Distributed Event-Based Control and Stability of Interconnected Systems

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***Abstract*— This paper presents sufficient conditions that characterize the stability properties of certain classes of interconnected systems. The considered classes of systems include autonomous, continuous and discrete time nonlinear systems coupled with linear or nonlinear interconnection terms. These conditions are then exploited for the decentralized event-based control of interconnected systems. Examples illustrate the theoretical results and simulations show the effectiveness of the proposed event-based techniques.**

I. Introduction

Vector Lyapunov functions were first introduced by Bellman [2] and Matrosov [15] and have extended the classical Lyapunov framework for the analysis of large-scale systems. Compared to a single scalar Lyapunov function, the use of vector Lyapunov functions has been proven to be more flexible for studying stability. Indeed, instead of using a single Lyapunov function to study the stability of the overall system, each component of the vector Lyapunov function can be used at different hierarchical levels. The advantages and flexibility of vector Lyapunov functions have been extensively used to study a variety of stability problems for complex or large-scale systems, see for instance [1], [6], [13], [14], [16], [17], [19], [25], [28] and references therein. Stabilization of nonlinear systems by vector control Lyapunov functions has also been considered in [11] and [18].

In this paper we present a novel distributed stability test for interconnected systems. In particular, similar to [1] and [19], we start by considering a weighted sum of Lyapunov functions $d_1V_1 + d_2V_2 + \ldots + d_NV_N$, $d_i > 0$, N being the number of subsystems, which will serve as a Lyapunov function for the overall interconnected system. Traditionally, the existence of the d_i's above that establish asymptotic or exponential stability of the interconnected system is based on M-matrices, namely, matrices with nonpositive off-diagonal elements and positive principal minors. In the proposed approach, instead of testing if a matrix is a M-matrix in a centralized way, i.e., calculating the principal minors and then finding suitable d_i's, we determine each d_i through the solutions of a quadratic inequality. In particular, we first associate the N subsystems to N different quadratic inequalities which are independent of each other. Then, the solutions of those N quadratic inequalities will form a stability criterion for the overall interconnected system and will provide different margins for the selection of the weights d_i, $i = 1, \ldots, N$ which also affect the region of attraction of the total system. We note that a distributed stability test by using optimization techniques was also presented in [8] for linear systems with symmetric interactions which is not the case in our work. The class of systems under consideration may result from the decomposition of large-scale systems into low-dimensional subsystems, structural perturbations, unmodeled dynamics or physical couplings, see for instance [19]. While we mainly focus on autonomous continuous-time systems, we also include an extension for the discrete-time case.

Finally, by leveraging techniques from [7], [10], [21], and [27], we exploit the previous sufficient conditions to study the event-triggered control of interconnected systems. More specifically, the solutions of the quadratic inequalities are used to define state-dependent triggering mechanisms for the cases of sampled feedback stabilization and state-broadcasting. Different mechanisms are used by each subsystem to update its control law and to broadcast its state at the same or different times. Note that [27] required global input-to-state stability assumptions while a uniform lower-bound on the inter-event period was not provided which is not the case in our work. Other non-trivial ISS designs and small gain conditions were also used in [7]. A small gain approach was also employed for sampled-data stabilization in [24].

The rest of the paper is organized as follows. Section II contains the notation and definitions. Section III contains the main result and its extensions to several classes of systems. These sufficient conditions are exploited in Section IV for the event-based control of interconnected systems.

II. Preliminaries

Notations. We recall first some basic concepts and definitions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$, if it is continuous and strictly increasing with $\alpha(0) = 0$. If in addition $\lim_{s\to\infty} \alpha(s) = \infty$, then α is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed t, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s it is decreasing to zero as $t \to \infty$. By $|x|$ we denote both the Euclidean norm of a vector $x \in \mathbb{R}^n$ and the absolute value for a scalar. By $\lambda_m(A)$ and $\lambda_M(A)$ we denote the minimum and maximum real part of the eigenvalues of $A \in \mathbb{R}^{n\times n}$, respectively. By $c(A)$ we denote the cardinality of a set A. With $\det(A)$ we denote the determinant of a matrix $A \in \mathbb{R}^{n\times n}$.

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Consider a system $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$, $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz, $0 \in \mathcal{D} \subset \mathbb{R}^n$ and $f(0) = 0$.

Definition 2.1: The equilibrium $x = 0$ of system $\dot{x} = f(x)$ is *locally asymptotically stable* if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the equilibrium $x = 0$ such that for any initial state $x(0) \in \mathcal{D}_0$ the solution exists for all $t \geq 0$ and satisfies $|x(t)| \leq \beta(|x(0)|, t)$. The equilibrium $x = 0$ of $\dot{x} = f(x)$ is *locally exponentially stable* if $\beta(|x(0)|, t) := \kappa |x(0)| e^{-\mu t}$, κ, $\mu > 0$.

Definition 2.2: Let $x = 0$ be an asymptotically stable equilibrium point of system $\dot{x} = f(x)$ and let $x(t; x_0)$ be the solution of the system. The region of attraction of the origin is $\mathcal{R} := \{x \in \mathcal{D} :\ x(t; x_0) \text{ is defined} \, \forall t \geq 0, \text{ and } x(t; x_0) \to 0 \text{ as } t \to \infty\}$.

