Decentralized Navigation Functions for Multiple Robotic Agents with Limited Sensing Capabilities*

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Abstract

The decentralized navigation function methodology, established in our previous work for navigation of multiple holonomic agents with global sensing capabilities is extended to the case of local sensing capabilities. Each agent plans its actions without knowing the destinations of the others and the positions of those agents lying outside its sensing neighborhood. The stability properties of the closed loop system are checked via Lyapunov stability techniques for nonsmooth systems. The collision avoidance and global convergence properties are verified through simulations.

Keywords: Decentralized Control, Autonomous Agents, Multi Agent Systems, Motion Planning.

Categories:(5)

*This work was partially presented in [5]

1 Introduction

Navigation of multiple agents is a field that has recently gained increasing attention in the robotics community, due to the need for autonomous control of more than one mobile robotic agents in the same workspace. While most approaches in the past had focused on centralized planning, specific real-world applications have lead researchers throughout the globe to turn their attention to decentralized concepts. The basic motivation of our work comes from two application domains: (i) decentralized conflict resolution in air traffic management ([13]) and (ii) the field of micro robotics ([20],[15]), where a team of autonomous micro robots must cooperate to achieve manipulation precision in the sub micron level.

The reduced computational complexity and increased robustness with respect to agent failures makes decentralized approaches are more appealing compared to the centralized ones. There have been many different approaches to the decentralized motion planning problem. Open loop approaches use game theoretic and optimal control theory to solve the problem taking the constraints of vehicle motion into account; see for example [2], [14], [27], [28]. On the other hand, closed loop approaches use tools from classical Lyapunov theory and graph theory to design control laws and achieve the convergence of the distributed system to a desired configuration both in the concept of cooperative ([8], [17], [12],) and formation control ([1], [10], [22], [25],[26]).

Closed loop strategies are apparently preferable to open loop ones, mainly because they provide robustness with respect to modelling uncertainties and agent failures and guaranteed convergence to the desired configurations. However, a common point of most work in this area is devoted to the case of point agents. Although this allows for variable degree of decentralization, it is far from realistic in real world applications, even in the field of microrobotics, where the non-zero volume of each robot cannot be disregarded due to the fact that the surrounding objects are of comparable size. Another example is conflict resolution in Air Traffic Management, where two aircraft are not allowed to approach each other closer than a specific "alert" distance. The construction of closed loop methods for decentralized non-point multi-agent systems is both evident and appealing.

A closed loop approach for single robot navigation was proposed by Koditschek and Rimon [16], [23] in their seminal work. This navigation functions' framework handled single, point-sized, robot navigation. In [18] this method was successfully extended to take into account the volume of each robot in a centralized multi-agent scheme, while a decentralized version of this work has been presented by the authors in [29],[7] for multiple holonomic agents with global sensing capabilities. In these papers, the decentralization factor lied in the fact that each agent had knowledge only of its own desired destination, but not of the desired destinations of the others. Each agent had global knowledge about the positions of every other member of the team at each time instant.

The degree of decentralization in a multiagent system generally depends on the knowledge each agent has about the state (position/velocity) and desired goals of each member of the rest of the team. In the current framework, the control design specification is to drive each agent to a desired configuration. Clearly, neglecting the desired destinations of the rest of the team as in [29],[7] is a first step towards decentralization.

Nevertheless, in practice, the sensing capabilities of each agent are limited. Consequently, each agent can not have knowledge of the positions and/or velocities of every agent in the workspace but only of the agents within its sensing zone at each time instant. As a sensing zone we define a circle of specified radius around an agent.

The rest of the paper is organized as follows: section 2 presents the multi-agent system in hand and defines the problem adressed in this paper. In section 3 the concept of decentralized navigation functions, introduced in [7],[29] to cope with navigation of multiple holonomic agents with global sensing capabilities, is reviewed and appropriately redefined in order to cope with the restrictions of the situation in hand. The convergence analysis of the multiagent feedback control strategy for the multiagent system presented in section 3 is provided in Appendix A. Section 4 contains some nontrivial computer simulations based on the proposed algorithm while section 5 summarizes the results and indicates some relevant future directions of research. A review of the nonsmooth stability analysis tools used in Appendix A are provided in Appendix B.

2 System and Problem Definition

Consider a system of N agents operating in the same workspace $W \subset \mathcal{R}^2$. Each agent i occupies a disc: $R_i = \{q \in \mathcal{R}^2 : \| q - q_i \| \leq r_i\}$ in the workspace where $q_i \in \mathcal{R}^2$ is the center of the disc and r_i is the radius of the agent. The configuration space is spanned by $q = [q_1, \ldots, q_N]^T$. Figure 1 shows a five-agent conflict situation. In the case of holonomic agents, the motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, \ i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

The desired destinations of the agents are respectively denoted by the index d: $q_d = [q_{d1}, \ldots, q_{dN}]^T$. We make the following assumptions:

1. Each agent i has knowledge of the position of only those agents located in a cyclic neighborhood of specific radius d_C at each time instant, where $d_C > max_{i,j \in \mathcal{N}}(r_i + r_j)$, so that it is guaranteed to be larger than the maximum sum of two agents radii. The disc $T_i = \{q : \|q - q_i\| \leq d_C\}$ is called the *sensing zone* of agent i.

2. Each agent has knowledge only of its own desired destination q_{di} but not of the others $q_{dj}, j \neq i$.

3. Each agent i knows the exact number N of agents in the workspace.

4. Spherical agents are considered.

5. The workspace is bounded and spherical.

The multi agent navigation problem treated in this paper can be stated as follows: "*under the prescribed assumptions, derive a set of control laws (one for each agent) that drives the team of agents from any initial configuration to a desired goal configuration avoiding, at the same time, collisions.*".

The first three assumptions reveal the decentralized nature of this framework, as well as its specific limitations. Each agent must know the existence of all agents in the workspace(ass. 3) but needs to know the exact position only of agents found within its sensing zone at each time instant(ass.2). Furthermore, knowledge of the desired destinations of the other agents is unnecessary(ass. 1). In this paper, the *navigation functions* ([16],[18],[29],[7]) tool is redefined in order to cope with assumptions 1,2.

3 Decentralized Navigation Functions for Agents with Limited Sensing Capabilities

3.1 Preliminaries

Navigation functions (NF's) are real valued maps realized through cost functions $\varphi(q)$, whose negated gradient field is attractive towards the goal configuration and repulsive with respect to obstacles [16]. It has been shown by Koditscheck and Rimon that strict global navigation (i.e. the system $\dot{q} = u$

under a feedback control law of the form $u = -K\nabla\varphi$ admits a globally attracting equilibrium state) is not possible, and a smooth vector field on any sphere world with a unique attractor, must have at least as many saddles as obstacles [16].

A navigation function is defined as follows:

Definition 1 [16]*: Let $F \subset R^{2N}$ be a compact connected analytic manifold with boundary. A map $\varphi : F \rightarrow [0,1]$ is a navigation function if:(1) it is analytic on F, (2) it has only one minimum at $q_d \in int(F)$, (3) its Hessian at all critical points (zero gradient vector field) is full rank, and (4) $lim_{q\rightarrow\vartheta F}\varphi(q) = 1$.*

In this definition, F represents the "free space" of robot movement, i.e. the subset of the workspace which is free of collisions.

Strictly speaking, the continuity requirements for the navigation functions are to be C^2. The first property of Definition 1 follows the intuition provided by the authors of [16], that it is preferable to use closed form mathematical expressions to encode actuator commands instead of "patching together" closed form expressions on different portions of space, so as to avoid branching and looping in the control algorithm. Analytic navigation functions, through their gradient provide a direct way to calculate the actuator commands, and once constructed they provide a provably correct control algorithm for every environment that can be diffeomorphically transformed to a sphere world. In this paper, we further relax this requirement by using a non-analytic, merely C^0 navigation function, in order to cope with the limited knowledge each agent has about the state of the other subsystems. The discontinuities however, take place outside of the region where critical points of the potential function occur, so it does not affect the navigation properties of the proposed function.

A function φ that has a unique minimum on F is called *polar*. By using a polar function on a compact connected manifold with boundary, all initial conditions will either be brought to a saddle point or to the unique minimum of the function.

