Scheduling Mode Switches in Distributed Plug-and-Play Observers

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Abstract: In this work, we study the problem of distributed state estimation of continuous-time linear systems. However, in contrast to existing studies, we consider that each observer node in the network has the following two modes of operation: i) to plug-in and engage/play, i.e., nodes can actively access partial outputs and exchange information, or ii) to remain on standby, i.e., nodes can only propagate in an open-loop fashion. These capabilities could allow the distributed observers to ration their energy/communication resources effectively. To facilitate these modes of the observer nodes, we modify the well-known Luenberger-based observer dynamics for state estimation and establish conditions on the switching signal that schedules mode changes at the observer nodes, and by extension, the switches in the underlying communication network. Consequently, we establish asymptotic omniscience of the distributed plug-and-play observers.

Keywords: Distributed state estimation, Scheduling observer modes, Switching networks, Plug-and-play observers

1. INTRODUCTION

Cyber-physical applications involving multi-vehicle or multi-robot systems and sensor networks often rely on spatially distributed observers for accurate state estimation in order to take appropriate corrective action. With increasing viability of such distributed systems, the problem of distributed state estimation, namely, estimating the state of a system while sensors/robots (or nodes in general) can only access partial outputs, has been at the receiving end of acute attention over the last decade, see studies by Açıkmeşe et al. (2014), Mitra and Sundaram (2018) for discrete-time observers and by Kim et al. (2016), Wang and Guay (2024) for continuous-time observers. In this paper, we focus on the continuous-time case.

Distributed state estimation for continuous-time linear systems was studied through augmented observer methods by Wang and Morse (2017) and through Luenberger-based observer methods, which arguably have relatively simpler structure, by Kim et al. (2016). Kim et al. (2016) studied Luenberger-based estimation over undirected networks which was subsequently refined by the authors Kim et al. (2019) and Han et al. (2018) to include directed directed networks thus generalizing the nature of network connectivity. Most of the literature mentioned so far focused on static networks.

Studies on switching networks is often motivated by practical phenomena such as selective addition or deletion of links in power grids, limited sensing radius in robotic applications, or malicious attacks on communication channels, see survey by Wen et al. (2020). Naturally, several articles studied the problem of distributed state estimation under switching networks. To begin with, Wang et al. (2020) assumed that the switches (in communication links) happened in a way that ensured strong connectivity of the underlying network at all times. This assumption on the switches was soon relaxed to ensure joint connectivity of the network over a finite time interval by Xu et al. (2021), Zhang et al. (2023), and by Liu and Huang (2024). The aforementioned articles primarily considered undirected networks; more recently, Basu and Sanfelice (2024) and Wang and Guay (2024) studied this problem for switching directed networks.

Although switching networks can address connectivity related concerns, they require the observer nodes to continuously monitor partial outputs. These studies do not allow for mode operations (such as plug-in and engage or remain on standby) at the observer nodes. In this context, we refer to the plug-and-play concept in control systems described by Bendtsen et al. (2011) as a tool to enable addition/deletion of new control laws to existing control laws as and when new sensors/actuators become available. For distributed estimation and control problems studied by Yang et al. (2023) and Kim et al. (2023), plug-and-play design implied that the nodes/agents can join or leave the network at any time during the operation (e.g., stabilization of the process) without resetting the states of other nodes/agents. This is accomplished through fully decentralized design of observer and controller dynamics and with the assumption that the network (connected or strongly connected always) of active controllers or observers involved offers full controllability or observability, respectively, of the overall system at any time. Although the plug-and-play feature offers flexibility and resiliency at the expense of challenging control design, it does not necessarily exploit joint connectivity property offered by switching networks.

⋆ This work is supported by the Swedish Foundation for Strategic Research project.

Statement of Contributions. In this work, we study the problem of continuous-time distributed state estimation where the observer nodes are capable of switching between two modes, namely, active and passive/idle. Active nodes are nodes that can access their partial system outputs and can exchange information with fellow active agents whereas passive nodes propagate their state estimates in an open-loop fashion. It is straightforward to see that this feature of the observer nodes differentiates the problem considered here from the aforementioned studies on distributed observers over switching networks by Xu et al. (2021); Basu and Sanfelice (2024); Wang and Guay (2024). Moreover, in contrast to studies on plug-n-play design by Yang et al. (2023) and Kim et al. (2023) that assume full controllability/observability, in this work, we relax this assumption to exploit joint connectivity and schedule mode changes on the nodes via periodic switching signals. Owing to these features at the observer nodes, and with slight abuse of terminology stemming from time-dependent mode changes, we refer to these observers as distributed *plug-n-play* observers.