III. Sufficient Conditions for Stability

Consider a group of $i = 1, \ldots, N$ systems described by

$$\dot{x}_i = f_i(x_i) \tag{1}$$

which are interconnected through functions $H_i(\cdot)$ to form the system

$$\dot{x}_i = f_i(x_i) + H_i(x), \ \ i = 1, \ldots, N \tag{2}$$

where $x_i \in \mathcal{D}_i \subset \mathbb{R}^{n_i}$, $x = (x_1^T, \ldots, x_N^T)^T \in \mathbb{R}^n$, $n = \sum_{i=1}^N n_i$ and f_i, H_i are locally Lipschitz. We assume that $x = 0$ is an equilibrium for (2), i.e., $f_i(0) = 0$, $H_i(0) = 0$ for all $i \in \mathcal{N} = \{1, \ldots, N\}$. We assume that the following holds:

Assumption A1: For each $i \in \mathcal{N}$, there exist C^1 functions $V_i : \mathcal{D}_i \to \mathbb{R}$ and constants a_{i1}, a_{i2}, a_{i3}, $a_{i4} > 0$ such that

$$a_{i1}|x_i|^2 \leq V_i(x_i) \leq a_{i2}|x_i|^2, \ x_i \in \mathcal{D}_i \tag{3a}$$

$$\nabla V_i(x_i) f_i(x_i) \leq -a_{i3}|x_i|^2, \ x_i \in \mathcal{D}_i \tag{3b}$$

$$\big| \nabla V_i(x_i) \big| \leq a_{i4}|x_i|, \ x_i \in \mathcal{D}_i \tag{3c}$$

Assumption A1 implies that each decoupled system (1) is exponentially stable. In the following, we consider the regions of attraction $\mathcal{R}_i$ for each system (1) and in particular we consider any bounded estimate $\mathcal{S}_i \subset \mathcal{R}_i \subset \mathcal{D}_i$ of the form

$$\mathcal{S}_i := \{x_i \in \mathcal{R}_i : V_i(x_i) \leq c_i\}, \ c_i > 0. \tag{4}$$

Note that Assumption A1 holds for all $x_i \in \mathcal{S}_i$ and several techniques in the literature provide such estimates $\mathcal{S}_i$, see for instance [5], [9]. Next, define the set $\mathcal{S} := \mathcal{S}_1 \times \ldots \times \mathcal{S}_N$ which is an estimate of the region of attraction of (2) under no interaction. For the terms $H_i(\cdot)$ we assume that

Assumption A2: There exist $\xi_{ij} \geq 0$ such that $|H_i(x)| \leq \sum_{j=1}^N \xi_{ij}|x_j|$, $\ x \in \mathcal{S}$.

Assumption A2 imposes a restriction on the interconnection term and it is a standard assumption when studying the stability of interconnected systems. Note however that contrary to [19], A2 needs not to hold globally.

Next we define the neighborhood $\Delta_i = \Xi_i \cup \Theta_i$ of the i-th subsystem (1), which includes the systems $j \in \mathcal{N}$ that directly drive system $i \in \mathcal{N}$ through the interconnection term $H_i(\cdot)$, $\Theta_i = \{j \in \mathcal{N} : \xi_{ij} \neq 0\}$, as well as the systems $j \in \mathcal{N}$ that are driven by system i, $\Xi_i = \{j \in \mathcal{N} : \xi_{ji} \neq 0\}$. Note that we do not exclude the case where the interconnection term $H_i(\cdot)$ also depends on the state x_i, i.e., $\xi_{ii} \neq 0$.

Traditionally, using the above assumptions one chooses a composite Lyapunov function $V = \sum d_i V_i$ to establish sufficient conditions for the stability of the interconnected system. In particular, for the derivative of this composite Lyapunov along system (2) it holds that $\dot{V}(x) \leq -\frac{1}{2}\phi^T(DW + W^T D)\phi$ where $D = diag(d_1, \ldots d_n)$, $\phi = (|x_1|, \ldots, |x_n|)^T$, $W = (w_{ij})$ with $w_{ij} = a_{i3} - a_{i4}\xi_{ii}$ if $i = j$ and $w_{ij} = -a_{i4}\xi_{ij}$ if $i \neq j$. Then, asymptotic stability of the origin of the interconnected system follows if there exists a positive diagonal matrix D such that $DW + W^T D$ is positive definite. The latter is established by the following lemma:

Lemma 3.1: ([1], [19]) Consider a $n \times n$ matrix $W = (w_{ij})$ with $w_{ij} \leq 0$ for all $i, j = 1, \ldots, n$, $i \neq j$. The following are equivalent:

(i) All leading principal minors of W are positive.

(ii) There exists a positive diagonal matrix $D = diag(d_1, \ldots d_n)$ such that $DW + W^T D$ is positive definite.

Matrices satisfying Lemma 3.1 are called M-matrices. Thus, the asymptotic stability of the origin of the interconnected system (2) is established if the interconnection matrix W above is an M-matrix. However, determining if the matrix W is an M-matrix and selecting an appropriate matrix D, are performed in a centralized way in the sense that manipulations are required using the whole interconnection matrix W. Namely, first test if the leading principal minors of W and then select $D = diag(d_1, \ldots d_N)$. Note that if W in Lemma 3.1 is positive diagonally dominant, then it is also a M-matrix, [1], [19]. However, finding the constants $d_i > 0$ still remains a global problem.

