Area Defense and Surveillance on Rectangular Regions Using Control Barrier Functions

Luis Guerrero-Bonilla[1], Magnus Egerstedt[2] and Dimos V. Dimarogonas[3]

***Abstract*— A formulation of the area defense and surveillance problem for one intruder and one defense and surveillance robot and its corresponding solution using control barrier functions is presented. The defense robot must follow the intruder as it moves through a rectangular region in the plane, ensuring that the position of the intruder is also within a rectangular region attached to the surveillance robot. The proposed reactive and closed-form control laws depend on the positions of the robots, their maximum speeds, and the size of the rectangular regions. We show the application and effectiveness of our results in experiments with real robots.**

I. INTRODUCTION

The work [1] introduced the target guarding problem, consisting of an *evader* or *intruder* which tries to reach a target location, and a *pursuer* or *defender* trying to intercept the evader before it reaches the target. In this paper, we study a version of the target guarding problem where a defense and surveillance robot, acting as the pursuer or defender, must protect a rectangular target region from undetected intrusions. The defense robot is equipped with a rectangular region around it which acts as a shield. Its goal is to move ensuring that the intruder remains within the shield region whenever it is on the target region.

The area surveillance problem can be considered in the general category of pursuit-evasion problems [2]. Approaches to solve these problems include computing the reachable sets of pursuers and evaders through the Hamilton-Jacobi equation [3]–[7], as well as the use of Voronoi-like partitions of the environment [8]–[10]. For area search and patrolling, continuous space [11] and graph-based [12] schemes have been studied, and techniques to calculate the boundaries of the area where the intruder might be located based on its velocity have been developed [13]. Closely related to area surveillance, algorithms for area coverage have been studied in the literature [14]–[16].

In our previous work [17], we study a solution to the perimeter surveillance problem for one intruder and multiple surveillance robots using control barrier functions. Our contribution in this paper is a formulation and solution to the area defense and surveillance problem for one intruder and one surveillance robot using set-invariance methods based on control barrier functions. The intruder is allowed to move throughout the plane with a continuously differentiable position and bounded speed. The surveillance robot is allowed to move on the plane with a bound on its speed that is less or equal than that of the intruder, and has an associated rectangular region attached to it. The goal of the surveillance robot is to ensure that the position of the intruder is always within its associated rectangular region whenever the intruder is inside a predefined and fixed rectangular target region on the plane. Compared to the reachability, planning, and coverage strategies in the literature, our solution consists of a reactive closed-form control law calculated from the position of the intruder and the surveillance robot, the speed bounds, and the geometric characteristics of the rectangular regions. Similar to recent approaches in coverage control [15], [16], the solution makes use of Zeroing Control Barrier Functions [18], [19].

The paper is structured as follows. Section II describes the problem formulation and states the control objective to be satisfied by our proposed control laws. Section III presents sufficient conditions to solve the control objective, and Section IV describes the proposed control laws. Our theoretical results are implemented in The Robotarium [20] in Section V.

II. PROBLEM FORMULATION

Let $\boldsymbol{x}_P \in \mathbb{R}^2$ denote the position of point P on the plane, and let $\boldsymbol{x}_A(t)$ denote the position of the intruder robot A. It is assumed that $\boldsymbol{x}_A(t)$ is continuously differentiable and can be measured, and that the velocity $\dot{\boldsymbol{x}}_A(t)$ is bounded by $v_A > 0$ such that $\|\dot{\boldsymbol{x}}_A(t)\| \leq v_A$, but is otherwise unknown.

Let $\boldsymbol{x}_D(t)$ be the position of the defense robot D, with dynamics given by

$$\dot{\boldsymbol{x}}_D(t) = \boldsymbol{u}(t) \tag{1}$$

The velocity of robot D is assumed to be bounded such that $\|\dot{\boldsymbol{x}}_D(t)\| \leq v_D$ with $0 < v_D \leq v_A$.

Let the *target region* $\mathcal{T} \subset \mathbb{R}^2$ be a rectangular region with vertices at $\boldsymbol{x}_{Vi}$ for $i \in \{1, 2, 3, 4\}$ numbered in counter-clockwise direction. To avoid using modulo arithmetic and for simplicity of the notation, let $\boldsymbol{x}_{V5} = \boldsymbol{x}_{V1}$. Each of the rectangle's sides is a boundary of $\mathcal{T}$. The vector going from one vertex to the other of the ith boundary is given by

$$\boldsymbol{b}_i = \boldsymbol{x}_{Vi+1} - \boldsymbol{x}_{Vi} \tag{2}$$

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TABLE I: Function $T_{A,i}\left(\boldsymbol{x}_A(t),\boldsymbol{x}_{Vi}\right)$

$$T_{A,i}\left(\boldsymbol{x}_A(t),\boldsymbol{x}_{Vi}\right)= \tag{3}$$
$$\begin{cases} \frac{1}{v_A}\left(\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i-L_i\right)^2+\epsilon^2}-\epsilon\right) & \text{for} \quad \left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i\geq L_i \\ \frac{1}{v_A}\left(\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\epsilon^2}-\epsilon\right) & \text{for} \quad 0<\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i<L_i \\ \frac{1}{v_A}\left(\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i\right)^2+\epsilon^2}-\epsilon\right) & \text{for} \quad \left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i\leq 0 \end{cases}$$

for $i\in\{1,2,3,4\}$ and $\boldsymbol{x}_{V5}=\boldsymbol{x}_{V1}$. The length L_i of the ith side of the rectangle is given by $L_i=\|\boldsymbol{b}_i\|$, and the corresponding unit vector is given by $\hat{\boldsymbol{b}}_i=\frac{\boldsymbol{b}_i}{\|\boldsymbol{b}_i\|}$. The vector perpendicular to $\hat{\boldsymbol{b}}_i$ and directed towards the inside of $\mathcal{T}$, denoted by $\hat{\boldsymbol{b}}_{\perp i}$, is given by $\hat{\boldsymbol{b}}_{\perp i}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\hat{\boldsymbol{b}}_i$. Since $\mathcal{T}$ is a rectangle, opposite sides are parallel and of the same length, and adjacent sides are perpendicular. Given these geometric constraints, we have $\hat{\boldsymbol{b}}_1=-\hat{\boldsymbol{b}}_3$, $L_1=\|\boldsymbol{b}_1\|=\|\boldsymbol{b}_3\|=L_3$, $\hat{\boldsymbol{b}}_2=-\hat{\boldsymbol{b}}_4$, $L_2=\|\boldsymbol{b}_2\|=\|\boldsymbol{b}_4\|=L_4$, $\hat{\boldsymbol{b}}_{\perp 1}=\hat{\boldsymbol{b}}_2$, $\hat{\boldsymbol{b}}_{\perp 2}=\hat{\boldsymbol{b}}_3$, $\hat{\boldsymbol{b}}_{\perp 3}=\hat{\boldsymbol{b}}_4$, and $\hat{\boldsymbol{b}}_{\perp 4}=\hat{\boldsymbol{b}}_1$. We use the notation $\hat{\boldsymbol{b}}_i=-\hat{\boldsymbol{b}}_{i+2}$ and $\hat{\boldsymbol{b}}_{\perp i}=\hat{\boldsymbol{b}}_{i+1}$ for $i\in\{1,2,3,4\}$, with $\hat{\boldsymbol{b}}_5=\hat{\boldsymbol{b}}_1$ and $\hat{\boldsymbol{b}}_6=\hat{\boldsymbol{b}}_2$.

