(\boldsymbol{x}) + \boldsymbol{w}_j \big) \| \leq M_j < \infty$ due to continuity of $\frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j}$, $f_j(\boldsymbol{x}_j)$, and $f_j^c(\boldsymbol{x})$ and the fact that $\Omega_{\boldsymbol{x}}$ and $\mathcal{W}_i$ are bounded. The lower bound $J_j \in \mathbb{R}_{\geq 0}$ arises naturally due to the norm operator as $0 \leq J_j \leq \| \frac{\partial \rho^{\psi_j}(\bar{\boldsymbol{x}}_j)}{\partial \boldsymbol{x}_j}) \|^2 < \infty$; (16) now implies

$$\dot{V}(\epsilon_i) \leq \alpha_i |\epsilon_i| \big(\hat{M}_i - |\epsilon_i| \hat{J}_i \big) \quad (17)$$

where $\hat{M}_i := \sum_{v_j \in \Xi_l} M_j + k_i$ and $\hat{J}_i := \sum_{v_j \in \Xi_l} \lambda_j J_j$. Note that $\hat{J}_i > 0$ since $\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \bar{\boldsymbol{x}}_i} = 0$ if and only if $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i) = \rho_i^{\text{opt}}$, which is excluded since (5) holds for all $t \in \mathcal{J}$ and $\rho_i^{\max} < \rho_i^{\text{opt}}$ (recall that $\rho^{\psi_j}(\bar{\boldsymbol{x}}_j)$ is concave). In other words, at least one J_j in $v_j \in \Xi_l$ is greater than zero. It holds that $\dot{V}(\epsilon_i) \leq 0$ if $\frac{\hat{M}_i}{\hat{J}_i} \leq |\epsilon_i|$. Hence, $|\epsilon_i|$ will be upper bounded as $|\epsilon_i(t)| \leq \max \Big(|\epsilon_i(0)|, \frac{\hat{M}_i}{\hat{J}_i} \Big)$, which implies that $\epsilon_i(t)$ is upper and lower bounded by some constants ϵ_i^u and ϵ_i^l, respectively, so that $\epsilon_i^l \leq \epsilon_i(t) \leq \epsilon_i^u$ for all $t \in \mathcal{J}$. By defining $\xi_i^l := -\frac{1}{\exp(\epsilon_i^l + 1)}$ and $\xi_i^u := -\frac{1}{\exp(\epsilon_i^u + 1)}$, $\xi_i(t)$ is bounded by $-1 < \xi_i^l \leq \xi_i(t) \leq \xi_i^u < 0$, which translates to $\xi_i(t) \in \Omega'_{\xi_i} := [\xi_i^l, \xi_i^u] \subset \Omega_\xi$ for all $t \in \mathcal{J}$. Note that if $\xi_i(t)$ evolves in a compact set, then $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t))$ will evolve in a compact set $\tilde{\Omega}'_i := [\rho_i^l, \rho_i^u]$ for some constants ρ_i^l and ρ_i^u. Due to Aubin and Frankowska (2009, Prop. 1.4.4) it again holds that

$$\Omega'_i := {\rho^{\psi_i}}^{-1}(\tilde{\Omega}'_i) = \{ \bar{\boldsymbol{x}}_i \in \Omega_i(0) | \rho_i^l \leq \rho^{\psi_i}(\bar{\boldsymbol{x}}_i) \leq \rho_i^u \}$$

is closed and also bounded since $\Omega'_i \subset \Omega_i(0)$. Select $v_{i_l} \in \Xi_l$ for each $l \in \{1, \ldots, L\}$ so that $\bar{\boldsymbol{x}}_{i_l}(t) \in \Omega'_{i_l} \subset \Omega_{i_l}(0)$ for all $t \in \mathcal{J}$ and all v_{i_l}. Define the compact sets $\Omega'_{\boldsymbol{x}} := \Omega'_{i_1} \times \ldots \times \Omega'_{i_L} \subset \mathbb{R}^n$, $\Omega'_{\boldsymbol{\xi}} := \Omega'_{\xi_1} \times \ldots \times \Omega'_{\xi_M} \subset \mathbb{R}^M$, and $\Omega'_{\boldsymbol{y}} := \Omega'_{\boldsymbol{x}} \times \Omega'_{\boldsymbol{\xi}} \subset \mathbb{R}^{n+M}$ for which it holds that $\boldsymbol{y}(t) \in \Omega'_{\boldsymbol{y}}$ for all $t \in \mathcal{J}$. It is also true that $\Omega'_{\boldsymbol{y}} \subset \Omega_{\boldsymbol{y}}$ by which it follows that there is no $t \in \mathcal{J} := [0, \tau_{\max})$ such that $\boldsymbol{y}(t) \notin \Omega'_{\boldsymbol{y}}$. By contradiction of Sontag (2013, Prop. C.3.6) it holds that $\tau_{\max} = \infty$, i.e., $\mathcal{J} = \mathbb{R}_{\geq 0}$. This means that (5) holds for all $v_i \in \mathcal{V}$ and for all $t \in \mathbb{R}_{\geq 0}$. By the choice of $\rho_i^{\max}$, r_i, and γ_i as in (7)-(11), it then holds that $0 < r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$ for each $v_i \in \mathcal{V}$. The control law $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)$ is continuous and bounded because $\rho^{\psi_i}(\bar{\boldsymbol{x}}_i)$, $\epsilon_i(\bar{\boldsymbol{x}}_i, t)$, and $g_i(\boldsymbol{x}_i)$ are continuous. Furthermore, γ_i is continuous with $0 < \gamma(t) < \infty$. Due to the compact domain $\Omega'_{\boldsymbol{x}}$, these functions are also bounded. ■

Remark 5 *If $L = M$, i.e., $|\mathcal{V}_i| = 1$ for each agent $v_i \in \mathcal{V}$, Theorem 1 trivially applies. If $m_i = n_i$ and $g_i(\boldsymbol{x}_i)$ is positive definite for all $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$, the control law* (14) *can be replaced by $\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t) := -\epsilon_i(\bar{\boldsymbol{x}}_i, t) \frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i}$, i.e., no knowledge of f_i, f_i^c, and g_i are needed.*

For a cluster $\Xi_l \in \bar{\Xi}$ that satisfies Case A in Problem 1, we next guarantee local satisfaction of ϕ_i for each $v_i \in \Xi_l$ while one or more clusters $\Xi_k \in \bar{\Xi} \setminus \Xi_l$ do not satisfy the assumption of Case A. We first derive a more general result by disregarding clusters. Therefore, consider a formula ϕ as in (1b) and the set of agents $\mathcal{V}_\phi \subset \mathcal{V}$ that participate in ϕ. Satisfaction of ϕ will be guaranteed when the trajectories of agents in $\mathcal{V} \setminus \mathcal{V}_\phi$ remain bounded.

