Online Abstractions for Interconnected Multi-Agent Control Systems ⋆

Dimitris Boskos * **Dimos V. Dimarogonas** *

* *ACCESS Linnaeus Center, School of Electrical Engineering and KTH Center for Autonomous Systems, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden.*
E-mail: {boskos, dimos}@kth.se

Abstract: In this paper, we aim at the development of an online abstraction framework for multi-agent systems under coupled constraints. The motion capabilities of each agent are abstracted through a finite state transition system in order to capture reachability properties of the coupled multi-agent system over a finite time horizon in a decentralized manner. In the first part of this work, we define online abstractions by discretizing an overapproximation of the agents' reachable sets over the horizon. Then, sufficient conditions relating the discretization and the agents' dynamics are provided, in order to quantify the agents' transition possibilities.

Keywords: Reachability analysis, verification and abstraction of hybrid systems, Multi-agent systems, Decentralized control.

1. INTRODUCTION

During the last decade there has been an emerging focus on the problem of high level planning for multi-agent systems by leveraging methods from formal verification (Loizou and Kyriakopoulos, 2004). In order to exploit these tools for dynamic agents, it is required to build a discretized model of the continuous system which allows for the algorithmic synthesis of high level plans. Specifically, the use of an appropriate abstract representation enables the conversion of discrete paths into sequences of feedback controllers which enable the continuous time model to implement the high level specifications. This control synthesis problem has lead to a significant research effort for the derivation of discrete state analogues of continuous control systems, also called abstractions, which can capture reachability properties of the original model. Abstractions for piecewise affine systems on simplices and rectangles were introduced in (Habets and van Schuppen, 2001) and have been further studied in (Brouke and Gannes, 2014). Closer related to the control framework that we adopt here for the derivation of the discrete models is the paper (Helwa and Caines, 2014) which builds on the notion of In-Block Controllability (Caines and Wei, 1995). Abstractions for nonlinear systems include (Reissig, 2011), which is focused on general discrete time systems and (Abate et al., 2009), where box abstractions are derived for polynomial and other classes of systems. Furthermore, abstractions for interconnected systems have been recently developed in (Tazaki and Imura, 2008; Pola et al., 2014, 2016; Rungger and Zamani, 2015; Meyer et al., 2015; Dallal and Tabuada, 2015) and are primarily based on small gain criteria.

⋆ This work was supported by the H2020 ERC Starting Grant BUCOPHSYS, the Knut and Alice Wallenberg Foundation, the Swedish Foundation for Strategic Research (SSF), and the Swedish Research Council (VR).

In this work we consider multi-agent systems and provide an online abstraction methodology which enables the exploitation of the system's dynamic properties over bounded reachable sets. Specifically, we focus on agents whose dynamics consist of decentralized feedback interconnection terms and additional bounded input terms which allow for the synthesis of high level plans under the coupled constraints. The analysis builds on parts of the framework introduced in our recent work (Boskos and Dimarogonas, 2015), which focused on the discretization of the whole workspace and required the assumption of global bounds for the dynamics of the agents. In this framework, the latter hypothesis is considerably weakened, since it is only required that the system is forward complete. In addition, it is also possible to obtain coarser discretizations, since (i) the transition system of each agent is updated at the end of the time interval and thus, heterogeneous discretizations are considered for different agents, and (ii) the dynamics bounds of each agent, which constitute a measure of "coarseness" for its discretization, are evaluated for overapproximations of the agent and its neighbors' reachable sets and can result in reduced size discrete models for agents with weaker couplings over the time horizon. A relevant abstraction approach can be also found in (Esmaeil Zadeh Soudjani and Abate, 2013) where local Lipschitz properties of probability densities for stochastic kernels are exploited for the efficient abstraction of probabilistic systems into finite Markov Chains.

The rest of the paper is organized as follows. Basic notation and preliminaries are introduced in Section 2. In Section 3, we formulate online abstractions for single integrator multi-agent systems, over a specified time horizon. Section 4 is devoted to the design of the controllers that are exploited for the derivation of the discrete transitions. Space-time discretizations that guarantee well posed abstractions and their reachability properties are quantified in Section 5 and we conclude in Section 6. Due to space constraints, the proofs have been omitted. However, they can be found in (Boskos and Dimarogonas, 2016).

2. PRELIMINARIES AND NOTATION

We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a subset S of $\mathbb{R}^n$, we denote by $\mathrm{int}(S)$ its interior and define the distance from a point $x \in \mathbb{R}^n$ to S as $d(x,S) := \inf\{|x-y| : y \in S\}$. Given $R > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x;R)$ the closed ball with center $x \in \mathbb{R}^n$ and radius R and $B(R) := B(0,R)$. Given two sets $A, B \subset \mathbb{R}^n$ their Minkowski sum is defined as $A + B := \{x + y \in \mathbb{R}^n : x \in A, y \in B\}$. We say that a continuous function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_+$ if it is positive and strictly increasing and that $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_+\mathcal{K}_+$, if $\beta(t,\cdot), \beta(\cdot,s) \in \mathcal{K}_+$ for all $t, s \geq 0$. Consider a multi-agent system with N agents. For each agent $i \in \mathcal{N} := \{1,\ldots,N\}$ we use the notation $\mathcal{N}_i$ for its neighbors' set and N_i for its cardinality. We also consider an ordering of the neighbors which is denoted by $j_1, \ldots, j_{N_i}$ and define the N_i-tuple $j(i) = (j_1(i), \ldots, j_{N_i}(i))$. Whenever it is clear from the context, the argument i will be omitted. The agents' network is represented by a directed graph $\mathcal{G} := (\mathcal{N}, \mathcal{E})$, with vertex set $\mathcal{N}$ and edge set $\mathcal{E}$ the ordered pairs (ℓ, i) with $i, \ell \in \mathcal{N}$ and $\ell \in \mathcal{N}_i$. The sequence $i_0 i_1 \cdots i_m$ with $(i_{\kappa-1}, i_\kappa) \in \mathcal{E}$, $\kappa = 1, \ldots, m$, forms *a path* (of length m) in $\mathcal{G}$. A path $i_0 i_1 \cdots i_m$ with $i_0 = i_m$ is called a *cycle*. Given nonempty index sets $\mathcal{I}_1, \ldots, \mathcal{I}_N$, their Cartesian product $\boldsymbol{\mathcal{I}} := \mathcal{I}_1 \times \cdots \times \mathcal{I}_N$ and an agent $i \in \mathcal{N}$ with neighbors $j_1, \ldots, j_{N_i}$, we define the map $\mathrm{pr}_i : \boldsymbol{\mathcal{I}} \to \boldsymbol{\mathcal{I}}_i := \mathcal{I}_i \times \mathcal{I}_{j_1} \times \cdots \times \mathcal{I}_{j_{N_i}}$ assigning to each N-tuple $(l_1, \ldots, l_N)$ the N_i+1-tuple $(l_i, l_{j_1}, \ldots, l_{j_{N_i}})$, i.e., the indices of agent i and its neighbors. Finally, a transition system is defined as a tuple $TS := (Q, Q_0, Act, \longrightarrow)$, where: Q is a set of states; $Q_0 \subset Q$ is a set of initial states; Act is a set of actions; $\longrightarrow$ is a transition relation with $\longrightarrow \subset Q \times Act \times Q$. The transition system is said to be finite, if Q and Act are finite sets. We also denote an element $(q, a, q') \in \longrightarrow$ as $q \overset{a}{\longrightarrow} q'$ and define $\mathrm{Post}(q;a) := \{q' \in Q : (q,a,q') \in \longrightarrow\}$, for every $q \in Q$ and $a \in Act$.