2. PRELIMINARIES

Denote $\mathbb{R}$ to be the field of real numbers, $\mathbb{Z}$ to be the set of integers; then $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. Let $|x|$ be the Euclidean norm of an n-dimensional vector $x \in \mathbb{R}^n$ and, for a square matrix $A \in \mathbb{R}^{n \times n}$, let $|A|$ denote its spectral norm. Let $\otimes$ denote the Kronecker product, $\mathbb{I}_n \in \mathbb{R}^{n \times n}$ denote the identity matrix of size $n \times n$ and $\mathbf{1}$ (resp., $\mathbf{0}$) denote either a matrix or a column vector of appropriate size with each entry as 1 (resp., 0). For vectors or matrices A_i, $i \in \{1, \cdots, N\}$, $\mathsf{blkdiag}\{A_1, \cdots, A_N\}$ denotes the block diagonal matrix of appropriate dimension.

2.1 Switched Systems

We present some results (with slight modifications) on switched systems that will be exploited at relevant instances in this work. To this end, consider a linear switched system

$$\dot{x}(t) = A^{\sigma} x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $A^{\sigma}, \forall \sigma \in \mathcal{M} = \{1, \cdots, M\}$, are the state matrices, and $\sigma(t) : [0, \infty) \to \mathcal{M}$ is the switching signal with dwell-times $\tau_k > 0$, $k \geq 0$ (namely, the value that the switching signal σ takes at t_k is held for at least τ_k units of time before the next switching instant t_{k+1}).

Lemma 1. (Xiang and Xiao (2014) [1]) Let $\mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$ denote the set of all switching signals with dwell-times $\tau_k \in [\tau_{\min}, \tau_{\max}]$, $k \geq 0$. For the system in (1), if $\forall \sigma(t) \in \mathcal{M}$, every state matrix A^{σ} is such that at least one of its eigenvalues lies in the right half-plane, then, there exists $\tau_{\min} \in (0, \tau_{\max}]$ for a given $\tau_{\max} > 0$ such that the switched system in (1) is asymptotically stable for any switching signal $\sigma(t) \in \mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$.

Lemma 2. (Sun and Ge (2005)) Let $\sigma(t)$ be a periodic switching signal with period $\mathcal{T}_s > 0$. If the convex combination $\sum_{c=1}^{M} \alpha_c A^c$ is Hurwitz and $\sum_{c=1}^{M} \alpha_c = 1$, $\alpha_c > 0, \forall c \in \mathcal{M}$, then there exists an upper bound $\mathcal{T}_{ub}$ such that for any $\mathcal{T}_s \in (0, \mathcal{T}_{ub}]$, the linear switched system in (1) is exponentially stable.

2.2 Switching Graphs

A switching graph $\mathcal{G}^{\sigma}$ is a pair $\mathcal{G}^{\sigma} = (\mathcal{V}, \mathcal{E}^{\sigma})$ where $\mathcal{V}$ is the index set of nodes and $\mathcal{E}^{\sigma} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between ordered pairs of nodes corresponding to the switching signal σ. Edges of the graph, given by ordered pairs $(i, j) \in \mathcal{E}^{\sigma}$, $\forall i, j \in \mathcal{V}$, $i \neq j$, denote exchange of information from nodes i to nodes j. A switching graph is undirected if at any time t, ordered pair $(i, j) \in \mathcal{E}^{\sigma} \iff (j, i) \in \mathcal{E}^{\sigma}$, i.e., information exchange is bidirectional. Let a non-negative matrix $\mathsf{A}^{\sigma} = [\mathsf{a}_{ij}^{\sigma}]_{N \times N}$ be the weighted adjacency matrix associated with $\mathcal{G}^{\sigma}$ where each positive entry $\mathsf{a}_{ij}^{\sigma} > 0$ corresponds to the undirected edge $(i, j) \in \mathcal{E}^{\sigma}$; otherwise, $\mathsf{a}_{ij}^{\sigma} = 0$, $\forall (i, j) \notin \mathcal{E}$. The weighted degree of a node i is defined as $\mathsf{d}_i^{\sigma} = \sum_{j \in \mathcal{V}} \mathsf{a}_{ij}^{\sigma}$; the corresponding degree matrix $\mathsf{D}^{\sigma} = \mathtt{diag}([\mathsf{d}_1^{\sigma} \cdots \mathsf{d}_N^{\sigma}])$ where $\mathtt{diag}(a)$ builds a diagonal matrix from vector a. A graph Laplacian $\mathsf{L}^{\sigma} \in \mathbb{R}^{N \times N}$ is defined as $\mathsf{L}^{\sigma} := \mathsf{D}^{\sigma} - \mathsf{A}^{\sigma}$ and, by construction, $\mathbf{1}_N$ lies in its null space. An undirected graph/network is connected if there is a path (namely, sequence of edges connecting nodes) between every pair of nodes in $\mathcal{G}^{\sigma}$. See article by Wen et al. (2020) for more information.