Assumption 5 *Given, for each $v_i \in \mathcal{V}_\phi$, a sufficiently regular control law $\boldsymbol{u}_i$, there exists, for each $v_k \in \mathcal{V} \setminus \mathcal{V}_\phi$, a sufficiently regular control law $\boldsymbol{u}'_k$ so that, for each $v_i \in \mathcal{V}$, there is a solution $\boldsymbol{x}_i : [0, \tau_{max}) \to \mathbb{R}^{n_i}$ to* (2) *with $\tau_{max} > 0$ so that, for each $v_k \in \mathcal{V} \setminus \mathcal{V}_\phi$, $\boldsymbol{x}_k(t)$ stays in a compact set. By 'sufficiently regular' we mean a controller so that the conditions in Sontag (2013, Thm. 54) are satisfied.*

Theorem 2 *Let each agent $v_i \in \mathcal{V}$ satisfy Assumption 1. Consider ϕ as in* (1b) *and let each $v_i \in \mathcal{V}_\phi$ be subject to $\phi_i := \phi$. Assume that for all $v_i, v_j \in \mathcal{V}_\phi$ it holds that: 1) $(v_i, v_j) \in \mathcal{E}$ and 2) $t_i^* = t_j^*$, $\rho_i^{max} = \rho_j^{max}$, $r_i = r_j$, and $\gamma_i = \gamma_j$ are chosen as in* (6)-(11). *Assume further that each $v_i \in \mathcal{V}_\phi$ applies* (14) *and that Assumptions 2 and 4* *hold. If Assumption 5 holds and all agents $v_k \in \mathcal{V} \setminus \mathcal{V}_\phi$ apply $\boldsymbol{u}'_k$, then it holds that $0 < r_i \leq \rho^\phi(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$.*

PROOF. Define $\boldsymbol{\xi} := \begin{bmatrix}\xi_{i_1} & \dots & \xi_{i_{|\mathcal{V}_\phi|}}\end{bmatrix}$ and $\boldsymbol{y} := \begin{bmatrix}\boldsymbol{x} & \boldsymbol{\xi}\end{bmatrix}$ where $v_{i_1}, \dots, v_{i_{|\mathcal{V}_\phi|}} \in \mathcal{V}_\phi$. Due to $\boldsymbol{u}_i$ and $\boldsymbol{u}'_k$ being sufficiently regular and similarly as in the proof of Theorem 1, there exists a maximal solution $\boldsymbol{y}(t) \in \Omega_{\boldsymbol{y}}$ for all $t \in \mathcal{J} := [0, \tau_{\max})$ with $\tau_{\max} > 0$ and for an open, non-empty, and bounded set $\Omega_{\boldsymbol{y}} \subset \mathbb{R}^{n+|\mathcal{V}_\phi|}$. Since $\boldsymbol{x}_k(t)$ evolves in a compact set for all $t \in \mathcal{J}$, it can again be shown that $\boldsymbol{y} \in \Omega'_{\boldsymbol{y}} \subset \Omega_{\boldsymbol{y}}$ for a compact set $\Omega'_{\boldsymbol{y}}$ so that $\tau_{\max} = \infty$. It follows that $0 < r_i \leq \rho^\phi(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$. ■

If all $v_i \in \mathcal{V}_\phi$ apply (14) under the conditions in Theorem 2, we say that the agents in $\mathcal{V}_\phi$ use *collaborative control* to satisfy ϕ. Theorem 2 solves Case A of Problem 1 with $\rho_i^{\text{opt}} > 0$ even when Theorem 1 does not apply, i.e., some other clusters do not satisfy Case A. Note also that Theorem 2 is more general than that as exemplified next.

Example 1 *Consider $\mathcal{V} := \{v_1, v_2, v_3, v_4\}$ with the tasks $\phi_1 := \phi_2 := F_{[0,5]}(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq 1)$, $\phi_3 := F_{[0,5]}(\|\boldsymbol{x}_3 - \boldsymbol{x}_2\| \leq 1 \wedge \|\boldsymbol{x}_3 - \boldsymbol{x}_4\| \leq 1)$, $\phi_4 := F_{[0,5]}(\|\boldsymbol{x}_4\| \leq 1)$, hence inducing only one maximal dependency cluster $\bar{\Xi} := \Xi_1$ to which Case B of Problem 1 applies. If agents v_1 and v_2 use collaborative control to satisfy $\phi_1 = \phi_2$, then Theorem 2 guarantees satisfaction of $\phi_1 = \phi_2$.*

Extensions of Theorems 1 and 2 to solve Case A when $\rho_i^{\text{opt}} \leq 0$ are stated in Corollary 1.

Corollary 1 *Assume that all assumptions of Theorem 1 (Theorem 2) hold except for Assumption 4 and the choice of ρ_i^{max} and r_i. If instead ρ_i^{max} and r_i are as in (12) and (13), respectively, then it holds that $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$ ($r_i \leq \rho^\phi(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{max}$).*

3.2 An Online Detection & Repair Scheme

Assume now that $\Xi_l \in \bar{\Xi}$ may not satisfy the assumption of Case A in Problem 1 and Case B applies. We propose that each agent $v_i \in \Xi_l$ initially applies (14) with properly chosen initial parameters t_i^*, $\rho_i^{\max}$, r_i, and γ_i; (14) consists of two components, one determining the control strength and one the control direction. The closer $\xi_i(\bar{\boldsymbol{x}}_i, t)$ gets to $\Omega_\xi := \{-1, 0\}$, the bigger will $\epsilon_i(\bar{\boldsymbol{x}}_i, t)$ and consequently $\|\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)\|$ become, i.e., $\|\boldsymbol{u}_i(\bar{\boldsymbol{x}}_i, t)\| \to \infty$ as $\xi_i(\bar{\boldsymbol{x}}_i, t) \to \Omega_\xi$. The control direction is given by $-\frac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i}$. We reason that applying (14) is hence a good initial choice. However, the resulting trajectory $\bar{\boldsymbol{x}}_i$ may still lead to $\xi_i(\bar{\boldsymbol{x}}_i(t)) = \{-1, 0\}$, which is equivalent to a violation of (5), for some $t \geq 0$ and result in critical events. The next two examples exhibit such behavior.

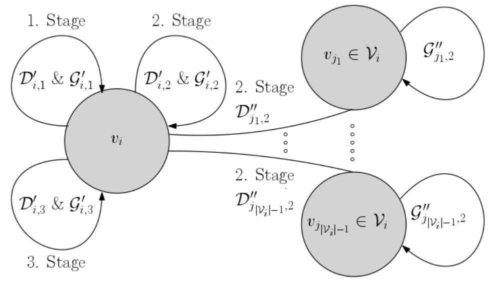

Fig. 1. Overview of the three repair stages.

Example 2 *Consider $\mathcal{V} := \{v_1, v_2, v_3, v_4\}$ with the collaborative formula $\phi_1 := G_{[0,15]}\big((\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq 25) \wedge (\|\boldsymbol{x}_1 - \boldsymbol{x}_3\| \leq 25) \wedge (\|\boldsymbol{x}_1 - \boldsymbol{x}_4\| \leq 25)\big)$ and the non-collaborative formulas $\phi_2 := G_{[10,15]}(\|\boldsymbol{x}_2 - \begin{bmatrix}10 & 90\end{bmatrix}\| \leq 5)$, $\phi_3 := G_{[10,15]}(\|\boldsymbol{x}_3 - \begin{bmatrix}10 & 10\end{bmatrix}\| \leq 5)$, and $\phi_4 := G_{[10,15]}(\|\boldsymbol{x}_4 - \begin{bmatrix}45 & 20\end{bmatrix}\| \leq 5)$. The set of formulas $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ is not globally satisfiable, although each formula is locally satisfiable. Under (14), agents v_2, v_3, and v_4 move to $\begin{bmatrix}10 & 90\end{bmatrix}$, $\begin{bmatrix}10 & 10\end{bmatrix}$, and $\begin{bmatrix}45 & 20\end{bmatrix}$, respectively. Agent v_1 can hence not satisfy ϕ_1 and will violate (5) for some $t \geq 0$. A solution is to decrease the robustness online so that $r_1 < 0$ to achieve $r_1 \leq \rho^{\phi_1}(\bar{\boldsymbol{x}}_1, 0) \leq 0$.*

Even if the set $\{\phi_1, \dots, \phi_M\}$ is globally satisfiable, the resulting trajectory may violate (5) for some $t \geq 0$.