We next present a sufficient condition to characterize the stability of an interconnected system where we only need to check the stability of each individual system. In particular we have

Proposition 3.1: Consider system (2) and assume that Assumptions A1 and A2 hold. Then,

(i) If for each $i \in \mathcal{N}$

$$a_{i3}^2 - c(\Theta_i) a_{i4}^2 \sum_{j=1}^N \xi_{ji}^2 > 0 \tag{5}$$

then the system is exponentially stable.

(ii) If there exists $i_0 \in \mathcal{N}$ such that (5) holds and

$$a_{i3}^2 - c(\Theta_i) a_{i4}^2 \sum_{j=1}^N \xi_{ji}^2 = 0, \ \forall i \in \mathcal{N} \setminus \{i_0\}; \tag{6a}$$

$$H_{i0}(x) \neq 0, \ x \neq 0, \tag{6b}$$

then the system is asymptotically stable.

(iii) The system is stable if one the following holds:

(a) (6a) holds and $H_{i0}(x) = 0$ for $x \neq 0$; or

(b) condition (6a) holds for all $i \in \mathcal{N}$.

Proof: For each $i \in \mathcal{N}$ and from Assumption A1, we consider the candidate composite Lyapunov function $V(x) = \sum_{i=1}^N d_i V_i(x_i)$ where d_i are positive constants to be selected appropriately, and note that $V(0) = 0$ and $V(x) > 0$, $x \neq 0$.

The derivative of V along the trajectories of the system is given by $\dot{V}(x) = \sum_{i=1}^{N} d_i \Big(\nabla V_i f_i(x_i) + \nabla V_i H_i(x)\Big)$. By taking into account Assumptions A1 and A2, we have that

$$\dot{V}(x) \leq -\sum_{i=1}^{N} d_i a_{i3}|x_i|^2 + \sum_{i=1}^{N} d_i a_{i4}|x_i| \sum_{j=1}^{N} \xi_{ij}|x_j|. \quad (7)$$

By applying the inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ with $x = d_i a_{i4}|x_i|$ and $y = \xi_{ij}|x_j|$ in the last term on the right hand side of (7) and by expanding the summations and rearranging terms we obtain $\sum_{i=1}^{N} d_i a_{i4}|x_i| \sum_{j=1}^{N} \xi_{ij}|x_j| \leq \frac{1}{2}\Big(\sum_{i=1}^{N} c(\Theta_i) d_i^2 a_{i4}^2 |x_i|^2 + \sum_{i=1}^{N} |x_i|^2 \sum_{j=1}^{N} \xi_{ji}^2\Big)$. Therefore, it follows from (7) that

$$\dot{V}(x) \leq \frac{1}{2}\sum_{i=1}^{N} |x_i|^2 \Big(c(\Theta_i) a_{i4}^2 d_i^2 - 2a_{i3} d_i + \sum_{j=1}^{N} \xi_{ji}^2\Big). \quad (8)$$

(i) Assume that (5) holds and notice that $\dot{V}(x)$ in (8) will be negative definite if for each $i \in \mathcal{N}$ there exists $d_i > 0$ such that

$$c(\Theta_i) a_{i4}^2 d_i^2 - 2a_{i3} d_i + \sum_{j=1}^{N} \xi_{ji}^2 = -q_i < 0, \;\; q_i > 0 \quad (9)$$

then, we would have from (8) and (9) that

$$\dot{V}(x) \leq -\sum_{i=1}^{N} \frac{q_i}{2}|x_i|^2, \quad (10)$$

which would imply exponential stability for system (2). Recall first that a_{i3}, a_{i4}, and $c(\Theta_i)$ are all positive constants. Define now

$$g_i(y) := \alpha_i y^2 + \beta_i y + \gamma_i, \quad (11)$$

where $\alpha_i := c(\Theta_i) a_{i4}^2$, $\beta_i := -2a_{i3}$, $\gamma_i := \sum_{j=1}^{N} \xi_{ji}^2$. Then, we have from (5) and (11) that there exist $y_{i1}, y_{i2} \in \mathbb{R}_{>0}$ such that $g_i(y) < 0$, $y \in (y_{i1}, y_{i2})$. Thus, inequality (9) is satisfied and consequently that (10) holds which implies exponential stability of $x = 0$. Finally, an estimate of the region of attraction $\mathcal{R}$ for the interconnected system is given as follows. Let $d_i \in (y_{i1}, y_{i2})$ and define $r := \min_{i \in \mathcal{N}} d_i c_i$, where $c_i > 0$ is given by (4). Then, an estimate of the region of attraction is given by $\Omega_r := \{x \in \mathbb{R}^n : V(x) \leq r\}$.