The ith half-plane corresponding to the ith boundary of $\mathcal{T}$ can be expressed as $\mathcal{H}_{\mathcal{T},i}=\{\boldsymbol{x}_P:(\boldsymbol{x}_P-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}\geq 0\}$. Then, $\mathcal{T}$ can be expressed as $\mathcal{T}=\bigcap_{i=1}^4\mathcal{H}_{\mathcal{T},i}$, such that

$$\mathcal{T}=\{\boldsymbol{x}_P:(\boldsymbol{x}_P-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}\geq 0,\ \forall i\in\{1,2,3,4\}\}. \tag{4}$$

Let the *shield region* $\mathcal{S}\subset\mathbb{R}^2$, be a rectangular region surrounding robot D, with perpendicular constant distances between its boundaries and $\boldsymbol{x}_D(t)$ denoted by the positive constants s_i, so that the position of the closest point on the ith boundary of $\mathcal{S}$ to $\boldsymbol{x}_D(t)$ is $\boldsymbol{x}_D(t)-s_i\hat{\boldsymbol{b}}_{i+1}$. The shield region can be defined by the intersection of the half-planes $\mathcal{H}_{\mathcal{S},i}=\{\boldsymbol{x}_P:(\boldsymbol{x}_P-\boldsymbol{x}_D(t))\cdot\hat{\boldsymbol{b}}_{i+1}+s_i\geq 0\}$, so that $\mathcal{S}$ can be expressed as $\mathcal{S}=\bigcap_{i=1}^4\mathcal{H}_{\mathcal{S},i}$, which is equal to

$$\mathcal{S}=\{\boldsymbol{x}_P:(\boldsymbol{x}_P-\boldsymbol{x}_D(t))\cdot\hat{\boldsymbol{b}}_{i+1}+s_i\geq 0,\ \forall i\in\{1,2,3,4\}\}. \tag{5}$$

Figure 1 shows an image of the regions and the robot D with parameters s_1, s_2, s_3 and s_4. The region $\mathcal{S}$ moves on the plane together with $\boldsymbol{x}_D(t)$. The objective of robot D is to move so that, if $\boldsymbol{x}_A(t)$ is within the target region $\mathcal{T}$, then it is also within the shield region $\mathcal{S}$. We formalize this as the following control objective:

Control Objective 1. *Given the continuously differentiable position $\boldsymbol{x}_A(t)$ of robot A with velocity bounded by $\|\dot{\boldsymbol{x}}_A(t)\|\leq v_A$, $v_A>0$, the position $\boldsymbol{x}_D(t)$ of robot D with dynamics given by* (1) *and velocity bounded by $\|\dot{\boldsymbol{x}}_D(t)\|\leq v_D$, $0<v_D\leq v_A$, and a target region $\mathcal{T}$ on the plane according to* (4)*, determine a shield region $\mathcal{S}$ with parameters s_1, s_2, s_3 and s_4 as in* (5) *and design a control law $\boldsymbol{u}(t)=\left[u_1(t),u_2(t)\right]^\intercal$ bounded by $\|\boldsymbol{u}(t)\|\leq v_D$ for robot D such that $\boldsymbol{x}_A(t)\in\mathcal{S}$ whenever $\boldsymbol{x}_A(t)\in\mathcal{T}$, that is, $(\boldsymbol{x}_A(t)-\boldsymbol{x}_D(t))\cdot\hat{\boldsymbol{b}}_{i+1}+s_i\geq 0\ \forall i\in\{1,2,3,4\}$ whenever $(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}\geq 0\ \forall i\in\{1,2,3,4\}$, $\forall t\geq 0$.*

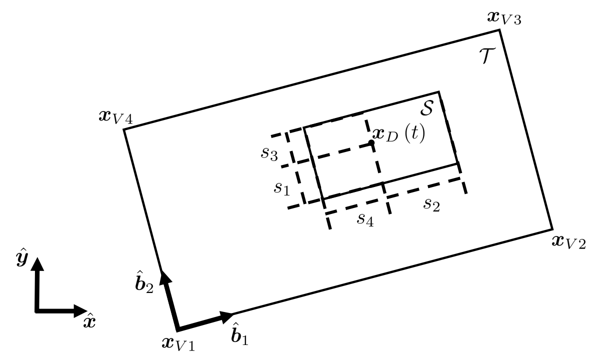

Fig. 1: The vertices $\boldsymbol{x}_{Vi}$ of the rectangular target region, and the distances s_i from $\boldsymbol{x}_D(t)$ to the edges of the shield region $\mathcal{S}$ are shown.

III. A SUFFICIENT CONDITION FOR RECTANGULAR AREA SURVEILLANCE

In this section, sufficient conditions to satisfy the Control Objective 1 are shown. The next Lemma will be used in later proofs.

Lemma 1. *If $\boldsymbol{x}_A(t)\in\mathcal{T}$, then $0\leq(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_i\leq L_i$ for all $i\in\{1,2,3,4\}$.*