Example 3 *Consider $\mathcal{V} := \{v_5, v_6, v_7\}$ with the collaborative formula $\phi_5 := F_{[5,10]}\big((\|\boldsymbol{x}_5 - \boldsymbol{x}_6\| \leq 10) \wedge (\|\boldsymbol{x}_5 - \boldsymbol{x}_7\| \leq 10) \wedge (\|\boldsymbol{x}_5 - \begin{bmatrix}110 & 20\end{bmatrix}\| \leq 5)\big)$ and the non-collaborative formulas $\phi_6 := F_{[5,15]}(\|\boldsymbol{x}_6 - \begin{bmatrix}50 & 20\end{bmatrix}\| \leq 5)$ and $\phi_7 := F_{[5,15]}(\|\boldsymbol{x}_7 - \begin{bmatrix}110 & 80\end{bmatrix}\| \leq 5)$. Under (14), v_6 and v_7 move to $\begin{bmatrix}50 & 20\end{bmatrix}$ and $\begin{bmatrix}110 & 80\end{bmatrix}$ by at latest 15 time units, respectively. However, v_5 is forced to move to $\begin{bmatrix}110 & 20\end{bmatrix}$ and be close to v_6 and v_7 by at latest 10 time units. This may lead to critical events where v_5 violates (5) for some $t \geq 0$. If v_6 and v_7 collaborate, satisfaction of ϕ_6 and ϕ_7 can be postponed and ϕ_5 can be locally satisfied first, e.g., by using collaborative control for ϕ_5.*

We propose an online detection & repair scheme by using a local hybrid control system $\mathcal{H}_i := (C_i, F_i, D_i, G_i)$ for each $v_i \in \Xi_l$. We first *detect* critical events, i.e., when (5) is violated, by using D_i. Then, a three-stage *repair* procedure is initiated as illustrated in Fig. 1 where repairs are according to G_i. In the first repair stage, detected by $\mathcal{D}'_{i,1}$, the design parameters t_i^*, $\rho_i^{\max}$, r_i, and γ_i are modified locally without communication among agents by the jump map $\mathcal{G}'_{i,1}$. If this is not successful, collaboration among agents will be considered in the second repair stage. Here, critical events of agent v_i are detected by $\mathcal{D}'_{i,2}$ and collaborative control is requested from agents $v_j \in \mathcal{V}_i \setminus \{v_i\}$ by $\mathcal{D}''_{j,2}$ to handle ϕ_i as in Example 3. If this second repair stage is not applicable, the third repair stage is detected by $\mathcal{D}'_{i,3}$ that successively decreases r_i by $\delta_i > 0$. First, define $\boldsymbol{p}_i^r := \begin{bmatrix}\mathfrak{n}_i & \mathfrak{c}_i\end{bmatrix}$

where $\mathfrak{n}_i \in \mathbb{N}$ indicates the number of repair attempts in the first repair stage, while $\mathfrak{c}_i$ is used in the second repair stage; $\mathfrak{c}_i \in \{1, \ldots, M\}$ indicates collaborative control for the formula $\phi_{\mathfrak{c}_i}$. If $\mathfrak{c}_i = 0$, then v_i tries to locally satisfy ϕ_i by itself and if $\mathfrak{c}_i = -1$, then v_i is free, i.e., not subject to a task. For convenience, we abbreviate $\boldsymbol{p}_i^\gamma := \begin{bmatrix}\gamma_i^0 & \gamma_i^\infty & l_i\end{bmatrix}$ and $\boldsymbol{p}_i^{\mathrm{f}} := \begin{bmatrix}t_i^* & \rho_i^{\max} & r_i & \boldsymbol{p}_i^\gamma\end{bmatrix}$ where $\boldsymbol{p}_i^{\mathrm{f}}$ defines the funnel (5). For agent v_i, define $\boldsymbol{z}_i := \begin{bmatrix}\boldsymbol{x}_i & \boldsymbol{p}_i^{\mathrm{f}} & \boldsymbol{p}_i^{\mathrm{r}}\end{bmatrix} \in \mathcal{Z}_i$ where $\mathcal{Z}_i := \mathbb{R}^{n_i} \times \mathbb{R}^6_{\geq 0} \times \mathbb{Z}^2$. The initial state is $\boldsymbol{z}_i(0,0) := \begin{bmatrix}\boldsymbol{x}_i(0) & \boldsymbol{p}_i^{\mathrm{f}}(0) & \mathbf{0}_2\end{bmatrix}$ with

$$\boldsymbol{p}_i^{\mathrm{f}}(0) := \begin{cases} \begin{bmatrix}t_i^* & \rho_i^{\max} & r_i & \gamma_i^0 & \gamma_i^\infty & l_i\end{bmatrix} & \text{if } \rho_i^{\mathrm{opt}} > 0 \\ \begin{bmatrix}t_i^* & \tilde{\rho}_i^{\max} & \tilde{r}_i & \gamma_i^0 & \gamma_i^\infty & l_i\end{bmatrix} & \text{if } \rho_i^{\mathrm{opt}} \leq 0 \end{cases}$$

where t_i^*, $\rho_i^{\max}$, r_i, γ_i^0, γ_i^∞, l_i, $\tilde{\rho}_i^{\max}$, and $\tilde{r}_i$ are chosen according to (6)-(13), respectively, and where $\tilde{\rho}_i^{\max} > \tilde{r}_i \geq \rho_i^{\mathrm{opt}} - \delta_i$ so that r_i is maximized up to a precision of $\delta_i > 0$. Additionally, $\boldsymbol{p}_i^{\mathrm{f}}(0) = \boldsymbol{p}_j^{\mathrm{f}}(0)$ if Case A holds for all agents $v_i, v_j \in \Xi_l$. The flow map is given by

$$F_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) := \begin{bmatrix}f_i(\boldsymbol{x}_i) + f_i^c(\boldsymbol{x}) + g_i(\boldsymbol{x}_i)\boldsymbol{u}_i^{\mathrm{int}} + \boldsymbol{w}_i & \mathbf{0}_8\end{bmatrix}$$

where the control law is given by

$$\boldsymbol{u}_i^{\mathrm{int}} := \begin{cases} -\epsilon_i\big(\bar{\boldsymbol{x}}_i, t\big) g_i(\boldsymbol{x}_i)^T \dfrac{\partial \rho^{\psi_i}(\bar{\boldsymbol{x}}_i)}{\partial \boldsymbol{x}_i} & \text{if } \mathfrak{c}_i = 0 \quad \text{(18a)} \\ -\epsilon_{\mathfrak{c}_i}\big(\bar{\boldsymbol{x}}_{\mathfrak{c}_i}, t\big) g_i(\boldsymbol{x}_i)^T \dfrac{\partial \rho^{\psi_{\mathfrak{c}_i}}(\bar{\boldsymbol{x}}_{\mathfrak{c}_i})}{\partial \boldsymbol{x}_i} & \text{if } \mathfrak{c}_i > 0. \quad \text{(18b)} \end{cases}$$