(ii) Assume now that there exists $i_0 \in \mathcal{N}$ such that (5) holds. Then, according to (i) there exists d_{i_0} such that (9) holds. Suppose now that for each $i \in \mathcal{N} \setminus \{i_0\}$, (6) holds. Then, it follows from (11) and (11) that for each $i \in \mathcal{N} \setminus \{i_0\}$, there exists $\hat{y} > 0$ such that $g_i(\hat{y}) = 0$. In particular, we can obtain that $\hat{y} = d_i = -\frac{\beta_i}{2\alpha_i} > 0$. With this selection of $d_i > 0$ we obtain from (8): $\dot{V}(x) \leq -\frac{1}{2} q_{i_0} |x_{i_0}|^2 \leq 0$. Define $E_c = \{x \in \mathcal{S} : \dot{V}(x) = 0\} = \{x \in \mathcal{S} : x_{i0} = 0\}$. Let $x(t)$ be a solution in E_c. Due to (6b) and the definition of E_c, the latter implies that

$$x_{i0}(t) \equiv 0 \Rightarrow \dot{x}_{i0}(t) \equiv 0 \Rightarrow$$
$$H_{i0}(x(t)) \equiv 0 \Rightarrow x(t) \equiv 0. \quad (12)$$

Thus, the only solution that lies in E_c is the zero solution and according to Barbashin-Krasovskii-LaSalle Theorem, [12] this implies asymptotic stability. (iii) (a) Assume first that (6a) holds and in addition that there exists $x \neq 0$ such that $H_{i0}(x) = 0$. Then, the third implication in (12) is not true and we we obtain $\dot{V}(x) \leq 0$ which establishes stability of (2). (b) Finally, suppose that for all $i \in \mathcal{N}$ condition (6a) holds. Then it follows from (11) and (11) that there exist $d_i > 0$ such that (10) holds with $q_i = 0$ for all $i \in \mathcal{N}$ which implies stability of (2). ■

Proposition (3.1) provides a condition to characterize the stability properties of an interconnected system of the form (2). More specifically, (5) implies the existence of two solutions to equation $g_i(y) = 0$ in (11). Those two solutions give a certain margin for the selection of the constants d_i in the weighted composite Lyapunov function $V = \sum d_i V_i$ to guarantee exponential stability of (2). In particular we obtain from (11) that for any $d_i \in (y_{i1}, y_{i2}) := \left(\frac{-\beta_i - \sqrt{\beta_i^2 - 2\alpha_i\gamma_i}}{2\alpha_i}, \frac{-\beta_i + \sqrt{\beta_i^2 - 2\alpha_i\gamma_i}}{2\alpha_i}\right)$ the system is exponentially stable. For values closer to y_{i2}, we decrease the rate of convergence, but we may increase the estimate of the region of attraction. Finally, note that exponential stability is guaranteed if (5) holds for all $i \in \mathcal{N}$. This condition is relaxed in property (ii) of Proposition 3.1 where only one system is required to satisfy (5). In this case, attractivity to zero is guaranteed by the Barbashin-Krasovskii-LaSalle Theorem only if we impose the additional condition (6b). Note that Proposition 3.1 (ii) requires at least one system to satisfy (6a) and (6b) in a neighborhood of zero to establish asymptotic stability. Finally, if condition (6b) does not hold, namely, if there exists $x \neq 0$ for which $H_{i0}(x) = 0$ it is not possible to apply the same arguments as in (12) and thus only stability of the interconnected system can be established.

Remark 3.1: Consider the interconnection of two identical systems $\dot{x}_1 = f(x_1) + H_1(x_2)$, $\dot{x}_2 = f(x_2) + H_2(x_1)$and suppose that Assumptions A1 and A2 hold globally with the same Lyapunov function. Then, according to (3.1), stability of the interconnected system follows if $\det(W) > 0$, or equivalently if $\xi_{12}\xi_{21} < \frac{a_{13}^2}{a_{14}^2}$. Notice now that according to Proposition 3.1(i), the system is exponentially stable if both $\xi_{12} < \frac{a_{13}}{a_{14}}$ and $\xi_{21} < \frac{a_{13}}{a_{14}}$. While condition (5) is more conservative than the traditional M-matrix test, it provides a systematic and distributed approach to characterize stability and calculate the weights d_i in the composite Lyapunov function.

Example 3.1: Consider the interconnection of systems

$$\dot{x}_i = f_i(x_i) + H_i(x_{i-1}, x_{i+1}), \;\; i \in \mathcal{N}$$

with $x_0, x_{N+1} \equiv 0$. Assume now that Assumptions A1 and A2 hold globally. Then, according to Proposition 3.1, the system is exponentially stable if, for $i = 1$, $a_{13}^2 > a_{14}^2 \xi_{21}^2$, $i = N$, $a_{N3}^2 > a_{N4}^2 \xi_{N-1,N}^2$, and for $i = 2, \ldots, N-1$, $a_{i3}^2 > 2a_{i4}^2(\xi_{i-1,i}^2 + \xi_{i+1,i}^2)$. In addition if (6) holds, the system is asymptotically stable. A similar system was also considered in [20] to study l_2 string stability and in [3] to study the robustness of spatially invariant large-scale systems. String stability and its properties are beyond the scope of this paper.