3. ABSTRACTION OF THE AGENTS REACH SETS

We focus on multi-agent systems with single integrator dynamics

$$\dot{x}_i = f_i(x_i, \mathbf{x}_j) + v_i, x_i \in \mathbb{R}^n, i \in \mathcal{N}, \qquad (1)$$

with $\mathbf{x}_j (= \mathbf{x}_{j(i)}) := (x_{j_1}, \ldots, x_{j_{N_i}}) \in \mathbb{R}^{N_i n}$. We assume that the agents are in general heterogeneous and consider decentralized control laws consisting of two terms, a locally Lipschitz feedback term $f_i(\cdot)$ which depends on the states of i and its neighbors, and a free input v_i. We assume that $v_i \in \mathcal{U}_i$, $i \in \mathcal{N}$ where $\mathcal{U}_i$ is a bounded subset of $L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ taking values in a compact set $U_i \subset \mathbb{R}^n$ for each i and define $\mathcal{U} := \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$. The online abstraction framework is based on the discretization of each agent's reachable set over a given time horizon and the selection of a time step δt which corresponds to the duration of the discrete transitions. We will consider specific types of space discretizations, called cell decompositions (see also (Grüne, 2002)). In particular, given a bounded domain D of $\mathbb{R}^n$, a *cell decomposition* $\mathcal{S} = \{S_l\}_{l \in \mathcal{I}}$ of D, is a finite family of bounded sets S_l, $l \in \mathcal{I}$ with nonempty interior, such that $\mathrm{int}(S_l) \cap \mathrm{int}(S_{\hat{l}}) = \emptyset$ for all $l \neq \hat{l}$ and $\cup_{l \in \mathcal{I}} S_l = D$. In addition, given a bounded domain D of $\mathbb{R}^n$, a cell decomposition $\mathcal{S}$ of D and a set $A \subset D$, we say that $\mathcal{S}$ is *compliant* with A, if for any $S \in \mathcal{S}$ with $S \cap A \neq \emptyset$ it holds that $S \subset A$.

In order to provide decentralized abstractions we follow parts of the approach employed in (Boskos and Dimarogonas, 2015) and design appropriate hybrid feedback laws in place of the v_i's in order to guarantee well posed transitions. We assume that system (1) is *forward complete*, i.e., that for every initial condition $x_0 \in \mathbb{R}^{Nn}$ and $v \in \mathcal{U}$ the solution $x(t, x_0; v)$ is defined for all $t \geq 0$. Hence, there exists a function $\beta \in \mathcal{K}_+\mathcal{K}_+$ (Karafyllis, 2005) such that $|x(t, x_0; v)| \leq \beta(t, |x_0|), \forall t \geq 0, x_0 \in \mathbb{R}^{Nn}, v \in \mathcal{U}$. Additionally, we assume that each free input v_i, $i \in \mathcal{N}$ is bounded by a positive constant $v_{\max}(i)$, i.e., that

$$\mathcal{U}_i = \{v_i \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n) : |v_i(t)| \leq v_{\max}(i), \forall t \geq 0\}. \qquad (2)$$

We will consider a fixed time horizon $[0, T]$ on which we aim to abstract the agents' dynamics through a finite state transition system. Thus, at time $t = 0$, given the agents' initial positions, we will discretize an overapproximation of their reachable set over $[0, T]$ and select a time step δt which exactly divides T, in order to capture the motion of the system over that time interval through a finite transition system. After employing a discrete plan over $[0, T]$, we repeat the same procedure for the positions of the agents at $t = T$ and the new horizon $[T, 2T]$, and proceed analogously with the horizons $[\kappa T, (\kappa+1)T]$, $\kappa \geq 2$. For the subsequent analysis, we will assume fixed the initial states $X_{10}, \ldots, X_{N0}$ of all agents at the beginning of the horizon $[0, T]$ and consider for each agent $i \in \mathcal{N}$ an *open overapproximation* $\mathcal{R}_i(t)$ of its reachable set at $t \geq 0$. We also define the union of the reachable sets $\mathcal{R}_i(t)$ over a time interval $[a, b] \subset [0, \infty)$ as $\mathcal{R}_i([a,b]) := \cup_{t \in [a,b]} \mathcal{R}_i(t)$ and their inflation by a certain constant $c > 0$ as $\mathcal{R}_i^c(t) := \mathcal{R}_i(t) + B(c)$, $\mathcal{R}_i^c([a,b]) := \cup_{t \in [a,b]} \mathcal{R}_i^c(t)$. By forward completeness, we may always assume that the sets $\mathcal{R}_i([a,b])$ are bounded. Thus, the feedback terms $f_i(\cdot)$, $i \in \mathcal{N}$ are bounded on the overapproximations of the reachable sets. In particular, there exist positive constants $M(i)$ such that

$$|f_i(x_i, \mathbf{x}_j)| \leq M(i), \qquad (3)$$

for all $x_i \in \mathcal{R}_i([0,T])$, $x_\kappa \in \mathcal{R}_\kappa([0,T])$ and $\kappa \in \mathcal{N}_i$. Apart from the time horizon $[0, T]$ we will consider for certain technical reasons an additional time duration $0 < \tau < T$ which corresponds to an upper bound on the time discretization step δt. Based on this time duration we consider for each $i \in \mathcal{N}$ the sets $\mathcal{R}_i^{c_i(\tau)}([0, T-\tau])$, where $c_i(\sigma) := (M(i) + v_{\max}(i))\sigma, \sigma > 0$, with $M(i)$ and $v_{\max}(i)$ as given in (3) and (2), respectively. We also assume that without any loss of generality it holds $\mathcal{R}_i^{c_i(\sigma)}([0, T-\tau]) \supset \mathcal{R}_i(T-\tau+\sigma), \forall \sigma \in (0, \tau]$. Given a time step $0 < \delta t < \tau$ we depict the overapproximations of the reachable sets $\mathcal{R}_i([0, T-\tau]) \subset \mathcal{R}_i([0, T-\delta t]) \subset \mathcal{R}_i([0,T])$ of agent i with the red areas in Fig. 1. They all contain the exact reachable set $\mathcal{R}_i^{\mathrm{exact}}([0, T-\tau])$ of i over $[0, T-\tau]$ and the initial condition X_{i0} of i. We also depict the inflation $\mathcal{R}_i^{c_i(\tau)}([0, T-\tau])$ of $\mathcal{R}_i([0, T-\tau])$ which contains $\mathcal{R}_i([0,T])$.