A scalar valued function φ whose Hessian at all critical points is full rank is called *Morse*. The corresponding critical points are called *non-degenerate*. The requirement in Definition 1 that a navigation function must be a Morse function, establishes that the initial conditions that bring the system to saddle points are sets of measure zero [21]. In view of this property, all initial conditions away from sets of measure zero are brought to the unique minimum.

The last property of Definition 1 guarantees that the resulting vector field is transverse to the boundary of the free space F. This establishes that the system always evolves in the interior of F, avoiding collisions and is safely brought to q_d,

3.2 DNF's vs MRNF's

In [18], the navigation functions method has been extended to the case of multiple mobile robots with the use of Multi-Robot navigation functions (MRNF's).

In the form of a centralized setup [18], where a central authority has knowledge of the current positions and desired destinations of all agents $i = 1, \ldots, N$, the sought control law $u = [u_1 \ldots u_N]$ is of the form: $u = -K\nabla\varphi(q)$ where K is a gain. In the decentralized case addressed in this work, each agent has only local knowledge of the current positions of the others, and not of their desired destinations. Hence each agent i has a different navigation law.

Following the procedure of [16],[18],[29],[7], we consider the following class of decentralized navigation functions(*DNF's*):

$$\varphi_i(q) = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}} \tag{2}$$

where k is a positive scalar parameter and $\gamma_{di} = \| q_i - q_{di} \|^2$ is the squared metric of the current agent's configuration q_i from its destination q_{di}. The definition of the function f_i will be given later. Function G_i has as arguments the coordinates of all agents, i.e. $G_i = G_i(q)$, is used to encapsulate all possible collision schemes of agent i with the others.

Figure 2 shows a plot of a DNF of an agent in an environment of 3 (other) moving agents denoted by A-i. The DNF is maximized on the boundary of the free space and minimized at the goal configuration. Using the notation $\tilde{q}_i \triangleq [q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N]^T$, the decentralized NF can be rewritten as

$$\varphi_i = \varphi_i(q_i, \tilde{q}_i)$$

3.3 Construction of the G function for Limited Sensing Zone

In [29],[7] the decentralization feature of the whole scheme lied in the fact that each agent didn't have knowledge of the desired destinations of the rest of the team. On the other hand, each one had global knowledge of the positions of the others at each time instant. This is far from realistic in real world applications where each agent is able of detecting and tracking those that are located within its sensing zone. The "Proximity Function" between two agents i, j in [29], [7] is

$$\beta_{ij} = \|q_i - q_j\|^2 - (r_i + r_j)^2$$

In this work we take the limited sensing capabilities of each agent into account. Specifically, each agent only knows the position of those agents which are within a cyclic neighborhood of specific radius d_C around its center. Therefore the Proximity Function between two agents has to be redefined in this case. We propose the following nonsmooth function:

$$\beta_{ij} = \begin{cases} \|q_i - q_j\|^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| \leq d_c \\ d_c^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| > d_c \end{cases} \tag{3}$$

This definition of the Proximity Function captures the fact that each agent has no knowledge about the whereabouts of those agents found outside its sensing zone. Figure 3 shows a plot of a Proximity Function.

For example in figure 4 we have $\|q_i - q_k\| < d_C$ therefore $\beta_{ik} = \|q_i - q_k\|^2 - (r_i + r_k)^2$, while $\|q_i - q_j\| > d_C$ therefore $\beta_{ij} = d_C^2 - (r_i + r_j)^2$.

Consider now a situation similar to the one in figure 1 where we have five agents. For an agent "R", we proceed to define function G_R. We denote by O_1, O_2, O_3, O_4 the remaining four agents in this scenario. To encode all possible inter-agent proximity situations, the multi-agent team is associated with an (undirected) graph whose vertices are indexed by the team members. The following are discussed in more detail in [6], [29],[7] .

Definition 2 *A* binary relation *with respect to an agent R is an edge between agent R and another agent.*

Definition 3 *A* relation *with respect to agent R is defined as a set of binary relations with respect to agent R.*

Definition 4 *The* relation level *is the number of binary relations in a relation with respect to agent R.*

We denote by $(R_j)_l$ the jth relation of level-l with respect to agent R. With this terminology in hand, the collision scheme of figure 5a is a level-1 relation (one binary relation) and that of figure 5b is a level-3 relation (three binary relations), always with respect to the specific agent R. We use the notation

$$(R_j)_l = \{\{R, A\}, \{R, B\}, \{R, C\}, \ldots\}$$

to denote the set of binary relations in a relation with respect to agent R, where $\{A, B, C, ...\}$ the set of agents that participate in the specific relation. For example, in figure 5b:

$$(R_1)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_3\}\}$$

where we have set arbitrarily $j = 1$.

The complementary set $(R_j^C)_l$ of relation j is the set that contains all the relations of the same level apart from the specific relation j. For example in figure 5b:

$$\left(R_1^C\right)_3 = \{(R_2)_3, (R_3)_3, (R_4)_3\}$$

where

$$(R_2)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_4\}\}$$

$$(R_3)_3 = \{\{R, O_1\}, \{R, O_3\}, \{R, O_4\}\}$$

$$(R_4)_3 = \{\{R, O_2\}, \{R, O_3\}, \{R, O_4\}\}$$

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent i and the other agents involved in the relation. Each relation has its own RPF. Let R_k denote the k^{th} relation of level l. The RPF of this relation is given by:

$$(b_{R_k})_l = \sum_{j \in (R_k)_l} \beta_{\{R,j\}} \tag{4}$$

where the notation $j \in (R_k)_l$ is used to denote the agents that participate in the specific relation of agent R. For example, in the relation of figure 5b we have

$$(b_{R_1})_3 = \sum_{m \in (R_1)_3} \beta_{\{R,m\}} = \beta_{\{R,O_1\}} + \beta_{\{R,O_2\}} + \beta_{\{R,O_3\}}$$

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k})_l = (b_{R_k})_l + \frac{\lambda (b_{R_k})_l}{(b_{R_k})_l + (B_{R_k^C})_l^{1/h}} \tag{5}$$

where λ, h are positive scalars and

$$(B_{R_k^C})_l = \prod_{m \in (R_k^C)_l} (b_m)_l$$

where as previously defined, $(R_k^C)_l$ is the complementary set of relations of level-l, i.e. all the other relations with respect to agent i that have the same number of binary relations with the relation R_k. Continuing with the previous example we could compute, for instance,

$$\left(B_{R_1^C}\right)_3 = (b_{R_2})_3 \cdot (b_{R_3})_3 \cdot (b_{R_4})_3$$

which refers to level-3 relations of agent R.

It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^C)_{n-1} = \emptyset$ and $(g_{R_k})_l = (b_{R_k})_l$ for $l = n-1$. The basic property that we demand from RVF is that it assumes the value of zero if a relation holds, while no other relations of the same or other levels hold. In other words it should indicate which of all possible relations holds. We have he following limits of RVF (using the simplified notation $g_{R_k}(b_{R_k}, B_{R_k^C}) \equiv g_i(b_i, \tilde{b}_i)$):

1. $\lim_{b_i \to 0} \lim_{\tilde{b}_i \to 0} g_i\left(b_i, \tilde{b}_i\right) = \lambda$

2. $\lim_{\substack{b_i \to 0 \\ \tilde{b}_i \neq 0}} g_i\left(b_i, \tilde{b}_i\right) = 0$

These limits guarantee that RVF will behave in the way we want it to, as an indicator of a specific collision.

The function G_i is now defined as

$$G_i = \prod_{l=1}^{n_L^i} \prod_{j=1}^{n_{R_l}^i} (g_{R_j})_l \tag{6}$$

where n_L^i the number of levels and $n_{R_l}^i$ the number of relations in level-l with respect to agent i. Hence G_i is the product of the RVF's of all relations wrt i.

The construction of the G_i function is done in such a way to ensure that the gradient motion imposed on agent i under the control strategy (9) is repulsive with respect to the boundary of the free space. This guarantees collision avoidance. More details can be found in [7].