Proof. If $\boldsymbol{x}_A(t)\in\mathcal{T}$ then, by (4), $(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}\geq 0$ for all $i\in\{1,2,3,4\}$ with $\hat{\boldsymbol{b}}_5=\hat{\boldsymbol{b}}_1$. Using (2), $\boldsymbol{x}_{Vi}=\boldsymbol{x}_{Vi+1}-\boldsymbol{b}_i=\boldsymbol{x}_{Vi+1}-L_i\hat{\boldsymbol{b}}_i$. Then, we have

$$\begin{aligned}0\leq(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}&=\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi+1}+L_i\hat{\boldsymbol{b}}_i\right)\cdot\hat{\boldsymbol{b}}_{i+1}\\&=(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi+1})\cdot\hat{\boldsymbol{b}}_{i+1}\end{aligned} \tag{6}$$

which is equivalent to $(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_i\geq 0$ for all $i\in\{1,2,3,4\}$ with $\boldsymbol{x}_{V5}=\boldsymbol{x}_{V1}$. Since, $0\leq(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_{i+1}$ implies $0\leq(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi+1})\cdot\hat{\boldsymbol{b}}_{i+2}$ with $\boldsymbol{x}_{V5}=\boldsymbol{x}_{V1}$, $\hat{\boldsymbol{b}}_5=\hat{\boldsymbol{b}}_1$, $\hat{\boldsymbol{b}}_6=\hat{\boldsymbol{b}}_2$ and $\hat{\boldsymbol{b}}_{i+2}=-\hat{\boldsymbol{b}}_i$, then

$$\begin{aligned}0\leq(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi+1})\cdot\hat{\boldsymbol{b}}_{i+2}&=\left(\boldsymbol{x}_A(t)-L_1\hat{\boldsymbol{b}}_i-\boldsymbol{x}_{Vi}\right)\cdot-\hat{\boldsymbol{b}}_i\\&=-(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi})\cdot\hat{\boldsymbol{b}}_i+L_i,\end{aligned} \tag{7}$$

which implies $(\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_i \leq L_i$ for all $i \in \{1,2,3,4\}$. Together, both results imply $0 \leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_i \leq L_i$ for all $i \in \{1,2,3,4\}$. $\square$

Let the continuously differentiable function $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi})$ be defined as in equation (3) in Table I. The function of $f(x) = \sqrt{x^2+\epsilon^2} - \epsilon \leq |x|$ is used as a continuously differentiable approximation to the absolute value function, so that the constant $\epsilon > 0$ must be selected as small as possible for a better approximation. Let $T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi})$ be defined as

$$T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) = \frac{(\boldsymbol{x}_D(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} - s_i}{v_D/\sqrt{2}}. \tag{8}$$

Theorem 1. *Assuming the conditions of the Control Objective 1 hold, if* $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0\ \forall i \in \{1,2,3,4\},\ \forall t \geq 0$, *then the Control Objective 1 is solved.*

Proof. Let $\boldsymbol{x}_A(t) \in \mathcal{T}$ so that, by (4), $(\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} \geq 0\ \forall i \in \{1,2,3,4\}$, and assume $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0\ \forall i \in \{1,2,3,4\}\ \forall t \geq 0$. By Lemma 1 and using (3), we have $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) = \frac{1}{v_A}\left(\sqrt{\left((\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1}\right)^2 + \epsilon^2} - \epsilon\right)$. Substituting together with (8) into $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0$, multiplying both sides by $v_D/\sqrt{2}$ and using $\frac{v_D}{v_A\sqrt{2}} < 1$ leads to

$$\begin{aligned}
0 &\leq \frac{v_D}{v_A\sqrt{2}}\left(\sqrt{\left((\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1}\right)^2 + \epsilon^2} - \epsilon\right) \\
&\qquad - \left((\boldsymbol{x}_D(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} - s_i\right) \\
&\leq \frac{v_D}{v_A\sqrt{2}}|(\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1}| \\
&\qquad - \left((\boldsymbol{x}_D(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} - s_i\right) \\
&\leq (\boldsymbol{x}_A(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} - (\boldsymbol{x}_D(t) - \boldsymbol{x}_{Vi}) \cdot \hat{\boldsymbol{b}}_{i+1} + s_i \\
&= (\boldsymbol{x}_A(t) - \boldsymbol{x}_D(t)) \cdot \hat{\boldsymbol{b}}_{i+1} + s_i
\end{aligned} \tag{9}$$

Therefore, $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0\ \forall i \in \{1,2,3,4\}\ \forall t \geq 0$ and $\boldsymbol{x}_A(t) \in \mathcal{T}$ imply $(\boldsymbol{x}_A(t) - \boldsymbol{x}_D) \cdot \hat{\boldsymbol{b}}_{i+1} + s_i \geq 0\ \forall i \in \{1,2,3,4\}$, which by (5) corresponds to $\boldsymbol{x}_A(t) \in \mathcal{S}$. Hence, if $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0\ \forall i \in \{1,2,3,4\}\ \forall t \geq 0$, $\boldsymbol{x}_A(t) \in \mathcal{S}$ whenever $\boldsymbol{x}_A(t) \in \mathcal{T}$. $\square$

IV. CONTROL LAWS FOR AREA SURVEILLANCE

We proceed to ensure the conditions of Theorem 1, i.e., $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \geq 0\ \forall i \in \{1,2,3,4\},\ \forall t \geq 0$. This is done by defining a set that contains the positions of robots A and D which satisfy the conditions, and ensuring its forward invariance. We use results from the literature on zeroing control barrier functions. A brief introduction is given next, but the reader is referred to [18]. Consider a system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \tag{10}$$

where $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^m$, with $\boldsymbol{f}$ and $\boldsymbol{g}$ locally Lipschitz continuous. For any initial condition $\boldsymbol{x}(0)$, there exists a maximum time interval $I(\boldsymbol{x}(0)) = [0, \tau_{max})$ such that $\boldsymbol{x}(t)$ is the unique solution to (10) on $I(\boldsymbol{x}(0))$. In the case when (10) is forward complete, $\tau_{max} = \infty$. A set $\mathcal{Z}$ is called forward invariant with respect to (10) if for every $\boldsymbol{x}(0) \in \mathcal{Z}$, $\boldsymbol{x}(t) \in \mathcal{Z}$ for all $t \in I(\boldsymbol{x}(0))$.

Let the set $\mathcal{C}$ be defined as

$$\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) \geq 0\} \tag{11}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. $\mathcal{C}$ is assumed to be non-empty and to have no isolated points.

Definition 1 (Definition 5, [18]). *Given a set* $\mathcal{C} \subset \mathbb{R}^n$ *as defined in* (11) *for a continuously differentiable function* h*, the function* h *is called a zeroing control barrier function* ($ZCBF$) *defined on a set* $\mathcal{D}$ *with* $\mathcal{C} \subseteq \mathcal{D} \subset \mathbb{R}^n$*, if there exists an extended class* $\mathcal{K}$ *function* α *such that*

$$\sup_{\boldsymbol{u} \in U}\left[L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u} + \alpha(h(\boldsymbol{x}))\right] \geq 0, \forall \boldsymbol{x} \in \mathcal{D}. \tag{12}$$