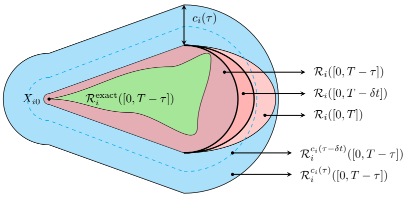

Fig. 1. Illustration of agent's i reachable sets over $[0,T]$.

Let $\{S_l^i\}_{l\in\mathcal{I}}$ be a cell decomposition of $\mathcal{R}_i([0,T])$. Then, we define the *product cell decomposition* $\{S_{\mathbf{l}}\}_{\mathbf{l}\in\boldsymbol{\mathcal{I}}}$ of $\mathcal{R}_1([0,T])\times\cdots\times\mathcal{R}_N([0,T])$ as the set $\mathcal{S}=\{S_{\mathbf{l}}\}_{\mathbf{l}\in\boldsymbol{\mathcal{I}}}:=\{S_l^1\}_{l\in\mathcal{I}_1}\times\cdots\times\{S_l^N\}_{l\in\mathcal{I}_N}$, with $\boldsymbol{\mathcal{I}}:=\mathcal{I}_1\times\cdots\times\mathcal{I}_N$. Given a cell decomposition $\{S_{\mathbf{l}}\}_{\mathbf{l}\in\boldsymbol{\mathcal{I}}}$ of $\mathcal{R}_1([0,T])\times\cdots\times\mathcal{R}_N([0,T])$, we use the notation $\mathbf{l}_i=(l_i,l_{j_1},\ldots,l_{j_{N_i}})\in\boldsymbol{\mathcal{I}}_i:=\mathcal{I}_i\times\mathcal{I}_{j_1}\times\cdots\times\mathcal{I}_{j_{N_i}}$ to denote the indices of the cells where agent i and its neighbors belong at a certain time instant and call it the cell configuration of i. Similarly, we use the notation $\mathbf{l}=(l_1,\ldots,l_N)\in\boldsymbol{\mathcal{I}}$ to specify the indices of the cells where all the agents belong and call it a *global* cell configuration. Thus, given a global cell configuration $\mathbf{l}$ it is possible to determine the cell configuration $\mathbf{l}_i=\mathrm{pr}_i(\mathbf{l})$ of agent i through the mapping $\mathrm{pr}_i:\boldsymbol{\mathcal{I}}\to\boldsymbol{\mathcal{I}}_i$ from Section 2. We next provide the class of hybrid feedback laws which are assigned to the free inputs v_i in order to obtain meaningful discrete transitions. The control laws are parameterized by the agents' initial conditions and a set of auxiliary parameters which are responsible for the agents' reachability capabilities. The specific control laws of this class which are exploited for the derivation of the discretizations are provided in the next section.

Definition 1. Consider an agent $i\in\mathcal{N}$, cell decompositions $\mathcal{S}_i=\{S_l^i\}_{l\in\mathcal{I}_i}$, $\mathcal{S}_\kappa=\{S_l^\kappa\}_{l\in\mathcal{I}_\kappa}$ of $\mathcal{R}_i([0,T])$ and $\mathcal{R}_\kappa([0,T])$, $\kappa\in\mathcal{N}_i$, respectively, a nonempty subset W_i of $\mathbb{R}^n$, and an initial cell configuration $\mathbf{l}_i$ of i. For each $x_{i0}\in S_{l_i}^i$ and $w_i\in W_i$, consider the mapping $k_{i,\mathbf{l}_i}(\cdot,\cdot,\cdot;x_{i0},w_i):[0,\infty)\times\mathbb{R}^{(N_i+1)n}\to\mathbb{R}^n$, parameterized by $x_{i0}\in S_{l_i}^i$ and $w_i\in W_i$. We say that $k_{i,\mathbf{l}_i}(\cdot)$ satisfies *Property (P)*, if: *(P1)* The map $k_{i,\mathbf{l}_i}(t,x_i,\mathbf{x}_j;x_{i0},w_i)$ is continuous on $[0,\infty)\times\mathbb{R}^{(N_i+1)n}\times S_{l_i}^i\times W_i$. *(P2)* The map $k_{i,\mathbf{l}_i}(t,\cdot,\cdot;x_{i0},w_i)$ is globally Lipschitz continuous on $(x_i,\mathbf{x}_j)$ (uniformly with respect to $t\in[0,\infty)$, $x_{i0}\in S_{l_i}^i$ and $w_i\in W_i$). $\triangleleft$

We next formalize our trasition requirement for each agent, based on the knowledge of its neighbors' discrete positions. In order to define the transitions, we will consider for each agent $i\in\mathcal{N}$ the following system with disturbances:

$$\dot{x}_i=g_i(x_i,\mathbf{d}_j)+v_i, \tag{4}$$

where $d_{j_1},\ldots,d_{j_{N_i}}:[0,\infty)\to\mathbb{R}^n$ (also denoted d_κ, $\kappa\in\mathcal{N}_i$) are continuous functions. The use of this auxiliary system is inspired by the approach in (Girard and Martin, 2012), where piecewise affine systems with disturbances are exploited for the construction of symbolic models for general nonlinear systems. The map $g_i(\cdot)$ constitutes a bounded Lipchitz extension of the restriction of $f_i(\cdot)$ on $\mathcal{R}_i([0,T])\times\mathcal{R}_{j_1}([0,T])\times\cdots\times\mathcal{R}_{j_{N_i}}([0,T])$ satisfying

$$|g_i(x_i,\mathbf{x}_j)|\le M(i),\forall(x_i,\mathbf{x}_j)\in\mathbb{R}^{(N_i+1)n} \tag{5}$$
$$|g_i(x_i,\mathbf{x}_j)-g_i(x_i,\mathbf{y}_j)|\le L_1(i)|\mathbf{x}_j-\mathbf{y}_i|, \tag{6}$$
$$|g_i(x_i,\mathbf{x}_j)-g_i(y_i,\mathbf{x}_j)|\le L_2(i)|x_i-y_i|, \tag{7}$$