Part (i) of Proposition (3.1) can be extended to non-autonomous interconnected systems. Consider the following non-autonomous system: $\dot{x}_i = f_i(t, x_i) + H_i(t, x)$ with f_i, H_i locally Lipschitz in x_i and $f_i(t, 0) = 0$ and $H_i(t, 0) = 0$. Then we have the following result whose proof is omitted due to space constraints:

Proposition 3.2: Assume that each decoupled system $\dot{x}_i = f_i(t, x_i)$ is exponentially stable and Assumptions A1, A2 hold uniformly in t. Further assume that for all $i \in \mathcal{N}$, condition (5) holds. Then, the interconnected system is exponentially stable. If in addition there exists $i \in \mathcal{N}$ such that (6a) holds, then the system is stable.

Finally, we extend the previous results to the case of discrete-time system

$$x_i(k+1) = f_i(x_i(k)) + H_i(x(k)),\ i \in \mathcal{N} \tag{13}$$

where $k \in \mathbb{Z}_{\geq 0}$, $x_i(k) \in \mathcal{D}_i \subset \mathbb{R}^{n_i}$, $0 \in \mathcal{D}_i$, $f_i : \mathcal{D}_i \to \mathbb{R}^{n_i}$ are locally Lipschitz with Lipschitz constant $L_i > 0$ and $f_i(0) = 0$, $H_i(0) = 0$.For each decoupled subsystem $x_i(k+1) = f_i(x_i(k))$ we assume that

Assumption A3: There exist functions $V_i : \mathcal{D}_i \to \mathbb{R}$ such that

$$a_{i1}|x_i|^2 \leq V_i(x_i) \leq a_{i2}|x_i|^2 \tag{14a}$$

$$\Delta_{f_i} V_i(x_i) = V_i(f_i(x_i)) - V_i(x_i) \leq -a_{i3}|x_i|^2 \tag{14b}$$

$$|V_i(x) - V_i(y)| \leq a_{i4}|x - y|(|x| + |y|)\ \forall x, y \in \mathcal{D}_i \tag{14c}$$

Assumption A3 provides exponential stability for discrete-time systems, [12, Exercise 4.68]. Note that (14c) is a Lipschitz property on the Lyapunov function. The following result extends Proposition 3.1 to the discrete-time case under the additional assumption

Assumption A4: The neighborhood Δ_i, $i \in \mathcal{N}$, is symmetric. Namely, it holds that $j \in \Theta_i \iff j \in \Xi_i$.

Assumption A4 implies that when a system j drives system i, then also i drives system j.

Proposition 3.3: Consider system (13) and assume that for each $i \in \mathcal{N}$ there exist V_i satisfying (14). Also, assume that A3 and A4 hold. Then,

(i) the system is exponentially stable if for each $i \in \mathcal{N}$

$$Q_i := a_{i3} - a_{i4} c(\Xi_i) \sum_{j=1}^{N} \xi_{ji}^2 > 0 \tag{15a}$$

$$Q_i^2 > 4c(\Xi_i) a_{i4}^2 L_i^2 \sum_{j=1}^{N} \xi_{ji}^2. \tag{15b}$$

(ii) If there exists $i_0 \in \mathcal{N}$ such that (15) holds and

$$Q_i^2 = 4c(\Xi_i) a_{i4}^2 L_i^2 \sum_{j=1}^{N} \xi_{ji}^2\ \forall i \in \mathcal{N} \setminus \{i_0\} \tag{16a}$$

$$H_{i0}(x) \neq 0,\ x \neq 0 \tag{16b}$$

then the system is asymptotically stable.

(iii) The system is stable if one the following holds:
   (a) (16a) holds and $H_{i0}(x) = 0$ for $x \neq 0$; or
   (b) condition (16a) holds for all $i \in \mathcal{N}$.

Proof: Due to space constraints the proof is omitted.

For the discrete time case, to guarantee the exponential stability of the interconnected system we require that $a_{i3} - a_{i4} N \sum_{j=1}^{N} \xi_{ji}^2 > 0$ and $(a_{i3} - a_{i4} c(\Xi_i) \sum_{j=1}^{N} \xi_{ji}^2)^2 > 4c(\Xi_i)(a_{i4} L_i)^2 \sum_{j=1}^{N} \xi_{ji}^2$ which imply that we require small interconnection terms and high degree of stability of each decoupled system $x_i(k+1) = f_i(x_i(k))$. Proposition 3.3 can also be partially extended to the non-autonomous case.

IV. Application to Event Triggered Control

The sufficient condition (5) in Proposition 3.1 is fulfilled when the degree of stability of each individual system is greater than the strength of the interconnection. Thus, by using local feedback laws it may be possible to fulfill this condition. For instance, consider that each decoupled system is of the form $\dot{x}_i = f_i(x_i, u_i)$ where $u_i = h_i(x_i)$ is a locally Lipschitz feedback law that exponentially stabilizes the closed-loop system $\dot{x}_i = f_i(x_i, h_i(x_i))$, namely, Assumption A1 holds. Then, if the feedback law can enhance the stability of the system, namely, if $a_{i3} > 0$ in (3b) can be selected arbitrarily large in such a way that condition (i) or (ii) of Proposition (3.1) hold, then we can establish asymptotic stability of the interconnected system. Note however that this is not always true, see [19].