The Lie derivative notation is used, so that $\dot{h}(\boldsymbol{x}) = \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) + \frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} = L_{\boldsymbol{f}} h(\boldsymbol{x}) + L_{\boldsymbol{g}} h(\boldsymbol{x})\boldsymbol{u}$. Given a ZCBF h, define the set $K = \{\boldsymbol{u} \in \mathcal{U} : L_{\boldsymbol{f}} h(\boldsymbol{x}) + L_{\boldsymbol{g}} h(\boldsymbol{x})\boldsymbol{u} + \alpha(h(\boldsymbol{x})) \geq 0\}$ for each $\boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2 (Corollary 2, [18]). *Given a set* $\mathcal{C} \subset \mathbb{R}^n$ *as defined in* (11) *for a continuously differentiable function* h*, if* h *is a ZCBF on* $\mathcal{D}$*, then any Lipschitz continuous controller* $\boldsymbol{u} : \mathcal{D} \to \mathcal{U}$ *for the dynamics* (10) *such that* $\boldsymbol{u}(\boldsymbol{x}) \in K$ *will render the set* $\mathcal{C}$ *forward invariant.*

These results can be applied to time varying systems, as described in [19]. In the following, the time and position dependencies are implied. Let the state $\boldsymbol{x} \in \mathbb{R}^4$ be defined as $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_D(t) & \boldsymbol{x}_A(t)\end{bmatrix}^\mathsf{T}$, with dynamics

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\boldsymbol{0} \\ \dot{\boldsymbol{x}}_A(t)\end{bmatrix} + \begin{bmatrix}I_{2\times 2} \\ 0_{2\times 2}\end{bmatrix}\boldsymbol{u} \tag{13}$$

where $\boldsymbol{u} \in \mathcal{U} = \{\boldsymbol{u} : \|\boldsymbol{u}\| \leq v_D\}$. The constraints from Theorem 1 are used to define the ZCBF candidates $h_i(\boldsymbol{x})$ for $i \in \{1,2,3,4\}$ as follows

$$h_i(\boldsymbol{x}) = T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi}) - T_{D,i}(\boldsymbol{x}_D(t), \boldsymbol{x}_{Vi}) \tag{14}$$

Since by Theorem 1 satisfying $h_i(\boldsymbol{x}) \geq 0$ for all $i \in \{1,2,3,4\}$ solves the Control Objective 1, we define the set $\mathcal{C}$ as

$$\mathcal{C} = \{\boldsymbol{x} : \bigcap_{i=1}^{4} h_i(\boldsymbol{x}) \geq 0\} \tag{15}$$

The time derivative of $T_{A,i}(\boldsymbol{x}_A(t), \boldsymbol{x}_{Vi})$, given by $\dot{T}_{A,i} = \frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A} \cdot \dot{\boldsymbol{x}}_A$ with $\frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A}$ shown in equation (16) in Table II, can be bounded by

$$-1 \leq -\|\frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A}\|v_A \leq \dot{T}_{A,i} \leq \|\frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A}\|v_A \leq 1. \tag{17}$$

TABLE II: $\frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A}$

$$\frac{\partial T_{A,i}}{\partial \boldsymbol{x}_A} = \begin{cases} \frac{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)\hat{\boldsymbol{b}}_{i+1}+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i}-L_i\right)\hat{\boldsymbol{b}}_{i}}{v_A\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i}-L_i\right)^2+\epsilon^2}} & \text{for} \quad \left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i \geq L_i \\ \frac{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)\hat{\boldsymbol{b}}_{i+1}}{v_A\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\epsilon^2}} & \text{for} \quad 0 < \left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i < L_i \\ \frac{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)\hat{\boldsymbol{b}}_{i+1}+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i}\right)\hat{\boldsymbol{b}}_{i}}{v_A\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i}\right)^2+\epsilon^2}} & \text{for} \quad \left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_i \leq 0 \end{cases} \tag{16}$$

The time derivative of $T_{D,i}$ is given by

$$\dot{T}_{D,i} = \frac{1}{v_D/\sqrt{2}}\hat{\boldsymbol{b}}_{i+1}\cdot\boldsymbol{u} \tag{18}$$

Let us define the coordinate frame of the target region, with origin at $\boldsymbol{x}_{V1}$ and with orthogonal axes given by $\hat{\boldsymbol{b}}_1$ and $\hat{\boldsymbol{b}}_2$. The rotation matrix that transforms coordinates from the target region frame $\mathcal{T}$ to the world frame is given by

$$^{\mathcal{W}}R^{\mathcal{T}} = \begin{bmatrix}\hat{\boldsymbol{b}}_1 & \hat{\boldsymbol{b}}_2\end{bmatrix} \tag{19}$$

and its inverse is equal to its transpose, $^{\mathcal{T}}R^{\mathcal{W}} = \left(^{\mathcal{W}}R^{\mathcal{T}}\right)^\intercal$. A position vector $\boldsymbol{x}_P$ in the frame of the target region $\mathcal{T}$ is denoted by $\boldsymbol{x}_P^{\mathcal{T}}$. Note that in the $\mathcal{T}$-frame, $\hat{\boldsymbol{b}}_1^{\mathcal{T}} = {}^{\mathcal{T}}R^{\mathcal{W}}\hat{\boldsymbol{b}}_1 = \begin{bmatrix}1 & 0\end{bmatrix}^\intercal$, $\hat{\boldsymbol{b}}_2^{\mathcal{T}} = {}^{\mathcal{T}}R^{\mathcal{W}}\hat{\boldsymbol{b}}_2 = \begin{bmatrix}0 & 1\end{bmatrix}^\intercal$, and $\hat{\boldsymbol{u}}^{\mathcal{T}} = \begin{bmatrix}u_1^{\mathcal{T}} & u_2^{\mathcal{T}}\end{bmatrix}^\intercal = {}^{\mathcal{T}}R^{\mathcal{W}}\boldsymbol{u}$. The coordinate frames are shown in Figure 1. Since in cartesian coordinates for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we have $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}^\intercal\boldsymbol{b}$, (18) can be expressed in the $\mathcal{T}$-frame as

$$\begin{aligned}\dot{T}_{D,i} &= \frac{1}{v_D/\sqrt{2}}\,^{\mathcal{W}}R^{\mathcal{T}}\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\cdot\,^{\mathcal{W}}R^{\mathcal{T}}\boldsymbol{u}^{\mathcal{T}} \\ &= \frac{1}{v_D/\sqrt{2}}\left(\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\right)^\intercal\left(^{\mathcal{W}}R^{\mathcal{T}}\right)^\intercal\,^{\mathcal{W}}R^{\mathcal{T}}\boldsymbol{u}^{\mathcal{T}}\left(t\right) \\ &= \frac{1}{v_D/\sqrt{2}}\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\cdot\boldsymbol{u}^{\mathcal{T}}.\end{aligned} \tag{20}$$