3. THE PROBLEM

Consider the following linear system:

$$\begin{aligned} \dot{x} &= Ax \\ y_i &= C_i x, \quad \forall i \in \{1, \cdots, N\}, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the system state, $y = [y_1^{\mathrm{T}}, \cdots, y_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^p$ is the concatenated output, $A \in \mathbb{R}^{n \times n}$ is the state matrix and $C = [C_1^{\mathrm{T}}, \cdots, C_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{p \times n}$ is the output matrix, respectively. In (2), $y_i \in \mathbb{R}^{p_i}$ denotes the component of output y that is accessible to observer node i such that $\sum_{i=1}^{N} p_i = p$ and let $\mathcal{V} = \{1, \cdots, N\}$ denote the index set of all the observer nodes. If $\forall i \in \mathcal{V}$, observer node i can independently estimate the full state x from its accessible partial output y_i, then collaboration among nodes is not necessary for state estimation. In this context, we make the following assumption that compels information exchanges among the observer nodes to accurately estimate x.

Assumption 1. (On joint observability) The pair (C, A) is observable whereas the pair (C_i, A) may not be observable for every $i \in \mathcal{V}$.

In this paper, we consider that every observer node in $\mathcal{V}$ features two modes, namely: i) active mode, i.e., nodes that can access their respective output channels and exchange information with fellow active neighbors, and ii) passive mode, i.e., nodes that propagate state estimates in an open-loop manner. At any time t, let $\sigma(t) : [0, \infty) \to \mathcal{M}$ denote the switching signal that designates/schedules mode changes for all the observer nodes in $\mathcal{V}$ where index set $\mathcal{M} = \{1, \cdots, M\}$, $M \geq 1$. For every $\sigma \in \mathcal{M}$ and at any time t, let $\mathcal{A}^{\sigma} \subseteq \mathcal{V}$ denote the set of all active nodes that communicate among themselves in an undirected manner; this set governs the set of edges $\mathcal{E}^{\sigma} \subseteq \mathcal{A}^{\sigma} \times \mathcal{A}^{\sigma}$ of the resultant switching graph $\mathcal{G}^{\sigma} = (V, \mathcal{E}^{\sigma})$. The

[1] We note that Xiang and Xiao (2014) also offer extensive sufficiency conditions to determine $\tau_{\min}$, $\tau_{\max}$. These will be discussed as a part of proof of Theorem 3. However, to avoid repetitions we refrain from stating these conditions explicitly in Lemma 1.

remainder set, namely, $\mathcal{V} \setminus \mathcal{A}^\sigma$, denotes the set of passive nodes which are isolated. In other words, the switching signal $\sigma(t)$ schedules mode switches in the observers nodes thereby influencing the connectivity within the underlying switching graph $\mathcal{G}^\sigma$.

Remark 1. Briefly, we note that the notion of scheduling these modes for the observers in the network gives rise to both switching in networks and switching in state estimation dynamics (discussed in Section 4.3) unlike existing distributed observers (such as articles by Basu and Sanfelice (2024), Liu and Huang (2024) and others), that are primarily focused on switching in networks.