If $\mathfrak{c}_i > 0$, then collaborative control for $\phi_{\mathfrak{c}_i}$ is used. External inputs contained in $\boldsymbol{u}_i^{\mathrm{ext}}$ are $\boldsymbol{w}_i$ and $\boldsymbol{x}_i^{\mathrm{ext}}$. By assuming $v_i \in \Xi_l$, we define $\mathfrak{c}_i^{\mathrm{ext}} := \begin{bmatrix}\mathfrak{c}_{j_1} & \ldots & \mathfrak{c}_{j_{|\Xi_l|-1}}\end{bmatrix}$ and $\boldsymbol{p}_i^{\mathrm{f,ext}} := \begin{bmatrix}\boldsymbol{p}_{j_1}^{\mathrm{f}} & \ldots & \boldsymbol{p}_{j_{|\Xi_l|-1}}^{\mathrm{f}}\end{bmatrix}$ such that $v_{j_1}, \ldots, v_{j_{|\Xi_l|-1}} \in \Xi_l \setminus \{v_i\}$. Note that $\mathfrak{c}_i^{\mathrm{ext}}$ and $\boldsymbol{p}_i^{\mathrm{f,ext}}$ contain states of all agents in the same dependency cluster Ξ_l. Ultimately, define the external input $\boldsymbol{u}_i^{\mathrm{ext}} := \begin{bmatrix}\boldsymbol{w}_i & \boldsymbol{x}_i^{\mathrm{ext}} & \mathfrak{c}_i^{\mathrm{ext}} & \boldsymbol{p}_i^{\mathrm{f,ext}}\end{bmatrix}$.

Detection: To detect a critical event, the jump set

$$\mathcal{D}_i' := \{(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathfrak{H}_i | \xi_i(t) \in \{-1, 0\},\ \mathfrak{c}_i = 0\}$$

is used. Throughout this section, we assume that agent v_i detects the critical event, while the agents with subscript j as $v_j \in \mathcal{V}_i \setminus \{v_i\}$ are possibly asked to collaborate. We introduce the notation $\{\hat{\boldsymbol{z}}_i \in \mathcal{Z}_i | \hat{\boldsymbol{z}}_i = \boldsymbol{z}_i\ ;\ \textit{exception}\}$ denoting the set of $\hat{\boldsymbol{z}}_i$ such that $\hat{\boldsymbol{z}}_i = \boldsymbol{z}_i$ after the jump except for the elements *exception* in $\hat{\boldsymbol{z}}_i$ that change during a jump and that are explicitly mentioned after the semicolon. The set $\mathcal{D}_i'$ is split into disjoint sets indicating repairs of the first, second, and third repair stage.

Repair Stage 1: The first repair stage is indicated by

$$\mathcal{D}_{i,1}' := \mathcal{D}_i' \cap \{(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathfrak{H}_i | \mathfrak{n}_i < N_i\}$$

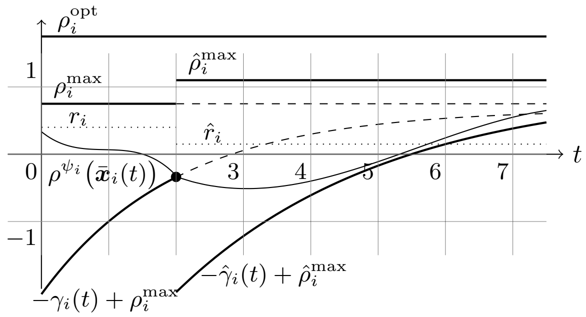

Fig. 2. Funnel repair in the first stage for $\phi_i := F_{[4,6]}\psi_i$.

where $N_i \in \mathbb{N}$ is a design parameter that represents the maximum number of repair attempts in the first stage. A good choice of N_i may in practice depend on the dynamics and tasks of the agents. If $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathcal{D}_{i,1}'$, we first relax the parameters t_i^*, $\rho_i^{\max}$, r_i, and γ_i.

Example 4 *Consider $\phi_i := F_{[4,6]}\psi_i$ with $r_i := 0.4$ (initial robustness), which is supposed to be achieved at $t_i^* := 4.5$. The initial funnel with ρ_i^{max} and $-\gamma_i + \rho_i^{max}$ is shown in Fig. 2. Without detection of a critical event, it would hold that $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \geq r_i$ since $\rho^{\psi_i}\big(\bar{\boldsymbol{x}}_i(t_i^*)\big) \geq r_i$. However, at $t_r := 2$, where t_r indicates the time where a critical event is detected, the trajectory $\rho^{\psi_i}\big(\bar{\boldsymbol{x}}_i(t)\big)$ touches the lower funnel boundary and repair action is needed. This is done by setting $\hat{t}_i^* := 6$ (time relaxation), $\hat{r}_i := 0.15$ (robustness relaxation), $\hat{\rho}_i^{max} := 1.1$ (upper funnel relaxation), and adjusting $\hat{\gamma}_i$ (lower funnel relaxation). The funnel is hence relaxed to $\hat{\rho}_i^{max}$ and $-\hat{\gamma}_i + \hat{\rho}_i^{max}$ as in Fig. 2. At the time t_r, the lower funnel is relaxed to $-\hat{\gamma}_i(t_r) + \hat{\rho}_i^{max}$ where we denote $\gamma_i^r := \hat{\gamma}_i(t_r)$. Due to repair action, $\bar{\boldsymbol{x}}_i$ locally satisfies ϕ_i as shown in Fig. 2.*

With Example 4 in mind, set

$$\mathcal{G}_{i,1}' := \big\{\hat{\boldsymbol{z}}_i \in \mathcal{Z}_i | \hat{\boldsymbol{z}}_i = \boldsymbol{z}_i\ ;\ \hat{t}_i^* = \mathcal{T}_i,\ \hat{\rho}_i^{\max} = \rho_i^{\max} + \zeta_i^{\mathrm{u}},\ \hat{r}_i \in \mathcal{R}_i,\ \hat{\boldsymbol{p}}_i^\gamma = \boldsymbol{p}_i^{\gamma,\mathrm{new}},\ \hat{\mathfrak{n}}_i = \mathfrak{n}_i + 1\big\}$$

where, to achieve time relaxation, we define

$$\mathcal{T}_i := \begin{cases} \{b_i\} & \text{if } \phi_i = F_{[a_i,b_i]}\psi_i \\ \{a_i\} & \text{if } \phi_i = G_{[a_i,b_i]}\psi_i \\ \{\underline{b}_i + \bar{a}_i\} & \text{if } \phi_i = F_{[\underline{a}_i,\underline{b}_i]}G_{[\bar{a}_i,\bar{b}_i]}\psi_i. \end{cases}$$

The parameter r_i is decreased to $\hat{r}_i \in (0, r_i)$ if $r_i > 0$ and set to $\hat{r}_i := r_i - \delta_i$ otherwise (robustness relaxation), i.e.,

$$\mathcal{R}_i := \left\{\hat{r}_i \in \mathbb{R} | \hat{r}_i \in \begin{cases} (0, r_i) & \text{if } r_i > 0 \\ r_i - \delta_i & \text{if } r_i \leq 0 \end{cases}\right\}.$$

The variable ζ_i^{u} relaxes the upper funnel and needs to be such that $\hat{\rho}_i^{\max} := \rho_i^{\max} + \zeta_i^{\mathrm{u}} < \rho_i^{\mathrm{opt}}$ (upper funnel relaxation) according to (7), i.e., let $\zeta_i^{\mathrm{u}} \in (0, \rho_i^{\mathrm{opt}} - \rho_i^{\max})$.