In this section, we exploit the stability margins q_i given by the solution of (9) in Proposition 3.1 for the event-based stabilization of the coupled system

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_i(t), u_i(t)) + H_i(x(t)) \\ u_i(t) &= h_i(x(t_k^i)),\ t \in [t_k^i, t_{k+1}^i) \end{aligned} \tag{17}$$

where t_k^i, $k \in \mathbb{Z}_{\geq 0}$ is the time the controller i is recomputed and updated.

Assumption A5: There exist locally Lipschitz feedback laws $u_i = h_i(x_i)$, $h_i : \mathcal{D}_i \to \mathbb{R}^m$, $h_i(0) = 0$, $i \in \mathcal{N}$, constants $a_{i1}, a_{i2}, a_{i3}, a_{i4} > 0$ and C^1 functions $V_i : \mathcal{D}_i \to \mathbb{R}_{\geq 0}$ such that $a_{i1}|x_i|^2 \leq V_i(x_i) \leq a_{i2}|x_i|^2$, $x_i \in \mathcal{D}_i$, $\nabla V_i(x_i) f_i(x_i, h_i(x_i)) \leq -a_{i3}|x_i|^2$, $x_i \in \mathcal{D}_i$ and $|\nabla V_i(x_i)| \leq a_{i4}|x_i|$, $x_i \in \mathcal{D}_i$.

In particular, we have

Proposition 4.1: Consider the interconnection of systems (17) under the Assumptions A2 and A5 and in addition let property (5) hold. Then, the triggering mechanism

$$L_i d_i a_{i4} |e_i| \leq \beta_i q_i |x_i| + \epsilon_i \tag{18}$$

with L_i, d_i, q_i, $\epsilon_i > 0$ and $\beta_i \in (0, 1)$ guarantees practical stability of the sampled system (17), where $e_i := x_i - \hat{x}_i$, $\hat{x}_i := x_i(t_k^i)$.

Proof: [Outline] Consider again the Lyapunov function $V = \sum_{i=1}^{N} d_i V_i$ for some positive constants d_i, $i \in \mathcal{N}$. Then, it follows by taking into account Assumption A1, A2, by adding and subtracting terms and by exploiting the Lipschitz properties of the system that $\dot{V}(x) =\leq \sum_{i=1}^{N} \frac{1}{2}\left(d_i^2 (c(\Theta_i) a_{i4}^2) - 2a_{i3} d_i + \sum_{i=1}^{N} \xi_{ji}^2\right)|x_i|^2 + L_i d_i a_{i4} |x_i||e_i|$ where $L_i > 0$ is the Lipschitz constant of f, h and $e_i = x_i - \hat{x}_i$. Since condition (5) holds for each $i \in \mathcal{N}$, it follows as in (11) that there exist $d_i > 0$ such that $1/2c(\Theta_i) a_{i4}^2 d_i^2 - a_{i3} d_i + 1/2 \sum_{j=1}^{N} \xi_{ji}^2 = -q_i < 0,\ q_i > 0$

and therefore, $\dot{V}(x) \leq \sum_{i=1}^{N} -q_i|x_i|^2 + L_i d_i a_{i4}|x_i||e_i|$. Finally, from the triggering condition (18) we obtain $\dot{V}(x) \leq \sum_{i=1}^{N} -q_i(1-\beta_i)|x_i|^2 + \epsilon_i|x_i| \leq \sum_{i=1}^{N} -q_i(1-\beta_i)|x_i|^2 + \frac{1}{2\delta_i}|x_i|^2 + \frac{\delta_i \epsilon_i^2}{2}$ where we have applied the inequality $xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon y^2}{2}$, $\varepsilon > 0$. Then for $\delta_i = \frac{2}{q_i(1-\beta_i)}$ it follows that $\dot{V}(x) \leq -\kappa_1 V(x) + \eta_1$ with $\kappa_1 = \frac{\min_{i\in\mathcal{N}}\{q_i(1-\beta_i)/2\}}{\max_{i\in\mathcal{N}}\{d_i a_{i2}\}}$ and $\eta_1 = \sum_{i=1}^{N} \frac{2}{q_i(1-\beta_i)}\epsilon_i^2$ which implies ultimate boundedness for sufficiently small $\epsilon_i > 0$. ■

Note that the triggering condition (18) requires first to solve $g_i(y) < 0$ in (11) for each subsystem $i \in \mathcal{N}$ and select d_i to obtain the constants $q_i > 0$. In particular, by appropriately selecting d_i in the interval (y_{i1}, y_{i2}), where y_{i1}, y_{i2} are the solutions of $g_i(y) = 0$ in (11), as well as the constant $\beta_i \in (0,1)$ we can regulate the rate of convergence of the system, the number of controller updates as well as the region of attraction $\mathcal{R}$. A lower-bound on the inter-event period can be obtained by exploiting the inequality $\frac{d}{dt}|e_i(t)| \leq |\dot{e}_i(t)| \leq 2L_i|x_i(t_k)| + L_i|e_i(t)| + h$ where $h = c(\Theta_i)\max_{j\in\Xi_i}\{\xi_{ij}|x_{j0}|\}$ and L_i the Lipschitz constant on the compact set $\mathcal{V} := \{x : V(x) \leq V(x_0)\}$, where $V(x) = \sum_{i=1}^{N} d_i V_i(x_i)$ and $x_0 = (x_{10}^T, \ldots, x_{N0}^T)^T$. Due to space constraints the details are omitted.