3.4 The f function

The key difference of the decentralized method with respect to the centralized case is that the control law of each agent ignores the destinations of the others. If we used $\varphi_i = \frac{\gamma_{di}}{\left((\gamma_{di})^k + G_i\right)^{1/k}}$ as a navigation function for agent i, there would be no "available potential" for i to cooperate in a possible collision scheme when its initial condition coincides with its final destination. In order to overcome this limitation,we need to add a function f_i to γ_i so that the cost function φ_i attains positive values in proximity situations even when i has already reached its destination. This function was introduced in [7]. Here, we modify the previous definitions to ensure that the destination point is a non-degenerate local minimum of φ_i with minimum requirements on assumptions. We define the function f_i by:

$$f_i(G_i) = \begin{cases} a_0 + \sum\limits_{j=1}^{3} a_j G_i^j, \; G_i \leq X \\ 0, \; G_i > X \end{cases} \tag{7}$$

where $X > 0, Y = f_i(0) > 0$. By definition, X is a parameter that "activates" the function f_i, while Y is the value of f_i when collision are bound to occur, namely when $G_i \to 0$. The parameters a_j are evaluated so that f_i is maximized when $G_i \to 0$ and minimized when $G_i = X$. We also require that

f_i is continuously differentiable at X. Therefore we have:

$$a_0 = Y, a_1 = 0, a_2 = \frac{-3Y}{X^2}, a_3 = \frac{2Y}{X^3}$$

We require that $Y \leq \frac{\Theta_1}{k}$ where Θ_1 is an arbitrarily large positive gain. This will help in obtaining a lower bound of k analytically in the stability analysis that follows. The parameter X serves as a sensing parameter that activates the f_i function whenever possible collisions are bound to occur. The only requirement we have for X is that it must be small enough to guarantee that f_i vanishes whenever the system has reached its equilibrium, i.e. when everyone has reached its destination. In mathematical terms:

$$X < G_i\left(q_{d1}, \ldots, q_{dN}\right) \forall i \tag{8}$$

That's the minimum requirement we have regarding knowledge of the destinations of the team.

3.5 Control Strategy

The proposed feedback control strategy for agent i is defined as

$$u_i = -K_i \frac{\partial \varphi_i}{\partial q_i} \tag{9}$$

where $K_i > 0$ a positive gain.

A key point in the discrimination between centralized and decentralized navigation functions is that the latter contain a time-varying part which depends on the movement of the other agents. Using the same procedure as in [18],[16] we first prove that the construction of each φ_i guarantees collision avoidance:

Proposition 1 *For each fixed $\tilde{q}_i$, the function $\varphi_i(q_i, \cdot)$ is a navigation function if the parameters h, k assume values bigger than a finite lower bound.*

For the complete proof see [6]. A crucial aspect of this Proposition is the fact that each φ_i is transverse to the boundary of the free space of the corresponding agent i. This guarantees collision avoidance.

On the other hand, the latter does not guarantee global convergence of the system state to the destination configuration. This is guaranteed by the following proposition:

Proposition 2 *The state of the system converges to q_d up to a set of initial conditions of measure zero if the parameters h, k assume values bigger than a finite lower bound.*

The proof of this proposition is based on nonsmooth analysis and is provided in Appendix A. The tools from nonsmooth stability theory used in the next section are reviewed in Appendix B.

4 Simulations

To demonstrate the navigation properties of our decentralized approach, we present two simulations of multiple holonomic agents that have to navigate from an initial to a final configuration, avoiding collisions with each other. Each agent has no knowledge of the positions of those agents lying outside its sensing zone, which is the big circle around its center of mass in Fig.6, Pic.1. In this picture A-i,T-i denote the initial position and desired destination of agent i respectively. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents. The following have been chosen for the simulation of figure 6:
Initial Conditions:

$$q_1(0) = \begin{bmatrix} -.1732 & -.1 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} .1732 & -.1 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} 0 & .2 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} 0 & -.2 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} .1732 & .1 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} - & .1732 & .1 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} 0 & -.1 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} 0 & .25 \end{bmatrix}^T$$

Parameters:

$$k = 110, r_1 = r_2 = r_3 = r_4 = .05, d_C = .11$$
$$\lambda = 1, h = 5, X = .001, Y = .01$$

Pictures 1-6 of Figure 6 show the evolution of the team configuration within a horizon of 6000 time units. One can observe that the collision avoidance as well as destination convergence properties are fulfilled.

The second simulation (Fig.7) involves seven holonomic agents. In screenshot A, A-i,T-i denote the initial condition and desired destination of each agent i respectively. Screenshots A-F of Figure 7 show the evolution in time of the seven agent team. One can observe that the collision avoidance and destination convergence properties are fulfilled in this case as well.

The following have been chosen for the simulation of figure 7:

Initial Conditions:

$$q_1(0) = \begin{bmatrix} -.35 & 0 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} -.13 & .075 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} 0 & .15 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} .13 & .075 \end{bmatrix}^T, q_5(0) = \begin{bmatrix} -.13 & -.075 \end{bmatrix}^T,$$
$$q_6(0) = \begin{bmatrix} 0 & -.15 \end{bmatrix}^T, q_7(0) = \begin{bmatrix} .13 & -.075 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} .13 & -.075 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} 0 & -.15 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} -.13 & -.075 \end{bmatrix}^T, q_{d5} = \begin{bmatrix} .13 & .075 \end{bmatrix}^T,$$
$$q_{d6} = \begin{bmatrix} 0 & .15 \end{bmatrix}^T, q_{d7} = \begin{bmatrix} -.13 & .075 \end{bmatrix}^T$$

Parameters:

$$k = 94, r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = r_7 = .05, d_C = .13$$
$$\lambda = 1, h = 5, X = .0000356, Y = .01$$

5 Conclusions

In this paper we extended the decentralized navigation method to the case of multiple holonomic agents with limited sensing capabilities. We proposed a nonsmooth extension of the navigation function of [7] and proved system convergence using tools from nonsmooth stability analysis. The effectiveness of the methodology is verified through computer simulations.

Current research includes applying this method to the case of distributed nonholonomic agents [19] as well as introducing new definitions of the sensing zone of an agent. Extensions of this method to 3-dimensional kinematics are also under investigation.

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A Stability Analysis

In this section we provide the proof of Proposition 2.

We immediately note that the result of this Proposition is existential rather than computational. We show that finite k, h that renders the system almost everywhere asymptotically stable *exist*, but we do not provide an analytical expression for this lower bound. However, practical values of k, h will be provided in the simulation section. In [6], we have used $\varphi = \sum_{i=1}^{n} \varphi_i$ as a Lyapunov function for the whole system. In this case this function is continuous everywhere, but nonsmooth whenever a switching occurs, i.e. whenever $\|q_i - q_j\| = d_c$ for some i, j. We define the *switching surface* as:

$$S = \{q : \exists i, j, i \neq j | \|q_i - q_j\| = d_c\} \tag{10}$$

We have proved that the derivative of $\varphi = \sum_{i=1}^{n} \varphi_i$ is negative definite across the trajectories of the system except from a set of initial conditions of measure zero whenever $q \notin S$ (see [6]). On the switching surface the Lyapunov function is no longer smooth so we must use stability theory for nonsmooth systems. In the case when $q \in S$ we shall make use of theorem B.6. First we must use the following lemma to ensure that φ is regular.

Lemma A.1 *The function φ is regular $\forall q \in S$.*

Proof of Lemma A.1: We show first that β_{ij} is regular whenever $\|q_i - q_j\| = d_C$. The directional derivative at d_C is

$$\beta'_{ij}(d_C; v) = \lim_{t \to 0} \frac{\beta_{ij}(d_C + tv) - \beta_{ij}(d_C)}{t} = \begin{cases} 0, v \geq 0 \\ c < 0, v < 0 \end{cases}$$

The generalized directional derivative is

$$\beta^0_{ij}(d_C; v) = \limsup_{\substack{t \to 0 \\ y \to d_C}} \frac{\beta_{ij}(y + tv) - \beta_{ij}(y)}{t} = \begin{cases} 0, v \geq 0 \\ c < 0, v < 0 \end{cases}$$

so that $\beta^0_{ij}(d_C; v) = \beta'_{ij}(d_C; v)\ \forall v$. It is easy to check that the terms $\frac{\partial b_i}{\partial \beta_{ij}}, \frac{\partial G_i}{\partial b_i}$ are nonnegative so by virtue of Theorem 2.3.9 (i), [4], the function G_i is regular at $q \in S$.