The objective of this work is to design: i) distributed observer node dynamics that accommodates active and passive modes, and ii) a switching signal $\sigma(t)$ that schedules these modes so that every observer node $i \in \mathcal{V}$ can accurately estimate the state $x(t)$, governed by (2), as $t \to \infty$.

4. MODE-DEPENDENT OBSERVERS

In this section, we introduce some assumptions on switching signals, switching networks, and present the modified Leunberger-based dynamics for observer nodes. Subsequently, for the convenience of analysis, we define observer errors, transformed observer errors, and their switching dynamics.

4.1 Periodic switching signals

To begin with, we make the following two assumptions.

Assumption 2. The switching signal $\sigma(t)$ is periodic with a period $\mathcal{T}_s > 0$, i.e., $\sigma(t + \mathcal{T}_s) = \sigma(t)$.

Under Assumption 2, let the switching signal $\sigma(t)$ with period $\mathcal{T}_s$ be defined as follows:

$$\sigma(t) = \begin{cases} 1 & \bmod (t, \mathcal{T}_s) \in [0, \alpha_1 \mathcal{T}_s) \\ 2 & \bmod (t, \mathcal{T}_s) \in [\alpha_1 \mathcal{T}_s, (\alpha_1 + \alpha_2)\mathcal{T}_s) \\ \vdots & \\ M & \bmod (t, \mathcal{T}_s) \in [\sum_{c=1}^{M-1} \alpha_c \mathcal{T}_s, \mathcal{T}_s) \end{cases} \tag{3}$$

where $\sum_{c=1}^{M} \alpha_c = 1$, for scalars $\alpha_c \geq 0$. Note that $\forall c \in \mathcal{M}$, $\alpha_c T_s$ marks the dwell-time for when signal $\sigma = c$.

Assumption 3. (On joint connectivity) There exists an increasing subsequence $\{i_k\}$ of sequence $\{i | i \in \mathbb{Z}_{\geq 0}\}$ with $t_{i_{k+1}} - t_{i_k} \leq \mathcal{T}_c$ for some positive $\mathcal{T}_c$, where t_i denote switching instants, such that the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \mathcal{G}^{\sigma(t_j)} = \bigcup_{c=1}^{M} \mathcal{G}^c$ is connected.

Together, Assumptions 2 and 3 imply that $t_{i_{k+1}} - t_{i_k} = \mathcal{T}_s \leq \mathcal{T}_c, \forall i \in \{0, 1, \cdots\}$, and the union graph $\bigcup_{c=1}^{M} \mathcal{G}^c = \mathcal{G}$ is *static* and connected. Let L denote the Laplacian matrix associated with $\mathcal{G}$.

Remark 2. (On Assumptions 2 and 3) We note that the state matrix A is assumed to be neutrally stable by Wang et al. (2020), Liu and Huang (2024), Basu and Sanfelice (2024), and Zhang et al. (2023); in these articles, the switching network is assumed to be jointly connected (or jointly strongly connected) but not necessarily static. In contrast, and with the intent of addressing distributed state estimation of systems with general state matrix A, we consider Assumptions 2 and 3 similar to articles by Xu et al. (2021) and Wang and Guay (2024) at the expense of employing jointly connected network $\mathcal{G}$ that is static.

4.2 Observability decomposition

Prior to discussing the observer node dynamics, we introduce the following transformation matrices. Let $T_i \in \mathbb{R}^{n \times n}$ be an orthonormal matrix such that the Kalman decomposition of the pair (C_i, A) at each node i is as follows:

$$T_i^{\mathrm{T}} A T_i = \begin{bmatrix} A_{io} & \mathbf{0} \\ A_{ir} & A_{iu} \end{bmatrix}, \quad C_i T_i = [C_{io} \ \mathbf{0}], \tag{4}$$

where $A_{io} \in \mathbb{R}^{\nu_i \times \nu_i}$ and $C_{io} \in \mathbb{R}^{p_i \times \nu_i}$ denote the state and the output matrices associated with the observable components of the state, respectively, and ν_i denotes the observability index, i.e., the rank of observability matrix of the pair (C_{io}, A_{io}). A_{ir} and A_{iu} are of appropriate dimensions. For every $i \in \mathcal{V}$, we also note that the transformation matrix T_i can be decomposed as $T_i = [T_{io} \ T_{iu}]$ where matrices $T_{io} \in \mathbb{R}^{n \times \nu_i}$ and $T_{iu} \in \mathbb{R}^{n \times n - \nu_i}$ are constructed using the observability matrix of the pair (C_{io}, A_{io}) and its kernel [2].