At t_r, the detection time of a critical event, we set $\gamma_i^r := \hat{\gamma}_i(t_r) := \hat{\rho}_i^{\max} - \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t_r)) + \zeta_i^{\mathrm{l}}$ with

$$\zeta_i^{\mathrm{l}} \in \begin{cases} \mathbb{R}_{>0} & \text{if } \hat{t}_i^* > t_r \\ (0, \rho^{\psi_i}(\bar{\boldsymbol{x}}_i(t_r)) - \hat{r}_i] & \text{otherwise,} \end{cases}$$

which resembles (9) (lower funnel relaxation). Let $\boldsymbol{p}_i^{\gamma,\mathrm{new}} := \left[\gamma_i^{0,\mathrm{new}} \ \gamma_i^{\infty,\mathrm{new}} \ l_i^{\mathrm{new}}\right]$ and select

$$\gamma_i^{\infty,\mathrm{new}} \in (0, \min(\gamma_i^r, \hat{\rho}_i^{\max} - \hat{r}_i)]$$
$$l_i^{\mathrm{new}} := \begin{cases} 0 & \text{if } -\gamma_i^r + \hat{\rho}_i^{\max} \geq \hat{r}_i \\ \frac{-\ln\left(\frac{\hat{r}_i + \gamma_\infty^{i,new} - \hat{\rho}_i^{\max}}{-(\gamma_i^r - \gamma_\infty^{i,new})}\right)}{\hat{t}_i^* - t_r} & \text{else} \end{cases}$$

similarly to (10) and (11). Finally, set $\gamma_i^{0,\mathrm{new}} := (\gamma_i^r - \gamma_i^{\infty,\mathrm{new}})\exp(l_i^{\mathrm{new}} t_r) + \gamma_i^{\infty,\mathrm{new}}$. The choices of $\hat{\rho}_i^{\max}$, $\hat{r}_i$, and $\hat{\gamma}_i$ follow the same intuition as in Remark 4.

Repair Stage 2: The jump set

$$\mathcal{D}'_{i,\{2,3\}} := \mathcal{D}'_i \cap \{(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathfrak{H}_i | \mathfrak{n}_i \geq N_i\}$$

detects repairs of the second or third stage. After N_i unsuccessful repair attempts, the second stage is initiated if some *timing constraints* (formalized in $\mathcal{D}'_{i,2}$) are satisfied. Collaborative control for ϕ_i by all agents in $\mathcal{V}_i$ will then be initiated and guarantee that there are no further critical events. The second stage is then detected as

$$\mathcal{D}'_{i,2} := \mathcal{D}'_{i,\{2,3\}} \cap \big\{(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathfrak{H}_i | \forall v_j \in \mathcal{V}_i \setminus \{v_i\}, (\mathfrak{c}_j = -1) \text{ or } \big(\mathfrak{c}_j = 0,\ b_i < \mathcal{T}_j\big)\big\}.$$

To use collaborative control to deal with ϕ_i, the in $\mathcal{D}'_{i,2}$ formalized timing constraints need to hold, i.e., each agent $v_j \in \mathcal{V}_i \setminus \{v_i\}$ is either not subject to a task or there is enough time to satisfy ϕ_j after ϕ_i has been collaboratively satisfied. In this respect, the control law switches from (18a) to (18b) for agent v_i. Therefore, set

$$\mathcal{G}'_{i,2} := \big\{\hat{\boldsymbol{z}}_i \in \mathcal{Z}_i | \hat{\boldsymbol{z}}_i = \boldsymbol{z}_i \ ; \ \hat{\rho}_i^{\max} = \rho_i^{\max} + \zeta_i^{\mathrm{u}}, \hat{r}_i \in \mathcal{R}_i,\ \hat{\boldsymbol{p}}_i^{\gamma} = \boldsymbol{p}_i^{\gamma,\mathrm{new}},\ \hat{\mathfrak{c}}_i = i\big\}$$

where $\hat{\mathfrak{c}}_i := i$ indicates collaborative control for $\phi_{\mathfrak{c}_i}$, while again relaxing the funnel parameters as in the first repair stage. Now changing the perspective to the participating agents $v_j \in \mathcal{V}_i \setminus \{v_i\}$, all agents v_j need to participate in collaborative control. Assume that $v_j \in \Xi_l$, then

$$\mathcal{D}''_{j,2} := \big\{(\boldsymbol{z}_j, \boldsymbol{u}_j^{\mathrm{int}}, \boldsymbol{u}_j^{\mathrm{ext}}) \in \mathfrak{H}_j | \mathfrak{c}_j \in \{-1, 0\}, \exists v_i \in \Xi_l \setminus \{v_j\}, v_j \in \mathcal{V}_i, \mathfrak{c}_i = i\big\},$$

is activated when agent v_i asks agent v_j for collaborative control (detected by $\mathcal{D}'_{i,2}$). If $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathcal{D}''_{j,2}$, the control law for each $v_j \in \mathcal{V}_i \setminus \{v_i\}$ switches to (18b) by

$$\mathcal{G}''_{j,2} := \big\{\hat{\boldsymbol{z}}_j \in \mathcal{Z}_j | \hat{\boldsymbol{z}}_j = \boldsymbol{z}_j \ ; \ \hat{\boldsymbol{p}}_j^{\mathrm{f}} = \boldsymbol{p}_i^{\mathrm{f}},\ \hat{\mathfrak{c}}_j = \mathfrak{c}_i\big\}$$

where $\hat{\mathfrak{c}}_j = \mathfrak{c}_i$ and $\hat{\boldsymbol{p}}_j^{\mathrm{f}} = \boldsymbol{p}_i^{\mathrm{f}}$ enforce that all conditions in Theorem 2 or Corollary 1 hold after the jump.

Remark 6 *The second repair stage is iniated solely based on $\mathfrak{c}_j$ and $\mathcal{T}_j$. Note that no future state predictions can be made since $f_j(\boldsymbol{x}_j)$ and $f_j^c(\boldsymbol{x})$ are unknown ($f_i(\boldsymbol{x}_i)$ and $f_i^c(\boldsymbol{x})$ are in general unknown) and that agent v_i only has knowledge of $\mathfrak{c}_j$ and a_j, b_j, or $\underline{b}_j + \bar{a}_j$ depending on whether an always, eventually, or always-eventually task is considered, so that ahead planning is not possible.*

Repair Stage 3: If the timing constraints in $\mathcal{D}'_{i,2}$ do not apply, repairs of the third stage are initiated by

$$\mathcal{D}'_{i,3} := \mathcal{D}'_{i,\{2,3\}} \setminus \mathcal{D}'_{i,2}.$$

Agent v_i reacts in this case by reducing the robustness r_i by $\delta_i > 0$ as illustrated in Example 2 and according to

$$\mathcal{G}'_{i,3} := \big\{\hat{\boldsymbol{z}}_i \in \mathcal{Z}_i | \hat{\boldsymbol{z}}_i = \boldsymbol{z}_i \ ; \ \hat{\rho}_i^{\max} = \rho_i^{\max} + \zeta_i^{\mathrm{u}}, \hat{r}_i = r_i - \delta_i,\ \hat{\rho}_i^{\max} = \rho_i^{\mathrm{opt}} + \sigma,\ \hat{\boldsymbol{p}}_i^{\gamma} = \boldsymbol{p}_i^{\gamma,\mathrm{new}}\big\}$$

where $\gamma_i^{\mathrm{r}} := \hat{\rho}_i^{\max} - \rho^{\psi_i}(\bar{\boldsymbol{x}}_i) + \delta_i$ is used to calculate $\boldsymbol{p}_i^{\gamma,\mathrm{new}}$ and $\sigma > 0$ is a small constant that avoids Zeno behavior.