Function φ_i is continuously differentiable wrt G_i. In this case the term $\frac{\partial \varphi_i}{\partial G_i}$ is nonpositive but we are fortunate that G_i is 1-dimensional. Following the proof of theorem 2.3.9 (ii),[4] we can see that the generalized derivative of φ_i satisfies the following inequality: $\varphi^0_i(q; v) \leq \frac{\partial \varphi_i}{\partial G_i} G^0_i(q; v) = \frac{\partial \varphi_i}{\partial G_i} G'_i(q; v) = \varphi'_i(q; v)$. But we always have $\varphi'_i(q; v) \leq \varphi^0_i(q; v)$, so that $\varphi'_i(q; v) = \varphi^0_i(q; v)$, ensuring the regularity of φ_i. The function φ is regular as the finite linear combination by nonnegative scalars of regular functions.$\diamondsuit$

We now proceed with the proof of Proposition 2. We make use of the following matrix theorems in our analysis:

Theorem A.2 [11]*: Given a matrix $A \in R^{n \times n}$ then all its eigenvalues lie* *in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\} \triangleq \bigcup_{i=1}^{n} R_i(A) \triangleq R(A)$$

Each of these discs is called a Gersgorin disc of A.

Corollary A.3 [11]*: Given a matrix $A \in R^{n \times n}$ and n positive real numbers $p_1, \ldots, p_n$ then all the eigenvalues of A lie in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \, |a_{ij}| \right\}$$

A key point of Corollary A.3 is that if we bound the first $n/2$ Gersgorin discs of a matrix A sufficiently away from zero, then there exist real numbers $p_1, \ldots, p_n$ rendering the remaining $n/2$ discs sufficiently close to the corresponding diagonal elements. Hence, by ensuring the positive definiteness of the eigenvalues of the matrix M corresponding to the first $n/2$ rows, then we can render the remaining ones sufficiently close to the corresponding diagonal elements. This fact will be made clearer in the analysis that follows.

Proof of Proposition 2: In the global sensing case, the Proximity function between agents i and j is given by:

$$\beta_{ij}(q) = \|q_i - q_j\|^2 - (r_i + r_j)^2 = q^T D_{ij} q - (r_i + r_j)^2$$

where the $2N \times 2N$ matrix D_{ij} is defined in [18]. :

$$D_{ij} = \begin{bmatrix} & & O_{2(i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & I_{2\times 2} & O_{2\times 2(j-i-1)} & -I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(j-i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & -I_{2\times 2} & O_{2\times 2(j-i-1)} & I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(N-j)\times 2N} & & \end{bmatrix}$$

We can also write $b_r^i = q^T P_r^i q - \sum\limits_{j \in P_r} (r_i + r_j)^2$,where $P_r^i = \sum\limits_{j \in P_r} D_{ij}$, and P_r denotes the set of binary relations in relation r. It can easily be seen that $\nabla b_r^i = 2P_r^i q, \nabla^2 b_r^i = 2P_r^i$. We also use the following notation for the r-th relation wrt agent i:

$$g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + \left(\tilde{b}_r^i\right)^{1/h}}, \tilde{b}_r^i = \prod_{\substack{s \in S_r \\ s \neq r}} b_s^i,$$
$$\nabla \tilde{b}_r^i = \sum_{\substack{s \in S_r \\ s \neq r}} \underbrace{\prod_{\substack{t \in S_r \\ t \neq s,r}} b_i^t}_{\tilde{b}_{s,r}^i} \cdot 2P_s^i q$$

where S_r denotes the set of relations in the same level with relation r. An easy calculation shows that

$$\nabla g_r^i = \ldots = 2\left[d_r^i P_r^i - w_r^i \tilde{P}_r^i\right] q \triangleq Q_r^i q, \tilde{P}_r^i \triangleq \sum_{\substack{s \in S_r \\ s \neq r}} \tilde{b}_{s,r}^i P_s^i$$

where $d_r^i = 1 + (1 - \frac{b_r^i}{b_r^i + (\tilde{b_r^i})^{1/h}}) \frac{\lambda}{b_r^i + (\tilde{b_r^i})^{1/h}}$, $w_r^i = \frac{\lambda b_r^i (\tilde{b_r^i})^{\frac{1}{h}-1}}{h(b_r^i + (\tilde{b_r^i})^{1/h})^2}$. The gradient of the G_i function is given by:

$$G_i = \prod_{r=1}^{N_i} g_r^i \Rightarrow \nabla G_i = \sum_{r=1}^{N_i} \underbrace{\prod_{\substack{l=1 \\ l \neq r}}^{N_i} g_l^i}_{\tilde{g}_r^i} \nabla g_r^i = \sum_{r=1}^{N_i} \tilde{g}_r^i Q_r^i q \triangleq Q_i q$$

where N_i all the relations with respect to agent i. We define

$$\nabla G \triangleq \begin{bmatrix} \nabla G_1 \\ \vdots \\ \nabla G_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} q \triangleq Qq$$

Remembering that $u_i = -K_i \frac{\partial \varphi_i}{\partial q_i}$ and that $\varphi_i = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}}, f_i = \sum\limits_{j=0}^{3} a_i G_i^j$

the closed loop dynamics of the system are given by:

$$\dot{q} = \begin{bmatrix} -K_1 A_1^{-(1+1/k)} \left\{ G_1 \frac{\partial \gamma_{d1}}{\partial q_1} + \sigma_1 \frac{\partial G_1}{\partial q_1} \right\} \\ \vdots \\ -K_N A_N^{-(1+1/k)} \left\{ G_N \frac{\partial \gamma_{dN}}{\partial q_N} + \sigma_N \frac{\partial G_N}{\partial q_N} \right\} \end{bmatrix} = \dots$$
$$= -A_K G \left(\partial \gamma_d\right) - A_K \Sigma Q q$$

where $\sigma_i = G_i \sigma(G_i) - \frac{\gamma_{di}+f_i}{k}$, $\sigma(G_i) = \sum\limits_{j=1}^{3} j a_j G_i^{j-1}$,$A_i = \left(\gamma_{di} + f_i\right)^k + G_i$ and the matrices

$$A_K \triangleq \underbrace{diag \begin{pmatrix} K_1 A_1^{-(1+1/k)}, K_1 A_1^{-(1+1/k)}, \dots \\ , K_N A_N^{-(1+1/k)}, K_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N \times 2N}$$

$$G \triangleq \underbrace{diag\left(G_1, G_1, \dots, G_N, G_N\right)}_{2N \times 2N}, \left(\partial \gamma_d\right) = \left[\frac{\partial \gamma_{d1}}{\partial q_1} \dots \frac{\partial \gamma_{dN}}{\partial q_N} \right]$$

$$\Sigma \triangleq \underbrace{\left[\underbrace{\Sigma_1}_{2N \times 2N}, \dots, \underbrace{\Sigma_N}_{2N \times 2N} \right]}_{2N \times 2N^2}, \Sigma_i = diag \begin{pmatrix} 0, 0, \dots, \underbrace{\sigma_i, \sigma_i}_{2i-1, 2i} \\ , \dots, 0, 0 \end{pmatrix}$$

By using $\varphi = \sum\limits_i \varphi_i$ as a candidate Lyapunov function we have $\varphi = \sum\limits_i \varphi_i \Rightarrow \dot{\varphi} = \left(\sum\limits_i \left(\nabla \varphi_i\right)^T \right) \dot{q}$,$\nabla \varphi_i = A_i^{-(1+1/k)} \left\{ G_i \nabla \gamma_{di} + \sigma_i \nabla G_i \right\}$ and after some trivial calculation

$$\sum_i \left(\nabla \varphi_i\right)^T = \dots = \left(\partial \gamma_d\right)^T A_G + q^T Q^T A_\Sigma$$

where $A_G = \underbrace{diag\begin{pmatrix} G_1 A_1^{-(1+1/k)}, G_1 A_1^{-(1+1/k)}, \ldots, \\ G_N A_N^{-(1+1/k)}, G_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N\times 2N}$ and

$$A_\Sigma = \underbrace{\begin{bmatrix} \underbrace{A_{\Sigma_1}}_{2N\times 2N} \\ \vdots \\ \underbrace{A_{\Sigma_N}}_{2N\times 2N} \end{bmatrix}}_{2N^2\times 2N}, A_{\Sigma_i} = \underbrace{diag\begin{pmatrix} A_i^{-(1+1/k)}\sigma_i, \ldots, \\ A_i^{-(1+1/k)}\sigma_i \end{pmatrix}}_{2N\times 2N}$$

The derivative of the candidate Lyapunov function is calculated as

$$\begin{aligned} \dot{\varphi} &= \left(\sum_i (\nabla\varphi_i)^T\right) \cdot \dot{q} = \ldots \\ &= -\begin{bmatrix} (\partial\gamma_d)^T & q^T \end{bmatrix} \underbrace{\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}}_{M} \begin{bmatrix} \partial\gamma_d \\ q \end{bmatrix} \end{aligned}$$

where $M_1 = A_G A_K G, M_2 = A_G A_K \Sigma Q, M_3 = Q^T A_\Sigma A_K G, M_4 = Q^T A_\Sigma A_K \Sigma Q$.