Remark 4.1: State-dependent mechanisms as in (18) have been widely used in the relative literature, see for instance [21], [23], [27] and references therein. While we do not require global input-to-state stability assumptions as in [7] and [27], we only obtain semi-global results. Several extensions can also be obtained by using time-dependent triggering mechanisms as in [10] and [23]. Note that for $\epsilon_i = 0$, $i \in \mathcal{N}$ in (18) we can establish exponential stability instead of practical. However the lower bound on the inter-event period may decrease as the state approaches the equilibrium as is the case in [27].

In (17) the state of each subsystem is transmitted continuously through the interconnection term (consider for instance collaborative manipulation or physically interconnected systems, [4], [10]). Motivated by networked environments where communication may take place over digital networks (see for instance [26]), we consider the interconnected system

$$\dot{x}_i(t) = f_i(x_i(t)) + H_i(x(t_k)) \tag{19}$$

where $x(t_k) = (x_1(t_k^1)^T, \ldots, x_{i-1}(t_k^{i-1})^T, x_{i+1}(t_k^{i+1})^T, \ldots, x_N(t_k^N)^T)^T$. In this case each subsystem $i \in \mathcal{N}$ sends a sampled version of its state to its neighbors, namely, $j \in \Xi_i$. We derive next a suitable mechanism to determine the broadcasting times and preserve the stability of the interconnected system. Due to space constraints the proof is omitted.

Proposition 4.2: Consider the system (19) and suppose that Assumptions A1 and A2 hold. In addition assume that condition (5) holds. Then, if each system $i \in \mathcal{N}$ broadcasts its state according to the rule

$$\delta_i \sum_{j=1}^{N} \xi_{ji}^2 |e_i|^2 \leq \beta_i Q_i |x_i|^2 \tag{20}$$

for some $\beta_i \in (0,1)$ with $Q_i := \frac{a_{i3}^2}{c(\Theta_i)a_{i4}^2}(1-\frac{1}{\delta_i}) - \sum_{j=1}^{N}\xi_{ji}^2$ and $\delta_i > \frac{a_{i3}^2}{a_{i3}^2 - c(\Theta_i)a_{i4}^2\sum_{i=1}^{N}\xi_{ji}^2}$, then the system (19) is exponentially stable.

Note that similar arguments as in Remark 4.1 hold for the broadcasting periods. While the bound on the period might decrease as the state approaches the equilibrium, Fig. 2 in Example 5.2 shows that this bound is conservative for certain systems, see also [27].

Two more cases can be considered. First, both the mechanisms presented before can be combined for the case of $\dot{x}_i(t) = f_i(x_i(t), u_i(t)) + H_i(x(b_k))$, $u_i(t) = h_i(x_i(t_k^i))$, $t \in [t_k^i, t_{k+1}^i)$, where t_k represent the controller update times and b_k represent the broadcasting times similar to [27].

Finally, another interesting case arises if we consider systems of the form $\dot{x}_i = \bar{f}_i(x_i, \mathbf{x}_j, u_i)$ with $\mathbf{x}_j = \{x_j : j \in \Xi_i\}$ and with the input $u_i(t) = h(x_i, \mathbf{x}_j)$ which decomposes the system to the form $\dot{x}_i = f_i(x_i, 0, h_i(x_i)) + H_i(x_i, \mathbf{x}_j)$ in such a way that Assumptions A5 and A2 hold. Then, by combining the approaches before, the following condition allows both controller updates and transmissions to occur at the same time: $\delta_i\left(L_i^2 + \sum_{j=1}^{N}\xi_{ji}^2\right)|e_i|^2 \leq \beta_i Q_i|x_i|^2 + \epsilon_i$, $\beta_i \in (0,1)$, $\epsilon_i > 0$ with $Q_i = \frac{a_{i3}^2}{(c(\Theta_i)+1)a_{i4}^2}(1-\frac{1}{\delta_i}) - \sum_{j=1}^{N}\xi_{ji}^2$ for sufficiently large $\delta_i > 0$. Such a case may occur in affine in the control systems under matching conditions. Finally, it is possible to extend the results of this section by considering data dropouts and delays using similar arguments as in [27] and by assuming globally bounded dynamics for the systems and interconnection terms.