The Overall System: It needs to be detected when $r_i \leq \rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \leq \rho_i^{\max}$. Define $\nu_i := \begin{cases} \mathfrak{c}_i & \text{if } \mathfrak{c}_i > 0 \\ i & \text{if } \mathfrak{c}_i = 0 \end{cases}$ and

$$\mathcal{D}_{i,\mathrm{sat}} := \big\{(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathfrak{H}_i | r_{\nu_i} \leq \rho^{\psi_{\nu_i}}(\bar{\boldsymbol{x}}_{\nu_i}) \leq \rho_{\nu_i}^{\max}, \mathfrak{c}_i \geq 0, t \in \mathcal{T}_i^{\mathrm{sat}}\big\} \setminus (\mathcal{D}'_i \cup \mathcal{D}''_{i,2}),$$

with

$$\mathcal{T}_i^{\mathrm{sat}} := \begin{cases} [a_{\nu_i}, b_{\nu_i}] & \text{if } \phi_{\nu_i} = F_{[a_{\nu_i}, b_{\nu_i}]}\psi_{\nu_i} \\ b_{\nu_i} & \text{if } \phi_{\nu_i} = G_{[a_{\nu_i}, b_{\nu_i}]}\psi_{\nu_i} \\ t^*_{\nu_i} + \bar{b}_{\nu_i} & \text{if } \phi_{\nu_i} = F_{[\underline{a}_{\nu_i}, \underline{b}_{\nu_i}]} G_{[\bar{a}_{\nu_i}, \bar{b}_{\nu_i}]}\psi_{\nu_i} \end{cases}$$

and where the set substraction of $\mathcal{D}'_i \cup \mathcal{D}''_{i,2}$ exludes the case where $\mathcal{D}'_i$ or $\mathcal{D}''_{i,2}$ apply simultaneously with $\mathcal{D}_{i,\mathrm{sat}}$. If $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\mathrm{int}}, \boldsymbol{u}_i^{\mathrm{ext}}) \in \mathcal{D}_{i,\mathrm{sat}}$ and in case of collaborative control, the agents $v_j \in \mathcal{V}_i \setminus \{v_i\}$ are then either not subject to a task or need to continue with ϕ_j. Hence, set

$$\mathcal{G}_{i,\mathrm{sat}} := \Bigg\{\hat{\boldsymbol{z}}_i \in \mathcal{Z}_i | \hat{\boldsymbol{z}}_i = \boldsymbol{z}_i \ ; \ \hat{t}_i^* = \mathcal{T}_i, \left[\hat{\rho}_i^{\max},\ \hat{r}_i\right] = \begin{cases} \left[\rho_i^{\max} \ r_i\right] & \text{if } \rho_i^{\mathrm{opt}} > 0 \\ \left[\tilde{\rho}_i^{\max} \ \tilde{r}_i\right] & \text{if } \rho_i^{\mathrm{opt}} \leq 0 \end{cases} \hat{\boldsymbol{p}}_i^{\gamma} = \boldsymbol{p}_i^{\gamma,\mathrm{new}},\ \hat{\mathfrak{c}}_i = \begin{cases} 0 & \text{if } \mathfrak{c}_i > 0 \text{ and } \mathfrak{c}_i \neq i \\ -1 & \text{if } \mathfrak{c}_i = 0 \text{ or } \mathfrak{c}_i = i \end{cases} \Bigg\}$$

where $\rho_i^{\max}$, r_i, $\tilde{\rho}_i^{\max}$, and $\tilde{r}_i$ are according to (7), (8), (12), and (13) but evaluated with $\bar{\boldsymbol{x}}_i(t)$ instead of $\bar{\boldsymbol{x}}_i(0)$. Finally, $\mathcal{H}_i$ is given by $D_i := \mathcal{D}'_i \cup \mathcal{D}''_{i,2} \cup \mathcal{D}_{i,\text{sat}}$, $C_i := \mathcal{Z}_i \setminus D_i$, and $F_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ as defined before. The corresponding jump map $G_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ is $\mathcal{G}'_{i,1}(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}'_{i,1}$, $\mathcal{G}'_{i,2}(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}'_{i,2}$, and $\mathcal{G}'_{i,3}(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}'_{i,3}$ for the detection of critical events. Furthermore, $G_i(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ is $\mathcal{G}''_{i,2}(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}''_{i,2}$ and $\mathcal{G}_{i,\text{sat}}(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}})$ if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}_{i,\text{sat}}$. It is crucial that the behavior of $\mathcal{H}_i$ does not exhibit two or more jump options at the same time, i.e., $\mathcal{H}_i$ should be deterministic with respect to jumps permitted by $\mathcal{H}_i$. Note that $\mathcal{D}'_i = \mathcal{D}'_{i,1} \cup \mathcal{D}'_{i,2} \cup \mathcal{D}'_{i,3}$ and that $\mathcal{D}'_{i,1}$, $\mathcal{D}'_{i,2}$, and $\mathcal{D}'_{i,3}$ are non-intersecting. Note also that the sets $\mathcal{D}'_i$ and $\mathcal{D}_{i,\text{sat}}$ as well as $\mathcal{D}''_{i,2}$ and $\mathcal{D}_{i,\text{sat}}$ are non-intersecting. However, $\mathcal{D}'_i$ and $\mathcal{D}''_{i,2}$ are intersecting. Therefore, if $(\boldsymbol{z}_i, \boldsymbol{u}_i^{\text{int}}, \boldsymbol{u}_i^{\text{ext}}) \in \mathcal{D}'_i \cap \mathcal{D}''_{i,2}$, we only execute the jump induced by $\mathcal{D}''_{i,2}$ to account for the logic modeled by the hybrid system. This can be achieved by modifying $\mathcal{D}'_i$ to $\mathcal{D}'_i \setminus \mathcal{D}''_{i,2}$.

Theorem 3 *Let each agent $v_i \in \mathcal{V}$ be controlled by $\mathcal{H}_i := (C_i, F_i, D_i, G_i)$, while Assumptions 1-3 and 5 are satisfied. For $v_i \in \Xi_l$ it holds that $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \geq r_i$ where either $r_i := r_i(0,0)$ (initial robustness) if Case A applies or r_i is lower bounded and maximized up to a precision of $\delta_i > 0$ if Case B applies. Zeno behavior is excluded.*