Let's return to the local sensing case.Let $S_1 = \{q : \exists i, j, i \neq j | (\|q_i - q_j\| = d_c) \bigwedge (\|q_k - q_l\| \neq d_c \forall k, l : k \neq i, j, l \neq i, j)\}$ denote the subset of S which corresponds to the simplest case of switching that involves only two agents. System dynamics are given by:

$$\dot{q} = f(q) = \left[-K_1 \frac{\partial\varphi_1}{\partial q_1}, ..., -K_n \frac{\partial\varphi_n}{\partial q_n}\right]^T$$

The vector function $f(q)$ is nonsmooth at S_1 so that $\dot{q} \in K[f](q), q \in S_1$. We have $K[f](q \in S_1) = \overline{co}\{f_{S_1}^-, f_{S_1}^+\}$ where $S_1^{-(+)} = \{q : \|q_i - q_j\| < (>) d_C\}$ and

$$f_{S_1}^{-(+)}(q \in S_1) = \lim_{\substack{q^* \to q, \\ q^* \in S_1^{-(+)}}} f(q^*)$$

Likewise, the generalized gradient of the candidate Lyapunov function at the discontinuity surface is given by $\partial\varphi(q \in S_1) = \overline{co}\{\nabla\varphi_{S_1}^-, \nabla\varphi_{S_1}^+\}$ where

$$\nabla\varphi_{S_1}^{-(+)}(q \in S_1) = \lim_{\substack{q^* \to q, \\ q^* \in S_1^{-(+)}}} \nabla\varphi(q^*)$$

Each $\rho \in \partial\varphi(q \in S_1)$ is the convex combination of the limit points of the convex hull: $\rho = \mu\left(\nabla\varphi_{S_1}^-\right) + (1-\mu)\left(\nabla\varphi_{S_1}^+\right), \mu \in [0,1]$. Similarly, each $\eta \in K[f](q \in S_1)$ as $\eta = \lambda f_{S_1}^- + (1-\lambda)f_{S_1}^+, \lambda \in [0,1]$, so that $\rho^T\eta = \lambda\mu\left(\nabla\varphi_{S_1}^-\right)^T f_{s_1}^- + (1-\lambda)\mu\left(\nabla\varphi_{S_1}^-\right)^T f_{s_1}^+ + \lambda(1-\mu)\left(\nabla\varphi_{S_1}^+\right)^T f_{s_1}^- + (1-\lambda)(1-\mu)\left(\nabla\varphi_{S_1}^+\right)^T f_{s_1}^+$. By virtue of theorem B.5 of Appendix B one has

$$\dot\varphi(q \in S_1) \in \bigcap_{\rho\in\partial\varphi(q\in S_1)} \rho^T\eta,\ \eta \in K[f](q \in S_1)$$

Going back to the previous analysis, it is easy to see that the matrices $A_G, A_K, G, \Sigma, A_\Sigma$ are continuous in the discontinuity surface. The matrix Q is discontinuous at S_1 and that's due to the nonsmoothness of the functions G_i, G_j. By using the notation $Q^-(q \in S_1) = \lim_{\substack{q^*\to q \\ q^*\in S_1^-}} Q(q^*)$, $Q^+(q \in S_1) = \lim_{\substack{q^*\to q \\ q^*\in S_1^+}} Q(q^*)$ and noting that $\bigcap_{\rho\in\partial\varphi(q\in S_1)} \rho^T\eta = \bigcap_{\mu\in[0,1]} \left\{\rho^T\eta | \lambda \in [0,1]\right\}$ we conclude after some trivial calculation that

$$\dot\varphi(q \in S_1) \in \bigcap_{\mu\in[0,1]} \left\{ \begin{array}{l} -\left[\begin{array}{cc} (\partial\gamma_d)^T & q^T \end{array}\right] M \left[\begin{array}{c} \partial\gamma_d \\ q \end{array}\right] \\ |\lambda \in [0,1] \end{array} \right\}$$

with $M = \left[\begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array}\right]$ where

$$M_1 = A_G A_K G, M_2 = A_G A_K \Sigma\left(\lambda Q^- + (1-\lambda)Q^+\right)$$

$$M_3 = \left(\mu\left(Q^-\right)^T + (1-\mu)\left(Q^+\right)^T\right) A_\Sigma A_K G$$

$$M_4 = \lambda\mu \left(Q^-\right)^T A_\Sigma A_K \Sigma Q^- + (1-\lambda)\,\mu \left(Q^-\right)^T A_\Sigma A_K \Sigma Q^+ +$$
$$\lambda (1-\mu) \left(Q^+\right)^T A_\Sigma A_K \Sigma Q^- + (1-\lambda)(1-\mu) \left(Q^+\right)^T A_\Sigma A_K \Sigma Q^+$$

We first proceed by examining the Gersgorin discs of the first half rows of the matrix M. We denote this procedure as $M_1 - M_2$, as the main diagonal elements of M_1 are "compared" with the corresponding raw elements of M_2. Note that the submatrices M_1, M_2 are both diagonal, therefore the only nonzero elements of raw i of the $4N \times 4N$ matrix M are the elements $M_{ii}, M_{i,2N+i}$ where of course $1 \leq i \leq 2N$ as we calculate the Gersgorin discs of the first half rows of the matrix M. With respect to corollary A.3, we have:

$$|z - M_{ii}| \leq \tfrac{1}{p_i} \sum_{j \neq i} p_j \, |M_{ij}|, 1 \leq i \leq 2N \Rightarrow$$
$$\Rightarrow |z - (M_1)_{ii}| \leq \tfrac{p_{2N+i}}{p_i} \, |(M_2)_{ii}|$$

where

$$(M_1)_{ii} = A_i^{-2(1+1/k)} K_i G_i^2$$

and

$$|(M_2)_{ii}| = \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i \cdot \left\{ \lambda \left(Q_{ii}^i\right)^+ + (1-\lambda) \left(Q_{ii}^i\right)^- \right\} \right|$$

Denote $\left| \lambda \left(Q_{ii}^i\right)^+ + (1-\lambda) \left(Q_{ii}^i\right)^- | \lambda \in [0,1] \right| \triangleq \left| \left(Q_{ii}^i\right)^\pm \right|$, where $Q_i i^i$ the ii-th element of Q_i. It is then obvious that $\left| \left(Q_{ii}^i\right)^\pm \right|_{\max} = \max \left\{ \left| \left(Q_{ii}^i\right)^- \right|_{\max}, \left| \left(Q_{ii}^i\right)^+ \right|_{\max} \right\}$, which is always bounded in a bounded workspace. Therefore we have:

$$\left| z - A_i^{-2(.)} K_i G_i^2 \right| \leq \tfrac{p_{2N+i}}{p_i} \left| A_i^{-2(.)} \sigma_i K_i G_i \left(Q_{ii}^i\right)^\pm \right|$$
$$\Rightarrow z \geq A_i^{-2(.)} K_i G_i^2 - \tfrac{p_{2N+i}}{p_i} \left| A_i^{-2(.)} \sigma_i K_i G_i \left(Q_{ii}^i\right)^\pm \right|$$

We examine the following three cases:

- $G_i < \varepsilon$ At a critical point in this region, the corresponding eigenvalue tends to zero, so that the derivative of the Lyapunov function could achieve zero values. However, the result of Lemma 6 in [6] indicates that φ_i is a Morse function, hence its critical points are isolated[16].