for all $x_i,y_i\in\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ and $\mathbf{x}_j,\mathbf{y}_j\in\mathcal{R}_{j_1}([0,T])\times\cdots\times\mathcal{R}_{j_{N_i}}([0,T])$, with $M(i)$ as given in (3) and with any constants $L_1(i)$ and $L_2(i)$ such that $|f_i(x_i,\mathbf{x}_j)-f_i(x_i,\mathbf{y}_j)|\le L_1(i)|\mathbf{x}_j-\mathbf{y}_i|$, $|f_i(x_i,\mathbf{x}_j)-f_i(y_i,\mathbf{x}_j)|\le L_2(i)|x_i-y_i|$, for all $x_i,y_i\in\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$, $\mathbf{x}_j,\mathbf{y}_j\in\mathcal{R}_{j_1}([0,T])\times\cdots\times\mathcal{R}_{j_{N_i}}([0,T])$. This auxiliary system is used in order to provide an overapproximation of each agent's discrete transition capabilities over the horizon, by exploiting the global bounds of the auxiliary vector field $g_i(\cdot)$. Conditions under which these transitions are also implementable by the original system (1) are given later in Lemma 7. Notice that the Lipschitz constants above are evaluated for x_i ranging in the inflated reachable set $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$. This requirement comes from the fact that the transition system of each agent will be based on reachability properties of the auxiliary system with disturbances over the time step $[0,\delta t]$, for initial cells lying in the overapproximation $\mathcal{R}_i([0,T-\delta t])$ of agent's i reachable set. Since these cells may in principle contain states which are outside the exact reachable state of the agent, and the disturbances do not necessarily coincide with trajectories of its neighbors over this time interval, it is possible that the solution of (4) lies outside $\mathcal{R}_i([0,T])$ over $[0,\delta t]$. However, by (2), (5) and the definition of $c_i(\cdot)$ it follows that it will lie in the larger set $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$.

Definition 2. Consider an agent $i\in\mathcal{N}$, cell decompositions $\mathcal{S}_i=\{S_l^i\}_{l\in\mathcal{I}_i}$, $\mathcal{S}_\kappa=\{S_l^\kappa\}_{l\in\mathcal{I}_\kappa}$ of $\mathcal{R}_i([0,T])$ and $\mathcal{R}_\kappa([0,T])$, $\kappa\in\mathcal{N}_i$, respectively, a time step $\delta t<\tau$ and assume that $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$. Also, consider a nonempty subset W_i of $\mathbb{R}^n$, a cell configuration $\mathbf{l}_i$ of i with $S_{l_i}^i\subset\mathcal{R}_i([0,T-\delta t])$, a control law

$$v_i=k_{i,\mathbf{l}_i}(t,x_i,\mathbf{x}_j;x_{i0},w_i) \tag{8}$$

as in Definition 1 that satisfies Property (P), and a cell decomposition $\mathcal{S}_i'=\{S_l^i\}_{l\in\mathcal{I}_i'}$ of $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ with $\mathcal{S}_i'\supset\mathcal{S}_i$, $\mathcal{I}_i'\supset\mathcal{I}_i$ and compliant with $\mathcal{R}_i([0,T])$. Given a vector $w_i\in W_i$ and a cell index $l_i'\in\mathcal{I}_i'$, we say that the *Consistency Condition* is satisfied if the following hold. There exists a point $x_i'\in S_{l_i'}^i$, such that for each initial condition $x_{i0}\in S_{l_i}^i$ and selection of continuous functions $d_\kappa:\mathbb{R}_{\ge0}\to\mathbb{R}^n$, $\kappa\in\mathcal{N}_i$ satisfying

$$\begin{aligned} d_\kappa(t)\in(S_{l_\kappa}^\kappa+B((M(\kappa)+v_{\max}(\kappa))t))\cap\mathcal{R}_\kappa([0,T]),\\ \forall\kappa\in\mathcal{N}_i,t\in[0,\delta t], \end{aligned} \tag{9}$$

the solution $x_i(\cdot)$ of the system with disturbances (4) with $v_i=k_{i,\mathbf{l}_i}(t,x_i,\mathbf{d}_j;x_{i0},w_i)$, satisfies $d(x_i(t),S_{l_i}^i)<(M(i)+v_{\max}(i))t,\forall t\in(0,\delta t]$. Furthermore, $x_i(\delta t)=x_i'\in S_{l_i'}^i$ and $|k_{i,\mathbf{l}_i}(t,x_i(t),\mathbf{d}_j(t);x_{i0},w_i)|<v_{\max}(i),\forall t\in[0,\delta t]$.$\triangleleft$

Notice that when the Consistency Condition is satisfied, agent i can be driven to cell $S_{l_i'}^i$ precisely in time δt under the auxiliary dynamics (4), with the feedback law $k_{i,\mathbf{l}_i}(\cdot)$ corresponding to the given parameter w_i in the definition. The latter is possible for all disturbances which satisfy (9) and capture the possibilities for the evolution of i's neighbors over the time interval $[0,\delta t]$, given the knowledge of its neighbors' cell configuration. Under some additional asumptions, which are provided in Lemma 7, the latter transitions can be also implemented by the original system (1) and the control law $k_{i,\mathbf{l}_i}(\cdot)$. We proceed with the definition of a well posed online abstraction for each agent in order to extract a finite transition system.

Definition 3. Consider cell decompositions $\mathcal{S}_i = \{S_l^i\}_{l\in\mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product decomposition $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$. *(i)* Given an agent $i \in \mathcal{N}$, a cell decomposition $\mathcal{S}_i' = \{S_l^i\}_{l\in\mathcal{I}_i'}$ of $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ with $\mathcal{S}_i' \supset \mathcal{S}_i$, $\mathcal{I}_i' \supset \mathcal{I}_i$ and compliant with $\mathcal{R}_i([0,T])$, an initial cell configuration $\mathbf{l}_i \in \boldsymbol{\mathcal{I}}_i$ of i with $S_{l_i}^i \subset \mathcal{R}_i([0,T-\delta t])$, and a cell index $l_i' \in \mathcal{I}_i'$, we say that *the transition* $l_i \xrightarrow{\mathbf{l}_i} l_i'$ *is well posed with respect to the space-time discretization* $\mathcal{S}-\delta t$, if there exist a feedback law $v_i = k_{i,\mathbf{l}_i}(\cdot,\cdot,\cdot;x_{i0},w_i)$ as in Definition 1 that satisfies Property (P), and a vector $w_i \in W_i$, such that the Consistency Condition of Definition 2 is fulfilled. *(ii)* We say that the *space-time discretization* $\mathcal{S}-\delta t$ *is well posed*, if for each agent $i \in \mathcal{N}$, cell decomposition $\mathcal{S}_i' = \{S_l^i\}_{l\in\mathcal{I}_i'}$ of $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ with $\mathcal{S}_i' \supset \mathcal{S}_i$, $\mathcal{I}_i' \supset \mathcal{I}_i$ and compliant with $\mathcal{R}_i([0,T])$, and cell configuration $\mathbf{l}_i$ of i, there exists a cell index $l_i' \in \mathcal{I}_i'$ such that the transition $l_i \xrightarrow{\mathbf{l}_i} l_i'$ is well posed with respect to $\mathcal{S}-\delta t$.