Substituting (17) and (20), $\dot{h}_i\left(\boldsymbol{x}\right)$ can be bounded by

$$\dot{h}_i\left(\boldsymbol{x}\right) = \dot{T}_{A,i} - \dot{T}_{D,i} \geq -1 - \frac{1}{v_D/\sqrt{2}}\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\cdot\boldsymbol{u}^{\mathcal{T}}. \tag{21}$$

Given (21), it can be ensured that $\dot{h}_i\left(\boldsymbol{x}\right) + \alpha\left(h_i\left(\boldsymbol{x}\right)\right) \geq 0$, with $\alpha\left(h_i\left(\boldsymbol{x}\right)\right) = \gamma_i h_i\left(\boldsymbol{x}\right)$ where γ_i a strictly positive constant, if the term $\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\cdot\boldsymbol{u}^{\mathcal{T}}$ satisfies the inequality

$$\hat{\boldsymbol{b}}_{i+1}^{\mathcal{T}}\cdot\boldsymbol{u}^{\mathcal{T}} \leq -\frac{v_D}{\sqrt{2}}\left(1-\gamma_i h_i\left(\boldsymbol{x}\right)\right) \tag{22}$$

for each i. The inequalities (22) for $i \in \{1,2,3,4\}$ take the form $u_2^{\mathcal{T}} \leq -\frac{v_D}{\sqrt{2}}\left(1-\gamma_1 h_1\left(\boldsymbol{x}\right)\right)$, $u_1^{\mathcal{T}} \geq \frac{v_D}{\sqrt{2}}\left(1-\gamma_2 h_2\left(\boldsymbol{x}\right)\right)$, $u_2^{\mathcal{T}} \geq \frac{v_D}{\sqrt{2}}\left(1-\gamma_3 h_3\left(\boldsymbol{x}\right)\right)$ and $u_1^{\mathcal{T}} \leq -\frac{v_D}{\sqrt{2}}\left(1-\gamma_4 h_4\left(\boldsymbol{x}\right)\right)$, which can be rewritten as

$$\frac{v_D}{\sqrt{2}}\left(1-\gamma_2 h_2\left(\boldsymbol{x}\right)\right) \leq u_1^{\mathcal{T}} \leq -\frac{v_D}{\sqrt{2}}\left(1-\gamma_4 h_4\left(\boldsymbol{x}\right)\right) \tag{23}$$

$$\frac{v_D}{\sqrt{2}}\left(1-\gamma_3 h_3\left(\boldsymbol{x}\right)\right) \leq u_2^{\mathcal{T}} \leq -\frac{v_D}{\sqrt{2}}\left(1-\gamma_1 h_1\left(\boldsymbol{x}\right)\right) \tag{24}$$

The existence of a $u_1^{\mathcal{T}}$ and $u_2^{\mathcal{T}}$ that satisfy equations (23) and (24) is discussed next. First, conditions to ensure that $\min_{\boldsymbol{x}}\left(h_1\left(\boldsymbol{x}\right)+h_3\left(\boldsymbol{x}\right)\right)$ and $\min_{\boldsymbol{x}}\left(h_2\left(\boldsymbol{x}\right)+h_4\left(\boldsymbol{x}\right)\right)$ are strictly positive are provided. Then, these minimum values are used to calculate constants γ_i to ensure the feasibility of (23) and (24).

Lemma 2. *For $h_j\left(\boldsymbol{x}\right)$ and $h_{j+2}\left(\boldsymbol{x}\right)$ as defined in* (14) *for $j \in \{1,2\}$, the value of $\min_{\boldsymbol{x}}\left(h_j\left(\boldsymbol{x}\right)+h_{j+2}\left(\boldsymbol{x}\right)\right)$ is given by*

$$\min_{\boldsymbol{x}}\left(h_j\left(\boldsymbol{x}\right)+h_{j+2}\left(\boldsymbol{x}\right)\right) = 2\left(\frac{\sqrt{\left(\frac{L_{j+1}}{2}\right)^2+\epsilon^2}-\epsilon}{v_A} - \frac{\frac{L_{j+1}}{2}-\frac{s_j+s_{j+2}}{2}}{v_D/\sqrt{2}}\right), \tag{25}$$

and it is strictly positive if

$$s_j + s_{j+2} > L_{j+1} - \frac{\sqrt{2}v_D}{v_A}\left(\sqrt{\left(\frac{L_{j+1}}{2}\right)^2+\epsilon^2}-\epsilon\right). \tag{26}$$

Proof. Direct substitution of (8) shows that $T_{D,j} + T_{D,j+2} = \frac{\sqrt{2}}{v_D}\left(L_{j+1}-s_j-s_{j+2}\right)$, and since $T_{A,i} \geq \frac{1}{v_A}\left(\sqrt{\left(\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1}\right)^2+\epsilon^2}-\epsilon\right)$ following (3), the sum $h_j\left(\boldsymbol{x}\right)+h_{j+2}\left(\boldsymbol{x}\right) = T_{A,j}+T_{A,j+2}-T_{D,j}-T_{D,j+2}$ can be bounded as follows

$$\begin{aligned}h_j\left(\boldsymbol{x}\right)+h_{j+2}\left(\boldsymbol{x}\right) \geq &-\frac{\sqrt{2}}{v_D}\left(L_{j+1}-s_j-s_{j+2}\right)+ \\ &\frac{1}{v_A}\left(\sqrt{\left(\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vj}\right)\cdot\hat{\boldsymbol{b}}_{j+1}\right)^2+\epsilon^2}-\epsilon\right. \\ &\left.+\sqrt{\left(\left(\boldsymbol{x}_A\left(t\right)-\boldsymbol{x}_{Vj+2}\right)\cdot\hat{\boldsymbol{b}}_{j+3}\right)^2+\epsilon^2}-\epsilon\right)\end{aligned} \tag{27}$$

with $\hat{\boldsymbol{b}}_5 = \hat{\boldsymbol{b}}_1$. As the terms of the right-hand side of (27) are 2-norms of affine functions in $\boldsymbol{x}_A\left(t\right)$ and constants, it can be verified that the right-hand side is a convex function of $\boldsymbol{x}_A(t)$. Its gradient is given by

$$\frac{1}{v_A}\left(\frac{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vj}\right)\cdot\hat{\boldsymbol{b}}_{j+1}\right)\hat{\boldsymbol{b}}_{j+1}}{\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vj}\right)\cdot\hat{\boldsymbol{b}}_{j+1}\right)^2+\epsilon^2}}\right.$$
$$\left.+\frac{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vj+2}\right)\cdot\hat{\boldsymbol{b}}_{j+3}\right)\hat{\boldsymbol{b}}_{j+3}}{\sqrt{\left(\left(\boldsymbol{x}_A(t)-\boldsymbol{x}_{Vj+2}\right)\cdot\hat{\boldsymbol{b}}_{j+3}\right)^2+\epsilon^2}}\right), \tag{28}$$

which can be verified to be equal to 0 at $\boldsymbol{x}_A(t)=\boldsymbol{x}_A^*=\boldsymbol{x}_{Vj}+\lambda_j\hat{\boldsymbol{b}}_j+\frac{1}{2}L_{j+1}\hat{\boldsymbol{b}}_{j+1}$ for all $\lambda_j\in\mathbb{R}$. Hence, $h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x})$ has a global minimum value at $\boldsymbol{x}^*=\begin{bmatrix}0 & \boldsymbol{x}_A^*\end{bmatrix}$ of $\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))=h_j(\boldsymbol{x}^*)+h_{j+2}(\boldsymbol{x}^*)$, which upon evaluation leads to equation (25).