Thus the set of initial conditions that lead to saddle points are sets of measure zero[21].

- $G_i > X$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$\begin{aligned}
&z > 0 \Leftarrow A_i^{-2(.)} K_i \left(G_i - \tfrac{p_{2N+i}}{p_i} \left|\sigma_i\right| \left|(Q_{ii}^i)^{\pm}\right| \right) > 0 \\
&\Leftarrow G_i \geq X > \tfrac{p_{2N+i}}{p_i} \left|\sigma_i\right| \left|(Q_{ii}^i)^{\pm}\right| = \tfrac{\gamma_{di}}{k} \tfrac{p_{2N+i}}{p_i} \left|(Q_{ii}^i)^{\pm}\right| \\
&\Leftarrow k > \tfrac{(\gamma_{di})_{\max}}{X} \tfrac{p_{2N+i}}{p_i} \left|(Q_{ii}^i)^{\pm}\right|_{\max}
\end{aligned}$$

- $0 < \varepsilon \leq G_i \leq X$ In [6], we prove that $|\sigma_i(\varepsilon)| \leq Y \left|\frac{1}{k} + \frac{8}{9}\right| + \left|\frac{\gamma_{di}}{k}\right|$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$\begin{aligned}
&z > 0 \Leftarrow \varepsilon > \left\{ Y \left|\tfrac{1}{k} + \tfrac{8}{9}\right| + \left|\tfrac{\gamma_{di}}{k}\right| \right\} \tfrac{p_{2N+i}}{p_i} \left|(Q_{ii}^i)^{\pm}\right|_{\max} \\
&\overset{Y \leq \frac{\Theta_1}{k}}{\Longleftarrow}
\end{aligned}$$

$$k > 2 \max \left\{ 2\sqrt{\frac{\Theta_1}{\varepsilon}}, \frac{16\Theta_1}{9\varepsilon}, \frac{(\gamma_{di})_{\max}}{\varepsilon} \right\} \frac{p_{2N+i}}{p_i} \left|\left(Q_{ii}^i\right)^{\pm}\right|_{\max}$$

A key point is that there is no restriction on how to select the terms $\frac{p_{2N+i}}{p_i}$. This will help us in deriving bounds that guarantee the positive definiteness of the matrix M.

We are now left to examine the Gersgorin discs of the second half rows of the matrix M. Likewise, we denote this procedure as $M_3 - M_4$. The discs of Corollary A.3 are evaluated:

$$\begin{aligned}
&|z - M_{ii}| \leq \sum_{j \neq i} \tfrac{p_j}{p_i} |M_{ij}|, 2N+1 \leq i \leq 4N, 1 \leq j \leq 4N \\
&\Rightarrow |z - (M_4)_{ii}| \leq R_i(M_3) + R_i(M_4)
\end{aligned}$$

where $R_i(M_3) = \sum\limits_{j=1}^{2N} \frac{p_j}{p_i} \left|(M_3)_{ij}\right|$,$R_i(M_4) = \sum\limits_{\substack{j=2N+1 \\ j\neq i}}^{4N} \frac{p_j}{p_i} \left|(M_4)_{ij}\right|$ and

$$(M_4)_{ii} = \sum_j \left(\begin{array}{l} K_i A_i^{-(1+1/k)} A_j^{-(1+1/k)} \sigma_j \sigma_i \cdot \\ \cdot \left\{ \begin{array}{l} \lambda\mu \left(Q_{ii}^i\right)^- \left(Q_{ii}^j\right)^- + (1-\lambda)\,\mu \left(Q_{ii}^i\right)^- \left(Q_{ii}^j\right)^+ + \\ \lambda\,(1-\mu) \left(Q_{ii}^i\right)^+ \left(Q_{ii}^j\right)^- + \\ (1-\lambda)\,(1-\mu) \left(Q_{ii}^i\right)^+ \left(Q_{ii}^j\right)^+ |\lambda \in [0,1] \end{array} \right\} \end{array} \right)$$

Following the same procedure as in [6], it can easily be shown that $R_i(M_3) \geq R_i(M_4) \forall i$.

The corresponding eigenvalue is guaranteed to be positive as long as:

$$\begin{array}{l} z > 0 \Leftarrow (M_4)_{ii} > R_i(M_3) + R_i(M_4) \\ \Leftarrow (M_4)_{ii} > \max\left\{2R_i(M_3), 2R_i(M_4)\right\} = 2R_i(M_3) \end{array}$$

Choosing without loss of generality $p_i = p, 2N+1 \leq i \leq 4N$, we have after some non-trivial calculations:

$$R_i(M_3) = \sum_{j=1}^{2N} \frac{p_i}{p} \underbrace{\left| \begin{array}{l} A_j^{-2(1+1/k)} \sigma_j K_j G_j \left(\mu \left(Q_{ii}^j\right)^- + (1-\mu) \left(Q_{ii}^j\right)^+ \right) + \\ (A_j A_i)^{-(1+1/k)} \sigma_i K_j G_j \left(\mu \left(Q_{jj}^i\right)^- + (1-\mu) \left(Q_{jj}^i\right)^+ \right) \end{array} \right|}_{\left|(M_3)_{ij}\right|}$$

The fact that $(M_4)_{ii} > 0$ is guaranteed by Lemma 2.3 in [6]. This lemma also guarantees that there is always a finite upper bound on the terms We have

$$\begin{array}{l} (M_4)_{ii} > 2R_i(M_3) = 2 \sum\limits_{j=1}^{2N} \frac{p_j}{p} \left|(M_3)_{ij}\right| \Leftarrow \\ p > \frac{4N}{(M_4)_{ii}} \max\limits_j \left\{ p_j \left|(M_3)_{ij}\right| \right\}, \\ 2N+1 \leq i \leq 4N, 1 \leq j \leq 2N \end{array}$$

We can now directly apply theorem B.6 to our case. We have proved that $v \leq 0 \; \forall v \in \dot{\tilde{\varphi}}$ and that the only invariant subset of the set $S = \{q | 0 \in \dot{\tilde{\varphi}}(q)\}$ is $\left\{ q_d = [q_{d1}, ..., q_{dn}]^T \right\}$. Hence the nonsmooth version of LaSalle's invariance principle guarantees convergence to the destination points. $\diamondsuit$

B Elements from Nonsmooth Analysis

In this section, we review some elements from nonsmooth analysis and Lyapunov theory for nonsmooth systems that we use in the stability analysis of Appendix A.

We consider the vector differential equation with discontinuous right-hand side:

$$\dot{x} = f(x) \tag{11}$$

where $f : R^n \to R^n$ is measurable and essentially locally bounded.

Definition B.1 [9]*: In the case when n is finite, the vector function $x(.)$ is called a solution of (11) in $[t_0, t_1]$ if it is absolutely continuous on $[t_0, t_1]$ and there exists $N_f \subset R^n, \mu(N_f) = 0$ such that for all $N \subset R^n, \mu(N) = 0$ and for almost all $t \in [t_0, t_1]$*

$$\dot{x} \in K[f](x) \equiv \overline{co}\{\lim_{x_i \to x} f(x_i) | x_i \notin N_f \cup N\}$$

Lyapunov stability theorems have been extended for nonsmooth systems in [24],[3]. The authors use the concept of *generalized gradient* which for the case of finite-dimensional spaces is given by the following definition:

Definition B.2 [4]*: Let $V : R^n \to R$ be a locally Lipschitz function. The generalized gradient of V at x is given by*

$$\partial V(x) = \overline{co}\{\lim_{x_i \to x} \nabla V(x_i) | x_i \notin \Omega_V\}$$

where Ω_V is the set of points in R^n where V fails to be differentiable.