Based on Definition 3(i), we define the discrete transition system which serves as an abstract model for the behavior of each agent. The transitions are established through the verification of the Consistency Condition which exploits the auxiliary system with disturbances (4).

Definition 4. Consider cell decompositions $\mathcal{S}_i = \{S_l^i\}_{l\in\mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product decomposition $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$. The *individual transition system* $TS_i := (Q_i, Q_{0i}, Act_i, \longrightarrow_i)$ of each agent $i \in \mathcal{N}$ is defined as: $Q_i := \mathcal{I}_i$ (the indices of the decomposition $\mathcal{S}_i$); $Q_{0i} := \{l_i \in \mathcal{I}_i : X_{i0} \in S_{l_i}^i\}$; $Act_i := \boldsymbol{\mathcal{I}}_i$ (the cell configurations of i); Transition relation $\longrightarrow_i \subset Q_i \times Act_i \times Q_i$ defined as follows. For any $l_i, l_i' \in Q$ and $\mathbf{l}_i = (l_i, l_{j_1}, \ldots, l_{j_{N_i}}) \in \boldsymbol{\mathcal{I}}_i$, $l_i \xrightarrow{\mathbf{l}_i}_i l_i'$, iff $l_i \xrightarrow{\mathbf{l}_i} l_i'$ is well posed (implying also that $S_{l_i}^i \subset \mathcal{R}_i([0,T-\delta t])$).

Remark 5. The auxiliary cell decomposition $\mathcal{S}_i'$ which is exploited for the verification of the Consistency Condition can provide according to Definition 3(i) well posed transitions which lead to a cell $S_{l_i'}^i$ outside $\mathcal{R}_i([0,T])$. These transitions are excluded from the definition of each agent's transition system, since they do not capture any possible behavior of the system over $[0,T]$. In particular, the transitions of possible interest over the horizon are the ones where the initial and final state of the agent lie in the exact reachable sets over $[0,T-\delta t]$ and $[0,T]$, respectively. In addition, for the case where the cells of an agent and its neighbors have nonempty intersection with the corresponding agents' reachable cells at certain time instant $t = m\delta t$ with $m \in \{0,\ldots,\ell-1\}$, it will be validated in Theroem 12 that there is always an outgoing transition for well posed discretizations.

In the subsequent analysis we will consider well posed discretizations which implies that their time step δt has been selected so that $T = \ell\delta t$ and will focus on transition sequences of length $m \le \ell$ originating from cells which contain the agents' initial positions X_{i0}, $i \in \mathcal{N}$. Such sequences are defined below for the individual transition system of each agent. In addition, it will be shown in the sequel that the projection of a transition sequence originating from the discrete state containing $\mathbf{X}_0$ in the product discrete model (of all agents) to the individual transition system of each agent will provide such a sequence of transitions for each agent, which can also be implemented by the continuous time system.

Definition 6. Consider cell decompositions $\mathcal{S}_i = \{S_l^i\}_{l\in\mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product decomposition $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$. Given an agent $i \in \mathcal{N}$, an integer $m \in \{1,\ldots,\ell\}$, cell configurations $\mathbf{l}^\kappa = (l_i^\kappa, l_{j_1}^\kappa, \ldots, l_{j_{N_i}}^\kappa) \in \boldsymbol{\mathcal{I}}_i$, $\kappa = 0,\ldots,m-1$ of i and a cell index $l_i^m \in \mathcal{I}_i$, we say that $\mathbf{l}_i^0\mathbf{l}_i^1\cdots\mathbf{l}_i^{m-1}l_i^m$ is *a strongly well posed transition sequence of order* m, if $X_{i0} \in S_{l_i^0}^i$ and $l_i^\kappa \xrightarrow{\mathbf{l}_i^\kappa}_i l_i^{\kappa+1}$. We also define $\mathbf{l}_i^0$ as *a strongly well posed transition sequence of order 0* if $X_{i0} \in S_{l_i^0}^i$.

The following lemma establishes that for well posed discretizations and cell configurations of all agents which intersect their exact reachable cells at a certain time instant $t = m\delta t$ with $m \in \{0,\ldots,\ell-1\}$ there exists a transition for each agent that can be implemented by the continuous time system (1).

Lemma 7. Consider cell decompositions $\mathcal{S}_i = \{S_l^i\}_{l\in\mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$ and that the space-time discretization $\mathcal{S}-\delta t$ is well posed. Also, consider a cell configuration $\mathbf{l} = (l_1,\ldots,l_N)$, an integer $m \in \{0,\ldots,\ell-1\}$, an input $v = (v_1,\ldots,v_N) \in \mathcal{U}$ and assume that each component $x_i(\cdot,\mathbf{X}_0;v)$ of the solution of (1) satisfies $x_i(m\delta t,\mathbf{X}_0;v) \in S_{l_i}^i$. *(i)* Then, it holds that $\mathrm{Post}_i(l_i;\mathrm{pr}_i(\mathbf{l})) \neq \emptyset$ for all $i \in \mathcal{N}$. In particular, $\mathrm{Post}_i(l_i;\mathrm{pr}_i(\mathbf{l})) = \{l_i' \in \mathcal{I}_i' : l_i \xrightarrow{\mathbf{l}_i} l_i' \text{ is well posed}\} \subset \mathcal{I}_i$, for any cell decomposition $\mathcal{S}_i' = \{S_l^i\}_{l\in\mathcal{I}_i'}$ of $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ with $\mathcal{S}_i' \supset \mathcal{S}_i$, $\mathcal{I}_i' \supset \mathcal{I}_i$ and compliant with $\mathcal{R}_i([0,T])$, and is uniquely defined, irrespectively of the cell decomposition $\mathcal{S}_i'$. *(ii)* In addition, for any selection of $l_i' \in \mathrm{Post}_i(l_i;\mathrm{pr}_i(\mathbf{l}))$, $i \in \mathcal{N}$, the following hold. There exist feedback laws $v_i = k_{i,\mathrm{pr}_i(\mathbf{l})}$ as in (8) and $w_i \in W_i$ for all $i \in \mathcal{N}$, such that the solution $\xi(\cdot)$ of the closed loop system (1), (8) with initial condition $\xi(0) = x(m\delta t,\mathbf{X}_0;v)$ satisfies $\xi_i(\delta t) \in S_{l_i'}^i$ and $|k_{i,\mathbf{l}_i}(t,\xi_i(t),\boldsymbol{\xi}_j(t);x_{i0},w_i)| \le v_{\max}(i)$ for all $t \in [0,\delta t]$ and $i \in \mathcal{N}$. Furthermore, there exists $u \in \mathcal{U}$ with $u(t) = v(t)$ for all $t \in [0,m\delta t)$, such that the solution of (1) satisfies $x_i((m+1)\delta t,\mathbf{X}_0;u) \in S_{l_i'}^i$ for all $i \in \mathcal{N}$.