Substituting (26) in (25) leads to $\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))>\frac{2}{v_A}\left(\sqrt{\left(\frac{L_{j+1}}{2}\right)^2+\epsilon^2}-\epsilon\right)-\frac{\sqrt{2}}{v_D}L_{j+1}+\frac{\sqrt{2}}{v_D}\left(L_{j+1}-\frac{\sqrt{2}v_D}{v_A}\left(\sqrt{\left(\frac{L_{j+1}}{2}\right)^2+\epsilon^2}-\epsilon\right)\right)=0.$ □

The next result provides values for γ_i following the constraints of Lemma 2 that ensure the existence of a $\boldsymbol{u}^{\mathcal{T}}$, bounded by $\|\boldsymbol{u}^{\mathcal{T}}\|\leq v_D$, that satisfies (23) and (24).

Lemma 3. *If $h_i(\boldsymbol{x})\geq 0$, L_i and s_i satisfy the conditions of Lemma 2 for all $i\in\{1,2,3,4\}$, and $\gamma_j=\gamma_{j+2}$ are given by*

$$\gamma_j=\gamma_{j+2}\geq\frac{2}{\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))} \tag{29}$$

for $j\in\{1,2\}$, then the inequalities (23) and (24) can be satisfied by some control input $\boldsymbol{u}^{\mathcal{T}}$ bounded by $\|\boldsymbol{u}^{\mathcal{T}}\|\leq v_D$.

Proof. Since $h_j(\boldsymbol{x})\geq 0$, multiplying (29) by $-h_j(\boldsymbol{x})$ and adding 1 on both sides leads to $1-\gamma_j h_j(\boldsymbol{x})\leq 1-\frac{2h_j(\boldsymbol{x})}{\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}$. Repeating the process with $h_{j+2}(\boldsymbol{x})$ leads to $1-\gamma_{j+2}h_{j+2}(\boldsymbol{x})\leq 1-\frac{2h_{j+2}(\boldsymbol{x})}{\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}$. Adding these two equations leads to

$$2-\gamma_j h_j(\boldsymbol{x})-\gamma_{j+2}h_{j+2}(\boldsymbol{x})\leq 2\left(1-\frac{(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}{\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}\right) \tag{30}$$

Since $\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))\leq(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))$, then $2\left(1-\frac{(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}{\min_{\boldsymbol{x}}(h_j(\boldsymbol{x})+h_{j+2}(\boldsymbol{x}))}\right)\leq 0$, and therefore

$$2-\gamma_j h_j(\boldsymbol{x})-\gamma_{j+2}h_{j+2}(\boldsymbol{x})\leq 0 \tag{31}$$

Evaluating at $j=1$ and $j=2$, rearranging terms and multiplying both sides by $v_D/\sqrt{2}$, the following inequalities can be obtained

$$\frac{v_D}{\sqrt{2}}(1-\gamma_2 h_2(\boldsymbol{x}))\leq-\frac{v_D}{\sqrt{2}}(1-\gamma_4 h_4(\boldsymbol{x})) \tag{32}$$

$$\frac{v_D}{\sqrt{2}}(1-\gamma_3 h_3(\boldsymbol{x}))\leq-\frac{v_D}{\sqrt{2}}(1-\gamma_1 h_1(\boldsymbol{x})) \tag{33}$$

Then, constants $\gamma_j=\gamma_{j+2}$ satisfying (29) ensure the feasibility of (32) and (33), which in turn ensure the existence of some values $\boldsymbol{u}_1^{\mathcal{T}}$ and $\boldsymbol{u}_2^{\mathcal{T}}$ satisfying (23) and (24).

To complete the proof, it is shown next that there exists such a $\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x})$ that, in addition, satisfies $\|\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x})\|\leq v_D$. Let us denote the leftmost and rightmost terms of (23) and (24) with the functions $m_1(\boldsymbol{x})$, $M_1(\boldsymbol{x})$, $m_2(\boldsymbol{x})$ and $M_2(\boldsymbol{x})$ respectively. For $h_i(\boldsymbol{x})\geq 0$, these can be bounded as follows:

$$m_1(\boldsymbol{x})=\frac{v_D}{\sqrt{2}}(1-\gamma_2 h_2(\boldsymbol{x}))\leq\frac{v_D}{\sqrt{2}}, \tag{34}$$

$$M_1(\boldsymbol{x})=-\frac{v_D}{\sqrt{2}}(1-\gamma_4 h_4(\boldsymbol{x}))\geq-\frac{v_D}{\sqrt{2}}, \tag{35}$$

$$m_2(\boldsymbol{x})=\frac{v_D}{\sqrt{2}}(1-\gamma_3 h_3(\boldsymbol{x}))\leq\frac{v_D}{\sqrt{2}}, \tag{36}$$

$$M_2(\boldsymbol{x})=-\frac{v_D}{\sqrt{2}}(1-\gamma_1 h_1(\boldsymbol{x}))\geq-\frac{v_D}{\sqrt{2}}. \tag{37}$$

Due to the bounds (34)-(37), (23) and (24) can be satisfied by control inputs bounded by $-\frac{v_D}{\sqrt{2}}\leq u_i^{\mathcal{T}}(\boldsymbol{x})\leq\frac{v_D}{\sqrt{2}}$, ensuring that $\|\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x})\|=\sqrt{u_1^{\mathcal{T}\,2}(\boldsymbol{x})+u_2^{\mathcal{T}\,2}(\boldsymbol{x})}\leq\sqrt{\left(\frac{v_D}{\sqrt{2}}\right)^2+\left(\frac{v_D}{\sqrt{2}}\right)^2}\leq v_D.$ □

Finally, the control action $\boldsymbol{u}(\boldsymbol{x})$ to satisfy the Control Objective 1 through the forward invariance of the set $\mathcal{C}$ is obtained next.