Lyapunov theorems for nonsmooth systems require the energy function to be *regular*. Regularity is based on the concept of *generalized derivative* which was defined by Clarke as follows:

Definition B.3 [4]*: Let f be Lipschitz near x and v be a vector in R^n. The generalized directional derivative of f at x in the direction v is defined*

$$f^0(x;v) = \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y+tv) - f(y)}{t}$$

Definition B.4 [4]*: The function $f : R^n \to R$ is called regular if*
1) $\forall v$, the usual one-sided directional derivative $f'(x;v)$exists and
2) $\forall v, f'(x;v) = f^0(x;v)$

The following chain rule provides a calculus for the time derivative of the energy function in the nonsmooth case:

Theorem B.5 [24]*: Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V : R^n \to R$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x(t))$$

We shall use the following nonsmooth version of LaSalle's invariance principle to prove the convergence of the prescribed system:

Theorem B.6 [24] *Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \to R$ be a time independent regular function such that $v \leq 0 \forall v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set,M, in the closure of S.*

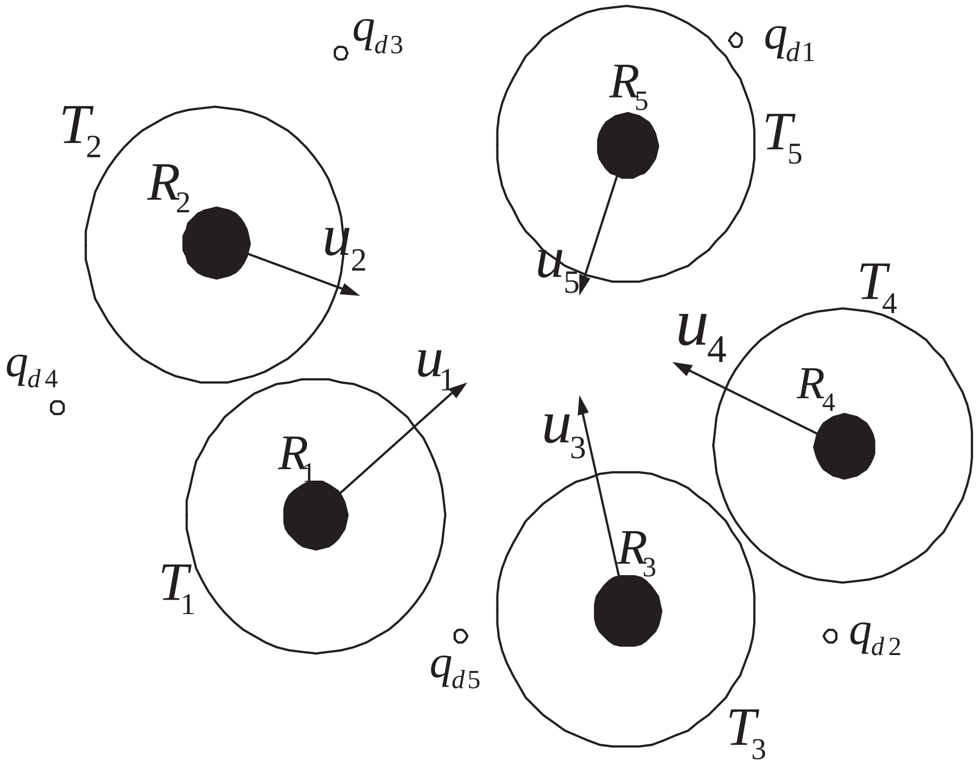

Figure 1: A conflict scenario with five agents. Each agent i occupies a disc R_i(black discs) of radius r_i centered at q_i. Each agent's sensing zone T_i(white discs) is centered at q_i and has radius d_C.

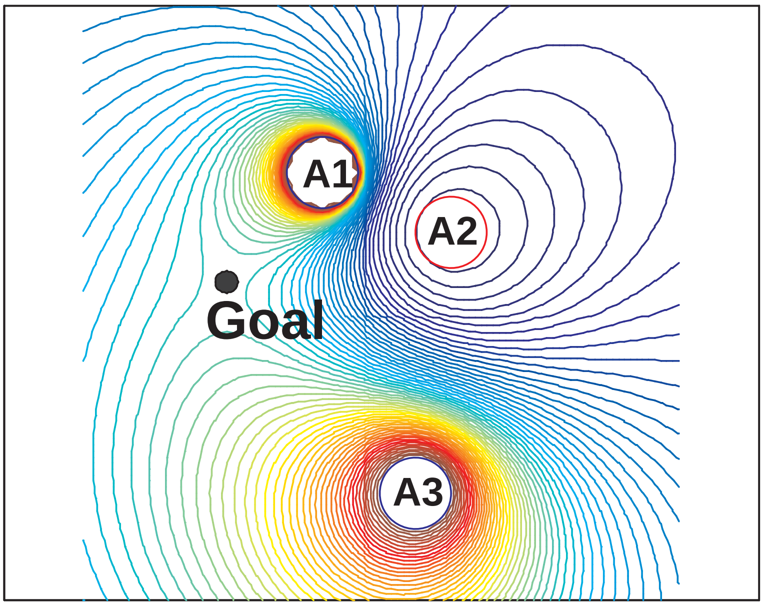

Figure 2: A DNF in an environment with three moving agents

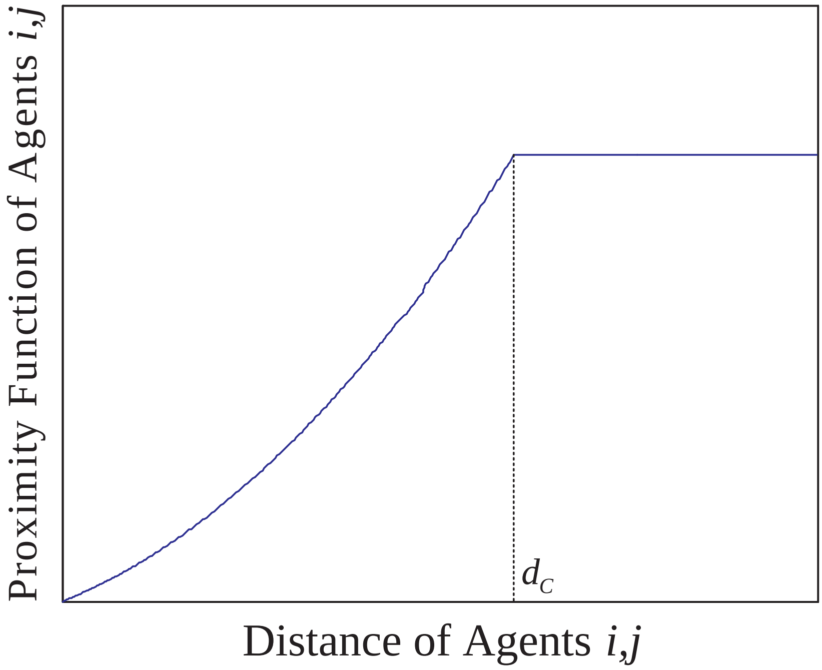

Figure 3: The function β_{ij} for $r_i + r_j = 1, d_c = 4$.

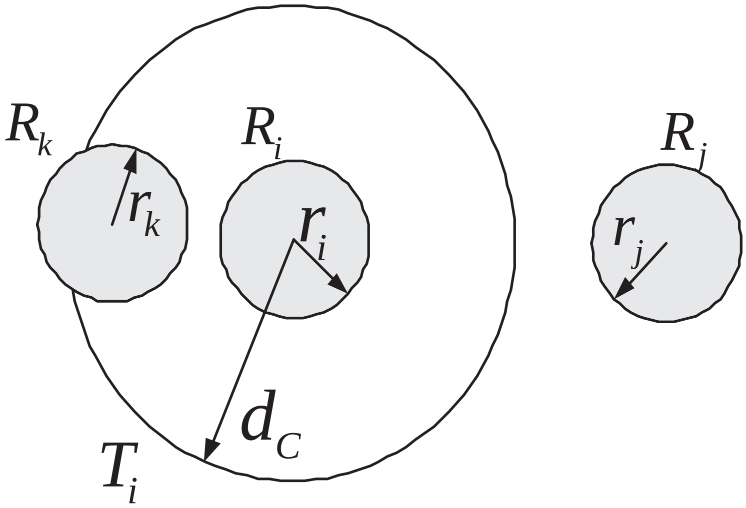

Figure 4: Agent k is within the sensing zone T_i of agent i, therefore $\beta_{ik} = \|q_i - q_k\|^2 - (r_i + r_k)^2$. Agent j is outside the sensing zone T_i of agent i, therefore $\beta_{ij} = d_C^2 - (r_i + r_j)^2$.