Based on Lemma 7, we show that consistent discrete sequences of all agents which project to strongly well posed individual transition sequences, have always outgoing transitions.

Proposition 8. Consider cell decompositions $\mathcal{S}_i = \{S_l^i\}_{l\in\mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$ and that the space-time discretization $\mathcal{S}-\delta t$ is well posed. Also, consider a sequence $\mathbf{l}^0\cdots\mathbf{l}^m$ of global cell configurations with $m \in \{0,\ldots,\ell\}$ such that $\mathrm{pr}_i(\mathbf{l}^0)\cdots\mathrm{pr}_i(\mathbf{l}^{m-1})l_i^m$ is a strongly well posed transition sequence of order m for each $i \in \mathcal{N}$. *(i)* Then, there exists $v \in \mathcal{U}$ such that each component $x_i(\cdot, \mathbf{X}_0; v)$ of the solution of (1) satisfies $x_i(\kappa\delta t, \mathbf{X}_0; v) \in S^i_{l^\kappa_i}$, for all $\kappa \in \{0, \dots, m\}$. *(ii)* If in addition $m < \ell$, then $\mathrm{Post}_i(l^m_i; \mathrm{pr}_i(\mathbf{l}^m)) \neq \emptyset$ for all $i \in \mathcal{N}$.

From Proposition 8 we can derive the desired properties of the product transition system corresponding to the space-time discretization, which will be defined recursively. In particular, given the product $\boldsymbol{\mathcal{I}} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_N$ of the cell indices corresponding to the decompositions of the sets $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, we define the operator $\mathcal{P} : \boldsymbol{\mathcal{I}} \to 2^{\boldsymbol{\mathcal{I}}}$ as $\mathcal{P}(\mathbf{l}) := \mathrm{Post}_1(l_1; \mathrm{pr}_1(\mathbf{l})) \times \cdots \times \mathrm{Post}_N(l_N; \mathrm{pr}_N(\mathbf{l}))$, $\mathbf{l} \in \boldsymbol{\mathcal{I}}$, where $\mathrm{Post}_i(\cdot;\cdot)$, $i \in \mathcal{N}$ are the post operators for the agent's individual transition systems. We also recursively define the operators $\mathcal{P}^\kappa : 2^{\boldsymbol{\mathcal{I}}} \to 2^{\boldsymbol{\mathcal{I}}}$, $\kappa \in \mathbb{N} \cup \{0\}$, as $\mathcal{P}^0(\mathcal{I}) := \mathcal{I}$; $\mathcal{P}^\kappa(\mathcal{I}) := \mathcal{P}(\mathcal{P}^{\kappa-1}(\mathcal{I})), \kappa \geq 1, \mathcal{I} \subset \boldsymbol{\mathcal{I}}$. We next provide the definition of the product transition system.

Definition 9. *(i)* Consider cell decompositions $\mathcal{S}_i = \{S^i_l\}_{l \in \mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0, T - \delta t])$. Also, consider for each agent $i \in \mathcal{N}$ its individual transition system TS_i as provided by Definition 4. The product transition system $TS^{\mathcal{P}} := TS_1 \otimes \cdots \otimes TS_N$ is the transition system $(Q, Q_0, \mathrm{Act}, \longrightarrow)$ with: $Q := \boldsymbol{\mathcal{I}} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_N$ (the indices of the product decomposition); $Q_0 := Q_{10} \times \cdots \times Q_{N0}$, $Q_{0i} := \{l_i \in \mathcal{I}_i : X_{i0} \in S_{l_i}\}$, $i \in \mathcal{N}$; *Act* $:= \{*\}$; Transition relation $\longrightarrow \subset Q \times Act \times Q$ defined as follows. For any $\mathbf{l}, \mathbf{l}' \in Q$, $\mathbf{l} \overset{*}{\longrightarrow} \mathbf{l}'$, iff there exists $m \in \{0, \dots, \ell - 1\}$ such that $\mathbf{l} \in \mathcal{P}^m(Q_0)$ and $\mathbf{l}' \in \mathcal{P}(\mathbf{l})$. *(ii)* A path of length $m \in \{0, \dots, \ell\}$ originating from $\mathbf{l}^0$ in $TS^{\mathcal{P}}$, is a finite sequence of states $\mathbf{l}^0\mathbf{l}^1 \cdots \mathbf{l}^m$ such that $\mathbf{l}^0 \in Q_0$ and $\mathbf{l}^{\kappa-1} \overset{*}{\longrightarrow} \mathbf{l}^\kappa$ for all $\kappa \in \{1, \dots, m\}$ (when $m \neq 0$).

We will show in the sequel that for well posed discretizations the sets $\mathcal{P}^m(Q_0)$, $m \in \{0, \dots, \ell\}$ in Definition 9 are always nonempty and that there exists an outgoing transition in the product transition system from any $\mathbf{l} \in \mathcal{P}^m(Q_0)$, $m \in \{0, \dots, \ell - 1\}$.

Proposition 10. Assume that the space-time discretization $\mathcal{S} - \delta t$ is well posed. Then, for each $m \in \{0, \dots, \ell - 1\}$ and $\mathbf{l} \in \mathcal{P}^m(Q_0) (\neq \emptyset)$ it holds $\mathrm{Post}(\mathbf{l}) = \mathcal{P}(\mathbf{l}) \neq \emptyset$.

The proposition below constitutes our main result in this section and guarantees the existence of paths of length m for any $m \in \{0, \dots, \ell\}$ originating from certain $\mathbf{l}^0 \in Q_0$ in $TS^{\mathcal{P}}$. Additionally, it is shown that any such path can be realized by a sampled trajectory of the continuous time system (1) initiated from $\mathbf{X}_0$ over the subinterval $[0, m\delta t]$ of the time horizon $[0, T] = [0, \ell\delta t]$.