Theorem 3. *Let $m_1(\boldsymbol{x})$, $M_1(\boldsymbol{x})$, $m_2(\boldsymbol{x})$ and $M_2(\boldsymbol{x})$ be defined as in equations (34)-(37). If the initial state $\boldsymbol{x}(0)\in\mathcal{C}$ with $\mathcal{C}$ as defined in (15) and the conditions of Lemma 2 and Lemma 3 hold, then the control law*

$$\boldsymbol{u}(\boldsymbol{x})={}^{\mathcal{W}}R^{\mathcal{T}}\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}) \tag{38}$$

where ${}^{\mathcal{W}}R^{\mathcal{T}}$ is given in (19), and $\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x})=\begin{bmatrix}u_1^{\mathcal{T}}(\boldsymbol{x}) & u_2^{\mathcal{T}}(\boldsymbol{x})\end{bmatrix}$ with

$$u_1^{\mathcal{T}}(\boldsymbol{x})=\max\{m_1(\boldsymbol{x}),0\}+\min\{0,M_1(\boldsymbol{x})\}, \tag{39}$$

$$u_2^{\mathcal{T}}(\boldsymbol{x})=\max\{m_2(\boldsymbol{x}),0\}+\min\{0,M_2(\boldsymbol{x})\}, \tag{40}$$

solves the Control Objective 1.

Proof. Equations (23) and (24) can be represented by $m_j(\boldsymbol{x})\leq u_j^{\mathcal{T}}(\boldsymbol{x})\leq M_j(\boldsymbol{x})$ for $j\in\{1,2\}$. Since $\boldsymbol{x}(0)\in\mathcal{C}$, then $h_i(\boldsymbol{x}(0))\geq 0$ for all $i\in\{1,2,3,4\}$. Then, given a γ_j following Proposition 3 which ensures $m_j(\boldsymbol{x})\leq M_j(\boldsymbol{x})$, we select a $u_j^{\mathcal{T}}$ as follows:

$$u_j^{\mathcal{T}}(\boldsymbol{x})=\begin{cases}m_j(\boldsymbol{x}) & \textit{for } m_j\geq 0\\ 0 & \textit{for } m_j(\boldsymbol{x})<0<M_j(\boldsymbol{x})\\ M_j(\boldsymbol{x}) & \textit{for } M_j(\boldsymbol{x})\leq 0.\end{cases} \tag{41}$$

An input $u_j^{\mathcal{T}}(\boldsymbol{x})$ as in (41) satisfies $m_j(\boldsymbol{x})\leq u_j^{\mathcal{T}}\leq M_j(\boldsymbol{x})$, and equations (39) and (40) are equivalent to (41). Since

$$\|\frac{\partial h_j(\boldsymbol{x})}{\partial\boldsymbol{x}}\|=\|\frac{\partial h_{j+2}(\boldsymbol{x})}{\partial\boldsymbol{x}}\|\leq\frac{\sqrt{v_A^2+\left(v_D/\sqrt{2}\right)^2}}{v_A\left(v_D/\sqrt{2}\right)}, \tag{42}$$

by Lemma 3.3 in [21], $h_j(\boldsymbol{x})$ and $h_{j+1}(\boldsymbol{x})$ are both globally Lipschitz continuous. Since the max and min functions are also globally Lipschitz continuous, it can be verified that (39) and (40) satisfy

$$\|u_j^{\mathcal{T}}(\boldsymbol{x}_1) - u_j^{\mathcal{T}}(\boldsymbol{x}_2)\| \leq \frac{4\gamma_j}{v_A\left(v_D/\sqrt{2}\right)}\sqrt{v_A^2 + \left(v_D/\sqrt{2}\right)^2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad (43)$$

and therefore

$$\begin{aligned}\|\boldsymbol{u}(\boldsymbol{x}_1) - \boldsymbol{u}(\boldsymbol{x}_2)\| &= \|\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_1) - \boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_2)\| \\ &\leq \|u_1^{\mathcal{T}}(\boldsymbol{x}_1) - u_1^{\mathcal{T}}(\boldsymbol{x}_2)\| + \|u_2^{\mathcal{T}}(\boldsymbol{x}_1) - u_2^{\mathcal{T}}(\boldsymbol{x}_2)\| \\ &\leq \frac{8\gamma_j}{v_A\left(v_D/\sqrt{2}\right)}\sqrt{v_A^2 + \left(v_D/\sqrt{2}\right)^2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad (44)\end{aligned}$$

showing that the control law is globally Lipschitz continuous. Furthermore, the right side of the state equation (13) is globally Lipschitz continuous since

$$\left\| \begin{bmatrix} {}^{\mathcal{W}}R^{\mathcal{T}}\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_1) \\ \dot{\boldsymbol{x}}_A(t) \end{bmatrix} - \begin{bmatrix} {}^{\mathcal{W}}R^{\mathcal{T}}\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_2) \\ \dot{\boldsymbol{x}}_A(t) \end{bmatrix} \right\| \leq \|\boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_1) - \boldsymbol{u}^{\mathcal{T}}(\boldsymbol{x}_2)\|, \quad (45)$$

and therefore by Theorem 3.2 in [21], there is a unique solution for all $t \geq 0$. By Theorem 2, the set $\mathcal{C}$ is forward invariant for all $t \geq 0$. Since $h_i(\boldsymbol{x}) \geq 0$ for all $i \in \{1, 2, 3, 4\}$ and for all $t \geq 0$, by Proposition 1 the Control Objective 1 is solved. □

In order to ensure that $\boldsymbol{x}(0) \in \mathcal{C}$, the inequalities $h_i(\boldsymbol{x}(0)) \geq 0$ must be satisfied. Using (3), (8) and (14), given an initial position $x_D(0) \in \mathcal{T}$ of robot D, it is sufficient that, for an initial position of robot A $\boldsymbol{x}_A(0) \notin \mathcal{T}$, the initial distance

$$d_{A,i}(0) = \sqrt{\left(v_A T_{A,i}\left(\boldsymbol{x}_A(0), \boldsymbol{x}_{Vi}\right) + \epsilon\right)^2 - \epsilon^2}, \quad (46)$$

which is the shortest distance from robot A to the ith boundary of $\mathcal{T}$, satisfies

$$d_{A,i}(0) \geq \sqrt{\left(\frac{v_A}{v_D/\sqrt{2}}\left(\left(\boldsymbol{x}_D(0) - \boldsymbol{x}_{Vi}\right)\cdot\hat{\boldsymbol{b}}_{i+1} - s_i\right) + \epsilon\right)^2 - \epsilon^2} \quad (47)$$

for each $i \in \{1, 2, 3, 4\}$, with $\hat{\boldsymbol{b}}_5 = \hat{\boldsymbol{b}}_1$.