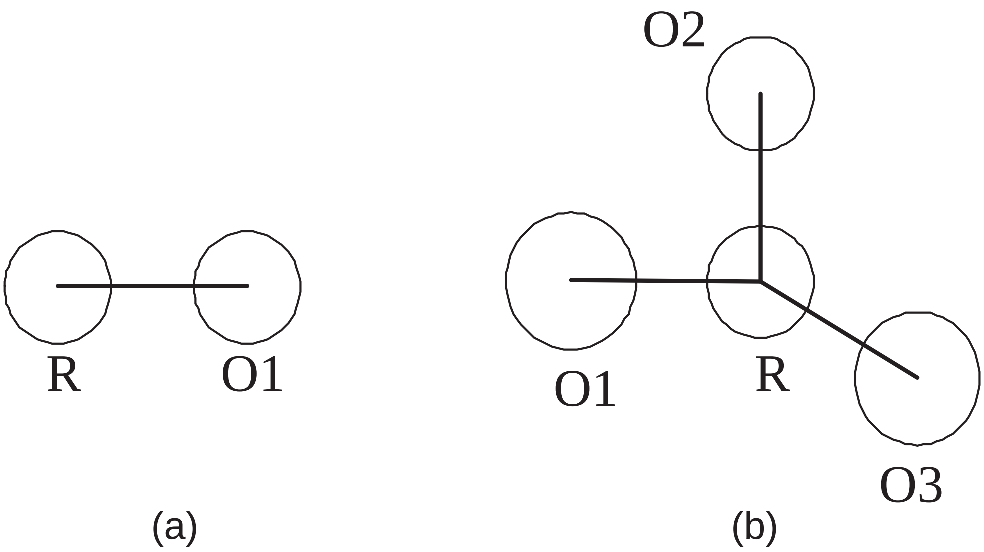

Figure 5: Part a represents a level-1 relation and part b a level-3 relation wrt agent R.

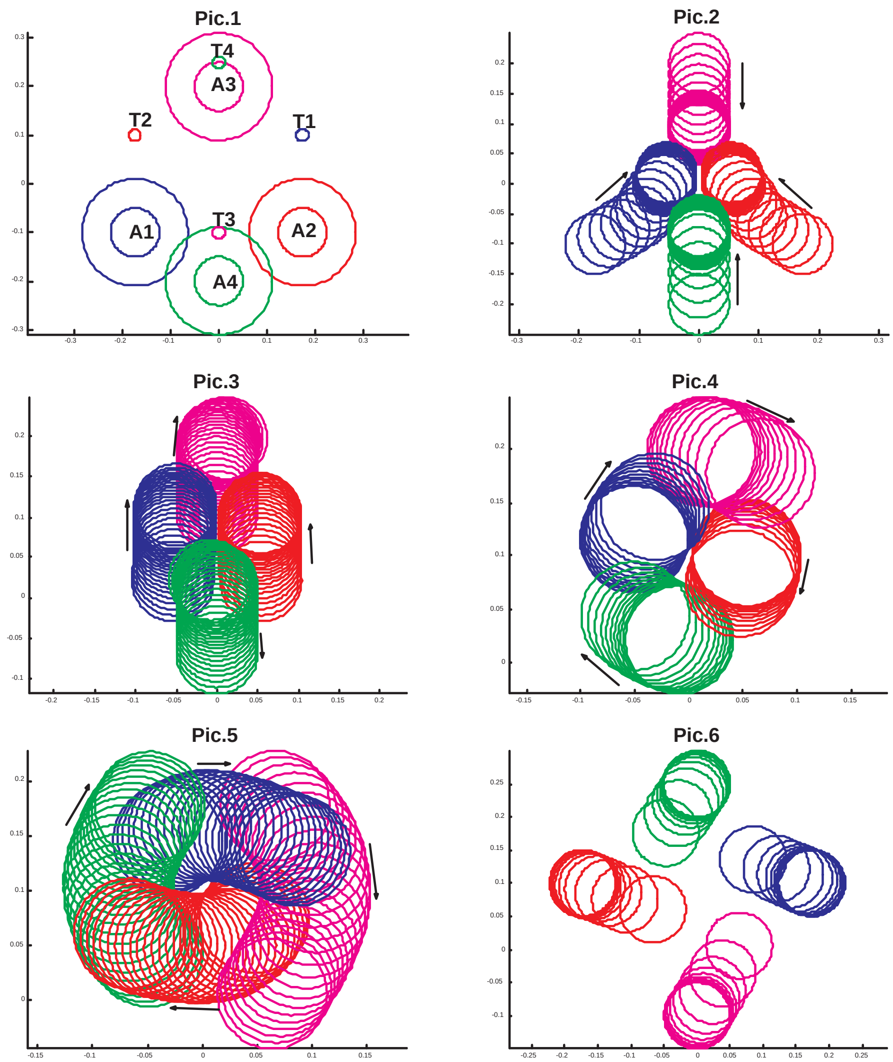

Figure 6: Simulation A

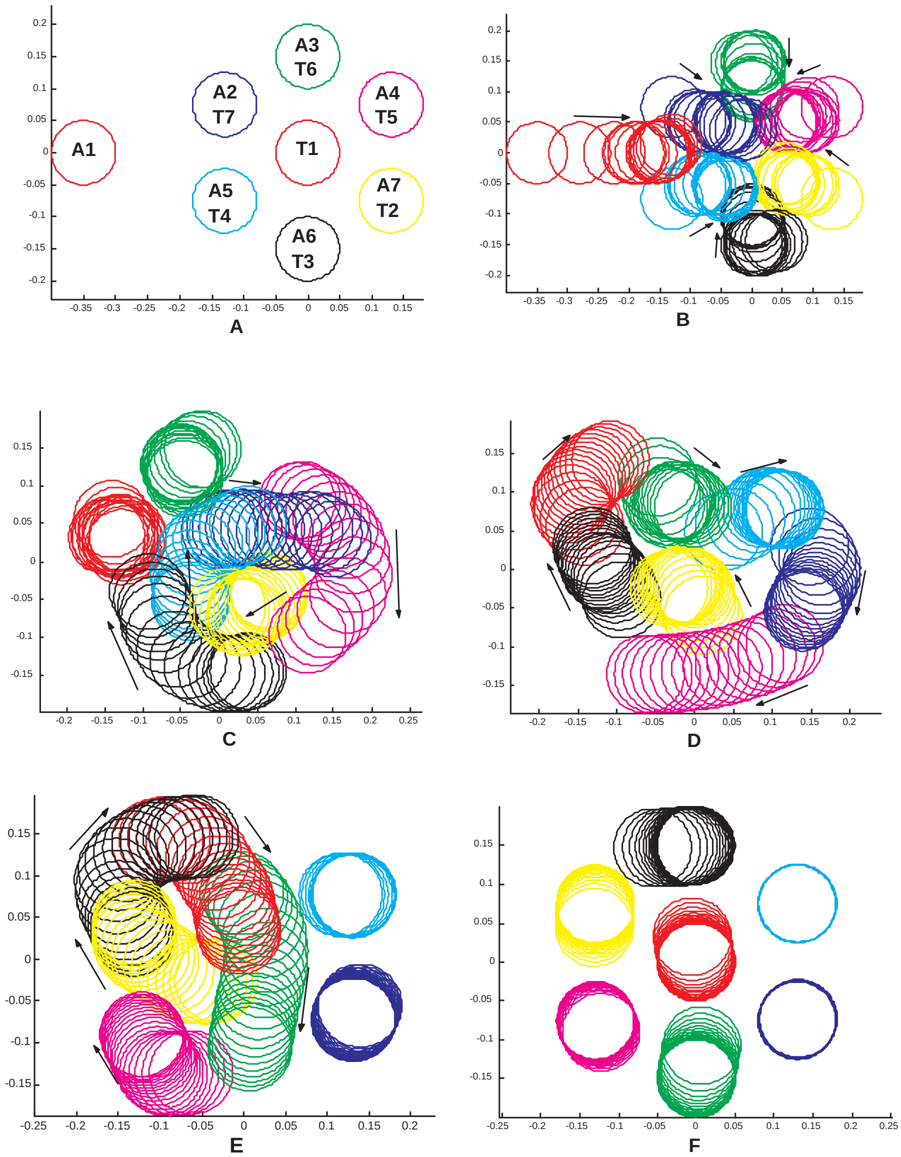

Figure 7: Simulation B