PROOF. Without critical events, it is guaranteed that either ϕ_i is locally satisfied if $\rho_i^{\text{opt}} > 0$ or a least violating solution with a given gap of ρ_i^{gap} is found if $\rho_i^{\text{opt}} \leq 0$ due to Theorem 2 and Corollary 1, respectively. In the first repair stage, the parameters t_i^*, $\rho_i^{\max}$, r_i, γ_i^0, γ_i^∞, and l_i are repaired in a way that still guarantees local satisfaction of ϕ_i if $\rho_i^{\text{opt}} > 0$ or, otherwise, r_i is reduced by δ_i. Zeno behavior is excluded for this stage since only a finite number of jumps, i.e., N_i jumps, are permitted. For the second repair stage, collaborative control for ϕ_i guarantees achieving the task ϕ_i with a robustness of r_i by Theorem 2 and Corollary 1. Afterwards, participating agents $v_j \in \mathcal{V}_i \setminus \{v_i\}$ have enough time to deal with their own local task ϕ_j, which is guaranteed by the timing constraints in $\mathcal{D}'_{i,2}$ that need to hold in order to initiate collaborative control. The third repair stage successively decreases r_i by δ_i. Note that r_i has to be lower bounded due to Assumption 2, which states that for local satisfaction of ϕ_i the state $\bar{\boldsymbol{x}}_i$ is bounded. Hence, all agents aim to stay within a bounded set. Consequently, successively reducing r_i will eventually lead to $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0) \geq r_i$, i.e. maximizing $\rho^{\phi_i}(\bar{\boldsymbol{x}}_i, 0)$ up to a precision of δ_i. This again means that only a finite number of jumps is possible when the lower funnel is touched. Touching the upper funnel will also only lead to a finite number of jumps since $\hat{\rho}_i^{\max} = \rho_i^{\text{opt}} + \sigma$ in $\mathcal{G}'_{i,3}$, exluding Zeno behavior.

4 Simulations

Consider $M := 9$ agents represented by three-wheeled omni-directional mobile robots as in Liu et al. (2008) with two states x_1 and x_2 indicating the robot's position and one state x_3 indicating the robot's orientation with respect to the x_1-axis. We denote $x_{i,k}$ with $k \in \{1,2,3\}$ as the k-th element of agent v_i's state and define $\boldsymbol{p}_i := \begin{bmatrix} x_{i,1} & x_{i,2} \end{bmatrix}$. Let $\boldsymbol{x}_i := \begin{bmatrix} \boldsymbol{p}_i & x_{i,3} \end{bmatrix} \in \mathbb{R}^3$ so that $\boldsymbol{x} := \begin{bmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_M \end{bmatrix} \in \mathbb{R}^{27}$. As in Remark 2, we use *induced dynamical couplings* $f_i^{\text{u}}(\boldsymbol{x}) := \begin{bmatrix} f_{i,1}^{\text{u}}(\boldsymbol{x}) & f_{i,2}^{\text{u}}(\boldsymbol{x}) & 0 \end{bmatrix}$ as a means of collision avoidance with $f_{i,k}^{\text{u}}(\boldsymbol{x}) := \sum_{j=1, j\neq i}^{9} \kappa_i \frac{x_{i,k} - x_{j,k}}{\|\boldsymbol{p}_i - \boldsymbol{p}_j\| + 0.000001}$ for $k \in \{1,2\}$ and where $\kappa_i := 10$. By chosing $f_i^{\text{u}}(\boldsymbol{x})$ as above, each agent v_i needs knowledge of the states of all agents, which can be prevented by only including the states of agents in the proximity of agent v_i. The dynamics are given by $\dot{\boldsymbol{x}}_i = f_i^{\text{u}}(\boldsymbol{x}) + \begin{bmatrix} \cos(x_{i,3}) & -\sin(x_{i,3}) & 0 \\ \sin(x_{i,3}) & \cos(x_{i,3}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(B_i^T\right)^{-1} R_i \boldsymbol{v}_i + \boldsymbol{w}_i$, where $R_i := 0.02$ is the wheel radius and $B_i := \begin{bmatrix} 0 & \cos(\pi/6) & -\cos(\pi/6) \\ -1 & \sin(\pi/6) & \sin(\pi/6) \\ L_i & L_i & L_i \end{bmatrix}$ describes geometrical constraints with $L_i := 0.2$ (radius of the robot body). The simulations have been performed in real-time on a two-core 1,8 GHz CPU with 4 GB of RAM. Using the forward Euler method with a sampling frequency of 500 Hz, calculations of the local control laws took on average 50 μs. The noise $\boldsymbol{w}_i$ is drawn from a truncated normal distribution with mean 0 and variance 100. The simulation example resembles Examples 2 and 3. We additionally add requirements on the robot's orientation. For the first cluster (as in Example 2), here denoted by Ξ_1, we let $\phi_1 := G_{[0,15]}\big((\|\boldsymbol{p}_1 - \boldsymbol{p}_2\| \leq 25) \wedge (\|\boldsymbol{p}_1 - \boldsymbol{p}_3\| \leq 25) \wedge (\|\boldsymbol{p}_1 - \boldsymbol{p}_4\| \leq 25)\big)$, $\phi_2 := G_{[10,15]}\big((\|\boldsymbol{p}_2 - \begin{bmatrix} 10 & 90 \end{bmatrix}\| \leq 5) \wedge (|x_{2,3} + 45| \leq 7.5)\big)$, $\phi_3 := G_{[10,15]}\big((\|\boldsymbol{p}_3 - \begin{bmatrix} 10 & 10 \end{bmatrix}\| \leq 5) \wedge (|x_{3,3} - 45| \leq 7.5)\big)$, and $\phi_4 := G_{[10,15]}\big(\|(\boldsymbol{p}_4 - \begin{bmatrix} 45 & 20 \end{bmatrix}\| \leq 5) \wedge (|x_{4,3} - 135| \leq 7.5)\big)$. We added the requirements that agents v_2, v_3, and v_4 should eventually be oriented with -45, 45, and 135 degrees and remain with this orientation from then on. For the second cluster (as in Example 3), denoted by Ξ_2, let $\phi_5 := F_{[5,10]}\big((\|\boldsymbol{p}_5 - \boldsymbol{p}_6\| \leq 10) \wedge (\|\boldsymbol{p}_5 - \boldsymbol{p}_7\| \leq 10) \wedge (\|\boldsymbol{p}_5 - \begin{bmatrix} 110 & 20 \end{bmatrix}\| \leq 5)\big)$, $\phi_6 := F_{[5,15]}\big((\|\boldsymbol{p}_6 - \begin{bmatrix} 50 & 20 \end{bmatrix}\| \leq 5) \wedge (|x_{6,3} - 45| \leq 7.5)\big)$, and $\phi_7 := F_{[5,15]}\big((\|\boldsymbol{p}_7 - \begin{bmatrix} 110 & 80 \end{bmatrix}\| \leq 5) \wedge (|x_{7,3} + 135| \leq 7.5)\big)$. We here added the requirements that agents v_6 and v_7 should eventually be oriented with 45 and -135 degrees; Ξ_1 and Ξ_2 correspond to Case B in Problem 1. For the third cluster, denoted by Ξ_3, let $\phi_8 := \phi_9 := F_{[5,15]}\big((x_{8,1} - x_{9,1} \leq 10) \wedge (x_{8,1} - x_{9,1} \geq$

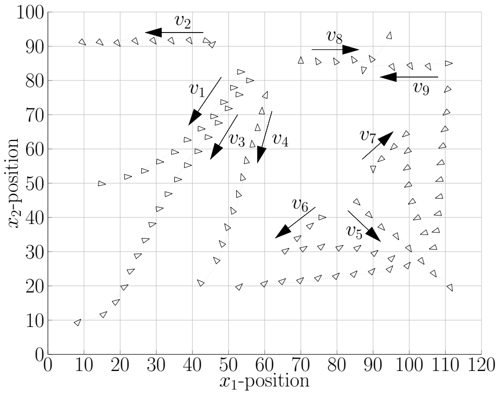

Fig. 3. State trajectory for the clusters Ξ_1, Ξ_2, and Ξ_3. Note that the triangles indicate the orientation of each agent.