Proposition 11. Consider cell decompositions $\mathcal{S}_i = \{S^i_l\}_{l \in \mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$, $i \in \mathcal{N}$, their product $\mathcal{S}$, a time step $\delta t < \tau$ with $T = \ell\delta t$, nonempty subsets W_i, $i \in \mathcal{N}$ of $\mathbb{R}^n$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0, T - \delta t])$ and that the space time discretization $\mathcal{S} - \delta t$ is well posed. Then: *(i)* For any $m \in \{0, \dots, \ell\}$ there exists a path $\mathbf{l}^0\mathbf{l}^1 \cdots \mathbf{l}^m$ of length m originating from $\mathbf{l}^0$ in the product transition system $TS^{\mathcal{P}}$. *(ii)* For any path $\mathbf{l}^0\mathbf{l}^1 \cdots \mathbf{l}^m$ of length m originating from $\mathbf{l}^0$ in $TS^{\mathcal{P}}$, there exists an input $v \in \mathcal{U}$ such that each component $x_i(\cdot, \mathbf{X}_0; v)$ of the solution of (1) satisfies $x_i(\kappa\delta t, \mathbf{X}_0; v) \in S^i_{l^\kappa_i}$, for all $\kappa \in \{0, \dots, m\}$.

4. DESIGN OF THE HYBRID CONTROL LAWS

In this section, we define the control laws that are exploited in order to derive well posed transitions in accordance to Definition 3. Consider for each agent i a cell decomposition $\{S^i_l\}_{l \in \mathcal{I}_i}$ of $\mathcal{R}_i([0,T])$ and a time step δt. We define the diameter $d_{\max}(i)$ of each cell decomposition $\{S^i_l\}_{l \in \mathcal{I}_i}$ as $d_{\max}(i) := \inf\{R > 0 : \forall l \in \mathcal{I}_i, \exists x \in S^i_l, S^i_l \subset B(x; \frac{R}{2})\}$ and select a reference point $x_{l_i,G}$ for every cell $S^i_{l_i}$, with $|x_{l_i,G} - x| \leq \frac{d_{\max}(i)}{2}, \forall x \in S^i_{l_i}, l_i \in \mathcal{I}_i, i \in \mathcal{N}$. For each agent i and cell configuration $\mathbf{l}_i$ of i, we define the family of feedback laws $k_{i,\mathbf{l}_i} : [0, \infty) \times \mathbb{R}^{(N_i+1)n} \to \mathbb{R}^n$ parameterized by $x_{i0} \in S^i_{l_i}$ and $w_i \in W_i$ as $k_{i,\mathbf{l}_i}(t, x_i, \mathbf{x}_j; x_{i0}, w_i) := k_{i,\mathbf{l}_i,1}(t, x_i, \mathbf{x}_j) + k_{i,\mathbf{l}_i,2}(x_{i0}) + k_{i,\mathbf{l}_i,3}(w_i)$, where $W_i := B(v_{\max}(i)) \subset \mathbb{R}^n$ and

$$k_{i,\mathbf{l}_i,1}(t, x_i, \mathbf{x}_j) := g_i(\chi_i(t), \mathbf{x}_{l_j,G}) - g_i(x_i, \mathbf{x}_j), \tag{10}$$

$$k_{i,\mathbf{l}_i,2}(x_{i0}) := \frac{1}{\delta t}(x_{l_i,G} - x_{i0}), k_{i,\mathbf{l}_i,3}(w_i) := \lambda(i)w_i. \tag{11}$$

The function $\chi_i(\cdot)$ in (10) is defined for all $t \geq 0$ through the solution of the initial value problem

$$\dot{\chi}_i = g_i(\chi_i, \mathbf{x}_{l_j,G}), \chi_i(0) = x_{l_i,G}, \tag{12}$$

with the globally Lipschitz function $g_i(\cdot)$ as given in (5). The parameter $\lambda(i)$ stands for the part of the free input that can be further exploited for motion planning. In particular, for each $w_i \in W$, the vector $\lambda(i)w_i$ provides the "velocity" of a motion that we superpose to the reference trajectory $\chi_i(\cdot)$ of agent i over $[0, \delta t]$. The latter allows the agent to reach all points inside a ball with center the position of the reference trajectory at time δt by following the curve $\bar{x}_i(t) := \chi_i(t) + \lambda(i)w_i t$, as depicted in Fig. 2 below. This ball has radius

$$r_i := \lambda(i)\delta t v_{\max}(i), \tag{13}$$

namely, the distance that the agent can cross in time δt by exploiting $k_{i,\mathbf{l}_i,3}(\cdot)$, which corresponds to the part of the free input that is selected for reachability purposes. Hence, it is possible to perform a well posed transition to any cell which has a nonempty intersection with $B(\chi_i(\delta t); r_i)$.

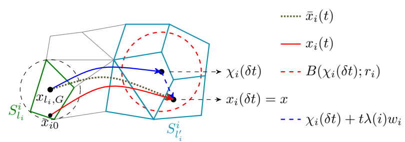

Fig. 2. Illustration of the reference trajectory and reachability capabilities of the control laws.

5. WELL POSED SPACE-TIME DISCRETIZATIONS

In this section, we exploit the controllers introduced in Section 4 to provide sufficient conditions for well posed space-time discretizations. Since the space discretization of each agent is affected by the local in time properties of its dynamics, it is convenient to consider different diameters for the decomposition of each agent, which require certain design constraints on the diameters of neighboring decompositions. In particular, for each agent's neighbors we impose the restriction that the diameters of their decompositions satisfy $d_{\max}(j) \leq \mu(j,i)d_{\max}(i)$. For these restrictions to be meaningful, we also impose the condition that $\mu(i_0,i_1)\mu(i_1,i_2)\cdots\mu(i_{m-1},i_m)\geq 1$, for all cycles $i_0 i_1\cdots i_m$ in the graph $\mathcal{G}$, which is always satisfied if we select $\mu(j,i)=1$ for all $i\in\mathcal{N}$ and $j\in\mathcal{N}_i$. For the acceptable values of the discretizations, it is also convenient to define for each agent the local network parameters $\boldsymbol{\mu}(i):=(\sum_{j\in\mathcal{N}_i}\mu(j,i)^2)^{\frac{1}{2}}$ and $\mathbf{M}(i):=(\sum_{j\in\mathcal{N}_i}(M(j)+v_{\max}(j))^2)^{\frac{1}{2}}$. Also, for each $i\in\mathcal{N}$ and cell configuration $\mathbf{l}_i\in\boldsymbol{\mathcal{I}}_i$ of i, consider the family of feedback laws in (10), (11), with the reference trajectory $\chi_i(\cdot)$ generated by (12). We now provide sufficient conditions for well posed space-time discretizations and their transition capabilities.