V. SIMULATIONS

Example 5.1: Consider the system $\dot{x}_i = -x_i\sin^2(x_i^2) + u_i\cos(x_i^2) + H_i(x), i = 1,2,3,4$ where $H_1(x) = 0.4x_2$, $H_2(x) = 3x_1x_3^2 + 0.8x_3x_4$, $H_3(x) = 0.2x_2^2 + 0.4x_4\sin(x_4)$, and $H_4(x) = 3x_2^2x_3^3$. The feedback law $u_i = -x_i\cos(x_i^2)$ exponentially stabilizes each decoupled system with Lyapunov function $V_i = \frac{1}{2}x_i^2$, and $a_{i3} = a_{i4} = 1$, $i = 1,2,3,4$. Then, for $|x_1| \leq 1.4$, $|x_2| \leq 1$, $|x_3| \leq 0.5$, $|x_4| \leq 1.2$ it follows that A2 holds with $\xi_{12} = 0.4$, $\xi_{21} = 0.9$, $\xi_{23} = 0.6$, $\xi_{24} = 0.4$, $\xi_{32} = 0.2$, $\xi_{34} = 0.5$, $\xi_{42} = 0.3$, $\xi_{43} = 0.3$ and (5) holds for each $i = 1,2,3,4$. Then, we can determine from (9) values for d_i to establish exponential stability. By using (11) we find that $d_1 \in (0.56, 1.43)$, $d_2 \in (0.21, 0.45)$, $d_3 \in (0.34, 0.66)$, $d_4 \in (0.29, 0.71)$ and let with $\beta_i = 0.8$, $i = 1,2,3,4$ for the triggering mechanism (18) and $\epsilon_i = 10^{-4}$. Also, let $x_{10} = 1.4$, $x_{20} = 1$, $x_{30} = -0.5$, and $x_{40} = -1.2$ be the initial conditions of the systems. Then, we can obtain on the compact sets $\{x \in \mathbb{R}, V_i(x) \leq V_i(x_{i0})\}$ that $L_1 = 4$, $L_2 = 1.14$, $L_3 = 1$, and $L_4 = 2.72$. Different values of d_i were selected as in Table 1. The simulation results and the number of updates for are depicted in Fig. 1 and Table 1 respectively. For smaller values of d_i the number of updates decreases but the target set becomes larger as shown in Fig. 1.

Example 5.2: Consider the systems $\dot{x}_i = -2x_i + y_i^2$, $\dot{y}_i = -x_iy_i - 3y_i + \sum_{j=1}^{3}\xi_{ij}y_j$ where $i, j = 1,2,3$ and $\xi_{ii} = 0$. For each system we consider the Lyapunov function $V_i = \frac{1}{2}(x_i^2 + y_i^2)$. It can be seen that each decoupled system is

TABLE 1

Case	1	2	3	4	5
Values of d_i					
d_1	1	0.78	1.21	1.42	0.57
d_2	0.33	0.27	0.39	0.44	0.22
d_3	0.5	0.42	0.57	0.65	0.35
d_4	0.5	0.39	0.60	0.7	0.3
No of Transmissions in 20s					
x_1	414	414	644	3557	2157
x_2	137	142	198	390	269
x_3	138	148	199	566	323
x_4	264	265	403	1365	616

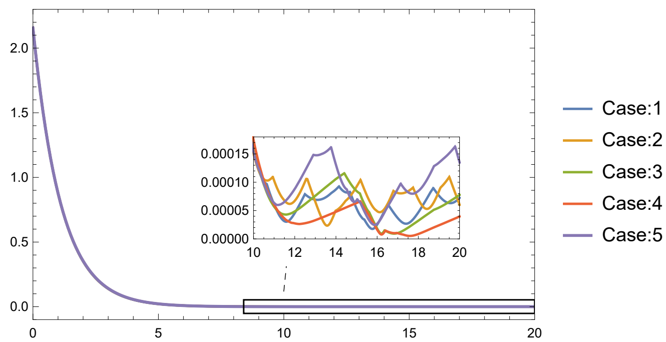

Fig. 1. Norm of states of systems of Example 5.1.

exponentially stable with $a_{i3} = 2$ and $a_{i4} = 1$. Also, let $\xi_{21} = 0.8$, $\xi_{31} = 1$, $\xi_{12} = 0.6$, $\xi_{32} = 1.2$, $\xi_{13} = 0.9$ and $\xi_{23} = 0.8$. Then, it follows that (5) is satisfied with $c(\Theta_i) = 2$ and thus the system is exponentially stable. Next, by considering (20) for $\delta_1 = 10$, $\delta_2 = 20$, $\delta_3 = 5$, we obtain, $Q_1 = 0.16$, $Q_2 = 0.1$, and $Q_3 = 0.15$. Also, let $\beta_1 = 0.7$, $\beta_2 = 0.8$, $\beta_3 = 0.6$, The simulations results are shown in Fig. 2. Note that in Fig. 2 (left) the broadcasting periods are lower bounded.

VI. Conclusion

In this paper we presented stability tests for certain classes of interconnected systems including autonomous, time-varying, continuous and discrete time nonlinear systems coupled with linear or nonlinear interconnection terms. These conditions were then exploited for the decentralized event-based control of interconnected systems with state dependent mechanisms. Future work will address the stabilization of interconnected systems by exploiting the sufficient conditions presented in this paper and extend the event-triggered control approaches of [22] and [23] to discrete time interconnected systems.

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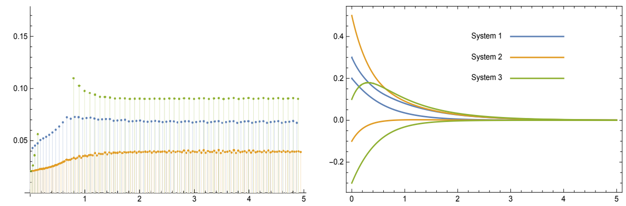

Fig. 2. Broadcasting periods and state of the systems in Example 5.2.

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