V. EXPERIMENTS

The control law in Theorem 3 ensures the solution to the Control Objective 1 while only requiring knowledge, with respect to the intruder, of its maximum speed and its current position. The control law does not require a further description of the motion of the intruder, such as its position as a function of time. To showcase the proposed control law in this paper, we implemented it in the Robotarium [20], and simulated an intruder using the position of a point on the plane following a predefined time function to ensure the maximum desired speed v_A. Let it be emphasized that the time function describing the position of the intruder was not

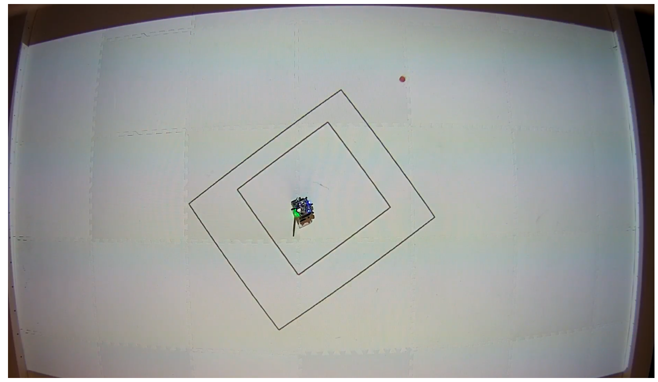

Fig. 2: A robot in the Robotarium as the defender, within the shield and target regions $\mathcal{S}$ and $\mathcal{T}$. The intruder is represented by the projection of a red dot.

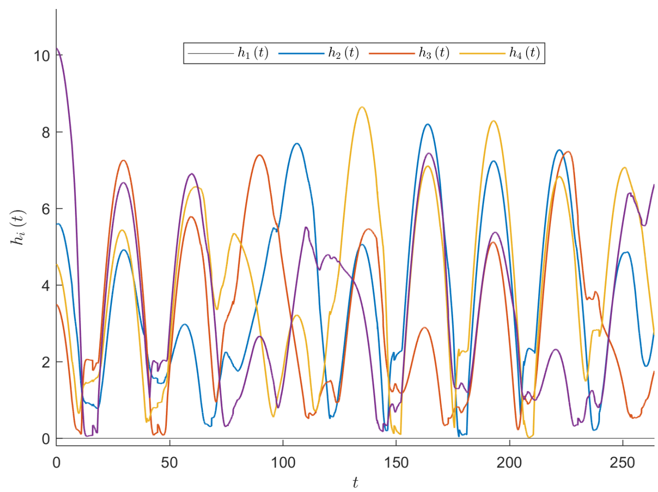

Fig. 3: The ZCBFs $h_i(t)$ corresponding to the sides of the region $\mathcal{T}$ have values always greater or equal than zero, ensuring the Control Objective 1.

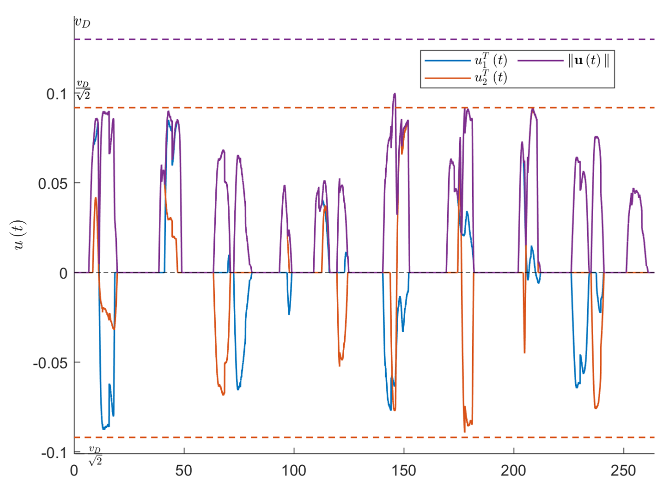

Fig. 4: The norm of the control action of robot D in the scenario from Figure 2 is bounded by $\|\boldsymbol{u}(t)\| \leq v_D$.

used to calculate the control law of the defender robot. It was only used to obtain the position $\boldsymbol{x}_A$ at each time instant t, which was then fed into the control loop. As long as the current position of the intruder and its maximum speed are known, and the conditions of Theorem 3 are satisfied, the Control Objective 1 is guaranteed to be solved. Similarly, as long as the conditions are satisfied, different rectangles and parameters can be selected to solve the control objective. The position $\boldsymbol{x}_D$ corresponds to a point $0.07m$ off of the axis of the wheels of the two-wheeled robots in the Robotarium, allowing to model and control the robots using a single integrator mathematical model.

Figure 2 shows a rectangular target region $\mathcal{T}$ with vertices at $\boldsymbol{x}_{V1} = [-0.1801, -0.6144]^{\mathsf{T}}, \boldsymbol{x}_{V2} = [0.6390, -0.0409]^{\mathsf{T}}, \boldsymbol{x}_{V3} = [0.1801, 0.6144]^{\mathsf{T}}, \boldsymbol{x}_{V4} = [-0.6390, 0.0409]^{\mathsf{T}}$. The intruder A moves according to $\boldsymbol{x}_A(t) = \begin{bmatrix}\cos(0.0959t + 6.5) & \sin(0.1151t + 7.7)\end{bmatrix}^{\mathsf{T}}$, ensuring a maximum speed of $v_A = 0.15m/s$. This function of time was not used for the control law, but only used to calculate the instantaneous position of robot A at every time t, which was then used in the controller. Robot D has an initial position at the origin, and maximum speed $v_D = 0.13m/s$. The initial position of robot A satisfies the initial condition described by equation (47) for each side of the rectangular region. The selected parameters s_i satisfying (26) are $s_1 = 0.2643, s_2 = 0.2913, s_3 = 0.2643, s_4 = 0.2913$, leading to the values of $\gamma_1 = \gamma_3 = 0.8446$ and $\gamma_2 = \gamma_4 = 0.9467$ satisfying (29). Figure 3 shows the values of the ZCBFs satisfy $h_i(t) \geq 0$, ensuring the satisfaction of the Control Objective. Figure 4 shows that the control action satisfies $\|\boldsymbol{u}(t)\| \leq v_D$.

VI. CONCLUSION

In this paper, control laws that ensure the defense and surveillance of a rectangular region in the plane are presented. The reactive and closed-form control laws based on control barrier functions can be used with robots of different speeds and geometric parameters defining the shield region. Future work will consider the case of multiple surveillance robots, multiple intruders with motion in the three-dimensional space, and regions with geometries different than rectangles.

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