$5) \wedge (x_{8,2} - x_{9,2} \leq 10) \wedge (x_{8,2} - x_{9,2} \geq 5)\big)$, hence satisfying Case A in Problem 1. We remark that all agents have been equipped with a formula $\psi_i^{\text{Ass.2}}$ as mentioned in Remark 3. We select $\eta := 1$ for which it holds that $\rho_1^{\text{opt}} = 23.9$, $\rho_2^{\text{opt}} = \rho_3^{\text{opt}} = \rho_4^{\text{opt}} = 4.92$, $\rho_5^{\text{opt}} = 4.97$, $\rho_6^{\text{opt}} = \rho_7^{\text{opt}} = 4.92$, and $\rho_8^{\text{opt}} = \rho_9^{\text{opt}} = 1.11$ so that we initially choose $r_i := 0.5$ for all $i \in \{1, \ldots, M\}$ and $\rho_1^{\max} := 20$, $\rho_2^{\max} := \rho_3^{\max} := \rho_4^{\max} := \rho_5^{\max} := \rho_6^{\max} := \rho_7^{\max} := 4.9$ and $\rho_8^{\max} := \rho_9^{\max} := 1.1$. For the parameters of the hybrid system, we set $\delta_i := 1.5$ and $N_i := 1$. The resulting trajectory is shown in Fig. 3. For Ξ_1, it can be seen that v_1 does not satisfy ϕ_1, but finds a least violating solution by staying as close as possible to v_2, v_3, and v_4. The latter agents independently satisfy their own formulas. For v_1 and v_2, the corresponding funnels are shown in Fig. 4a and 4b. It can be seen that v_1 first tries to repair the parameters in the first repair stage and then successively decreases the robust r_i by δ_i in the third repair stage. For v_5, v_6, and v_7, it can be seen that all agents satisfy their formulas. In particular, v_5 uses collaborative control together with v_6 and v_7 in the second repair stage after an unsuccessul repair attempt in the first stage. This can be seen in Fig. 4d, while Fig. 4e and 4f show the behavior of the collaborating agents v_6 and v_7. The third cluster Ξ_3 with v_8 and v_9 satisfies collaboratively their formulas according to Theorem 2. The funnel for v_8 can be seen in Fig. 4c. All tasks, except of ϕ_1, are satisfied with a robustness of $r_i := 0.5$. Our method is hence robust with respect to additive noise and with respect to the formula, where the designer can impose a robustness r_i. Note also in Fig. 3 that collisions are avoided. A comparison with the methods in Raman et al. (2014) and Pant et al. (2018) was not possible due to their high computational complexity rendering the entailed optimization programs intractable. Note that Raman et al. (2014) and Pant et al. (2018) focus on single-agent systems so that a high-dimensional centralized multi-agent system with 27 states had to be used.

5 Conclusion

A framework for the control of multi-agent systems under local signal temporal logic tasks has been presented. The local tasks may depend on the behavior of other agents and may hence be conflicting. In a first step, we identified conditions under which a local feedback control law guarantees the satisfaction of the local tasks if they are satisfiable. For not satisfiable tasks, a least violating solution can be found. If the identified conditions do not hold, we proposed to combine the previously developed local feedback control law with an online detection & repair scheme. This detection & repair scheme is expressed as a local hybrid control system. Critical events are detected and repaired in a three-stage procedure so that cases with locally conflicting formulas can be resolved.

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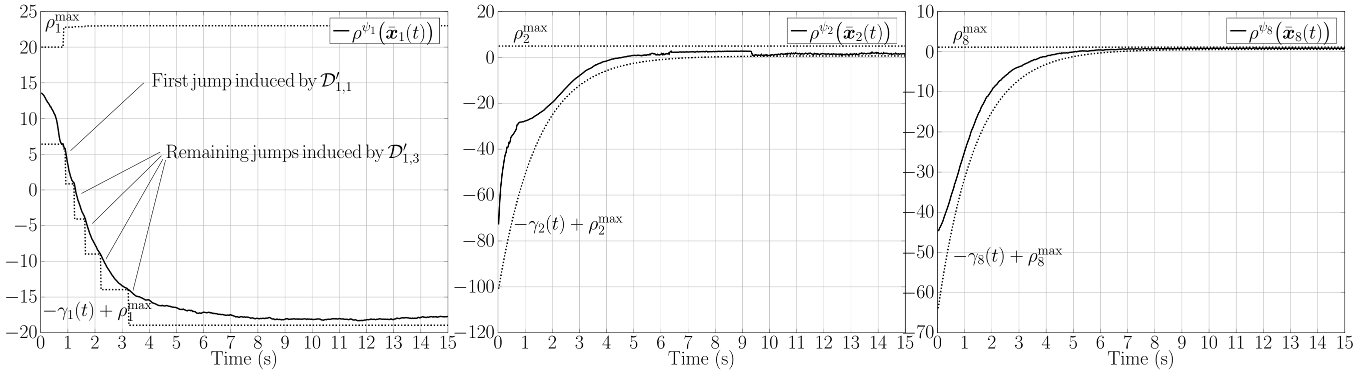
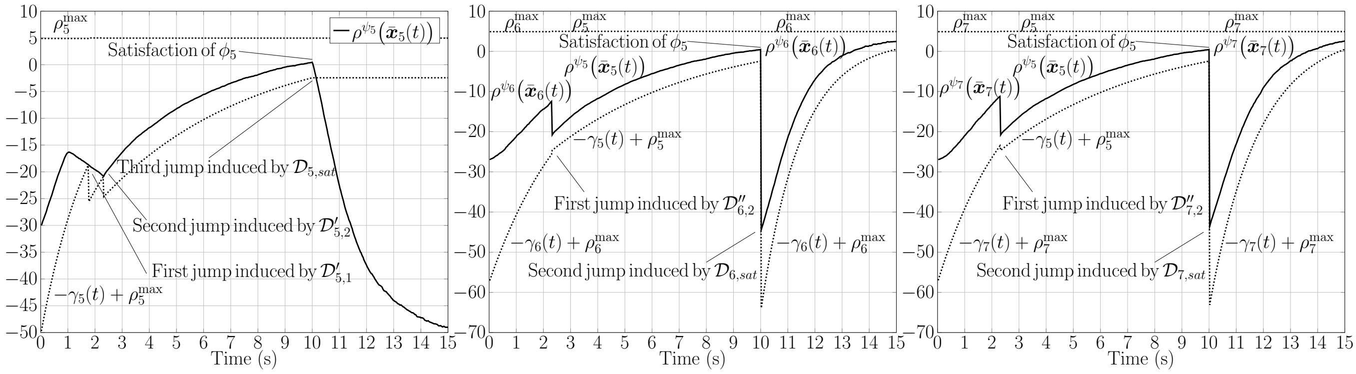


(a) Funnel repairs for agent v_1. First and third repair stage are activated.

(b) No funnel repairs for agent v_2. The formula ϕ_2 is satisfied without repairs.

(c) No funnel repairs for agent v_8.The formula ϕ_8 is satisfied without repairs.



(d) Funnel repairs for agent v_5, who requests collaborative help from v_6 and v_7.

(e) Agent v_6 starts collaborating with agent v_5 at around 2.3 seconds.

(f) Agent v_7 starts collaborating with agent v_5 at around 2.3 seconds.

Fig. 4. Funnel repairs for the agents v_1, v_2, v_8, v_5, v_6, and v_7.

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