Theorem 12. Consider cell decompositions $\mathcal{S}_i=\{S_l^i\}_{l\in\mathcal{I}_i}$ of the sets $\mathcal{R}_i([0,T])$ with diameters $d_{\max}(i)$, their product $\mathcal{S}$, a time step δt, the constant r_i in (13), the parameters $\lambda(i)\in(0,1)$ and assume that each $\mathcal{S}_i$ is compliant with $\mathcal{R}_i([0,T-\delta t])$. We also assume that $d_{\max}(i)$, $i\in\mathcal{N}$ satisfy the requirements above, $\ell\delta t=T$ for certain $\ell\in\mathbb{N}$ and that $\delta t\in(0,\frac{(1-\lambda(i))v_{\max}(i)}{L_1(i)\mathbf{M}(i)+L_2(i)\lambda(i)v_{\max}(i)})$, $d_{\max}(i)\in(0,\min\{\frac{2(1-\lambda(i))v_{\max}(i)\delta t}{1+(L_1(i)\mu(i)+L_2(i))\delta t},(2(1-\lambda(i))v_{\max}(i)\delta t-2(L_1(i)\mathbf{M}(i)+L_2(i)\lambda(i)v_{\max}(i))\delta t^2)/(1+L_1(i)\boldsymbol{\mu}(i)\delta t)\})$, with $L_1(i)$, $L_2(i)$ and $v_{\max}(i)$ as given in (6), (7) and (2), respectively, and $\boldsymbol{\mu}(i)$, $\mathbf{M}(i)$ as defined above. Then, the space-time discretization is well posed for system (1). In particular: *(i)* For each agent $i\in\mathcal{N}$, cell decomposition $\mathcal{S}_i'=\{S_l^i\}_{l\in\mathcal{I}_i'}$ of $\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ with $\mathcal{S}_i'\supset\mathcal{S}_i$, $\mathcal{I}_i'\supset\mathcal{I}_i$ and compliant with $\mathcal{R}_i([0,T])$, and cell configuration $\mathbf{l}_i$ of i with $S_{l_i}^i\subset\mathcal{R}_i([0,T-\delta t])$ it holds $B(\chi_i(\delta t);r_i)\subset\mathcal{R}_i^{c_i(\tau)}([0,T-\tau])$ and $l_i\xrightarrow{\mathbf{l}_i}l_i'$ is well posed for all $l_i'\in\{l\in\mathcal{I}_i':S_l^i\cap B(\chi_i(\delta t)r_i)\neq\emptyset\}$, with the reference trajectory $\chi_i(\cdot)$ as given by (12) and r_i as defined in (13). *(ii)* For each agent $i\in\mathcal{N}$, cell configuration $\mathbf{l}_i$ of i, integer $m\in\{0,\ldots,\ell-1\}$ and input $v\in\mathcal{U}$ such that each component $x_\kappa(\cdot,\mathbf{X}_0;v)$, $\kappa\in\mathcal{N}_i\cup\{i\}$ of the solution of (1) satisfies $x_\kappa(m\delta t,\mathbf{X}_0;v)\in S_{l_\kappa}^\kappa$, it holds $\mathrm{Post}(l_i;\mathbf{l}_i)\supset\{l\in\mathcal{I}_i':S_l^i\cap B(\chi_i(\delta t);r_i)\neq\emptyset\}$.

6. CONCLUSIONS AND FUTURE WORK

We have provided a distributed online abstraction framework for forward complete multi-agent systems under coupled constraints. The derived abstractions provide for each agent an individual discrete model for an overapproximation of its reachable set over a finite time horizon. In addition, the composition of the individual agent models provides transitions which capture the evolution of the continuous time system over the horizon. Future work directions include the study of specific network structures and quantifying the tradeoff between the depth of the planning horizon and the depth of the required information in the network graph for the investigation of the agents' reachability properties.

REFERENCES

Abate, A., Tiwari, A., and Sastry, S. (2009). Box invariance in biologically-inspired dynamical systems. 7(45), 1601–1610.

Boskos, D. and Dimarogonas, D.V. (2015). Decentralized abstractions for feedback interconnected multi-agent systems. *in Proceedings of the 54th IEEE Conference on Decision and Control*, 282–287.

Boskos, D. and Dimarogonas, D.V. (2016). Online abstractions for interconnected multi-agent control systems. *arXiv:1611.03067*.

Brouke, M.E. and Gannes, M. (2014). Reach control on simplices by piecewise affine feedback. *SIAM Journal on Control and Optimization*, 5(52), 3261–3286.

Caines, P.E. and Wei, Y.J. (1995). The hierarchical lattices of a finite machine. *Systems and Control Letters*, (25), 257–263.

Dallal, E. and Tabuada, P. (2015). On compositional symbolic controller synthesis inspired by small-gain theorems. *in Proceedings of the 54th IEEE Conference on Decision and Control*, 6133–6138.

Esmaeil Zadeh Soudjani, S. and Abate, A. (2013). Adaptive and sequential gridding procedures for the abstraction and verification of stochastic processes. *SIAM Journal on Applied Dynamical Systems*, 12(2), 921–956.

Girard, A. and Martin, S. (2012). Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices. *IEEE Transactions on Automatic Control*, 57(4), 1046–1051.

Grüne, L. (2002). *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization*, volume 1783. Springer-Verlag, Berlin, Germany.

Habets, L.C. and van Schuppen, J.H. (2001). Control of piecewise-linear hybrid systems on simplices and rectangles. In *International Workshop on Hybrid Systems: Computation and Control*, 261–274. Springer.

Helwa, M.K. and Caines, P.E. (2014). In-block controllability of affine systems on polytopes. *in Proceedings of the 53rd IEEE Conference on Decision and Control*, 3936–3942.

Karafyllis, I. (2005). Non-uniform in time robust global asymptotic output stability. *Systems & control letters*, 54(3), 181–193.

Loizou, S.G. and Kyriakopoulos, K.J. (2004). Automatic synthesis of multi-agent motion tasks based on ltl specifications. *in Proceedings of the 43rd IEEE Conference on Decision and Control*, 153–158.

Meyer, P., Girard, A., and Witrant, E. (2015). Safety control with performance guarantees of cooperative systems using compositional abstractions. *in Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems*, 317–322.

Pola, G., Pepe, P., and di Benedetto, M.D. (2014). Symbolic models for networks of discrete-time nonlinear control systems. *in Proceedings of the American Control Conference*, 1787–1792.

Pola, G., Pepe, P., and di Benedetto, M.D. (2016). Symbolic models for networks of control systems. *IEEE Transactions on Automatic Control*, TAC.2016.2528046.

Reissig, G. (2011). Computing abstractions of nonlinear systems. *IEEE Transactions on Automatic Control*, 56(11), 2583–2598.

Rungger, M. and Zamani, M. (2015). Compositional construction of approximate abstractions. *in Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, 68–77.

Tazaki, Y. and Imura, J.i. (2008). Bisimilar finite abstractions of interconnected systems. In *International Workshop on Hybrid Systems: Computation and Control*, 514–527. Springer.