4.3 Observer node dynamics

Let $\hat{x}_i \in \mathbb{R}^n$ denote the observer state estimate of state x at node i and let $\hat{x} = [\hat{x}_1^{\mathrm{T}} \cdots \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$ denote the concatenated state estimates. As stated in Section 3, the nodes in $\mathcal{V}$ switch between active and passive modes. As a result, the switched observer dynamics at each node $i \in \mathcal{V}$ can be expressed as follows:

$$\dot{\hat{x}}_i = A\hat{x}_i + \delta_i^{\sigma(t)} \Big(L_i(C_i \hat{x}_i - y_i) - \gamma M_i (\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n) \hat{x} \Big),$$

$$\delta_i^{\sigma(t)} = \begin{cases} 1, & i \in \mathcal{A}^\sigma \\ 0, & i \in \mathcal{V} \setminus \mathcal{A}^\sigma \end{cases} \tag{5}$$

where $\mathsf{L}_i^{\sigma(t)}$ is the Laplacian row vector, the observer gain L_i and the gain matrix M_i are considered as per those designed by Kim et al. (2019):

$$L_i = T_i \begin{bmatrix} L_{io} \\ \mathbf{0} \end{bmatrix}, \quad M_i = T_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_{n-\nu_i} \end{bmatrix} T_i^{\mathrm{T}} \tag{6}$$

Here T_i is as per (4) and $L_{io} \in \mathbb{R}^{\nu_i \times p_i}$ is such that $A_{io} + L_{io} C_{io}$ is Hurwitz.

Remark 3. (On observer dynamics in (5)) The scalar $\delta_i^{\sigma(t)}$ in (5) governs the modes associated with each node i. When a node is active, its state estimate propagates along the Luenberger-based observer dynamics employed by Kim et al. (2019). Furthermore, with each switching, the set of active nodes in the network that communicate with each other might change; this is reflected in $\mathsf{L}_i^{\sigma(t)}$. Meanwhile, if a node is passive, its state estimates are propagated along $A\hat{x}_i$ in open-loop; recall that we make no assumptions on the nature of A. This switching between modes introduces additional complexities in analysis compared to existing studies on switching networks.

[2] For further details on transformation matrices, readers can refer to literature on distributed state estimation such as Kim et al. (2016).

Definition 1. A distributed observer achieve omniscience asymptotically if for all initial conditions on (2) and (5), we have

$$\lim_{t\to\infty} |\hat{x}_i(t) - x(t)| = 0, \forall i \in \mathcal{V}. \tag{7}$$

In short, the objective of this work is to guarantee asymptotic omniscience with switched observer dynamics in (5).

4.4 Observer error dynamics

To evaluate the features of the dynamics in (5), we define $e_i = \hat{x}_i - x \in \mathbb{R}^n$ to be the observer error corresponding to node $i \in \mathcal{V}$ and let $e = [e_1^{\mathrm{T}} \cdots e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$ denote the concatenated observer errors. The switched error dynamics for each node i is as follows:

$$\dot{e}_i = Ae_i + \delta_i^{\sigma(t)}\Big(L_iC_ie_i - \gamma M_i(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)e\Big). \tag{8}$$

For the sake of the analysis, let $\tilde{e}_i = [\tilde{e}_{io}^{\mathrm{T}} \;\; \tilde{e}_{iu}^{\mathrm{T}}]^{\mathrm{T}} = T_i^{\mathrm{T}} e_i \in \mathbb{R}^n$ be the transformed observer error where $\tilde{e}_{io} \in \mathbb{R}^{\nu_i}$ and $\tilde{e}_{iu} \in \mathbb{R}^{n-\nu_i}$ are the transformed errors corresponding to the observerable and unobservable parts of x at node i, respectively. Using the definitions of L_i, M_i in (6), the transformed observer error dynamics is as follows:

$$\begin{aligned}
\dot{\tilde{e}}_i &= T_i^{\mathrm{T}}AT_i\tilde{e}_i + \delta_i^{\sigma(t)}\big(T_i^{\mathrm{T}}L_iC_iT_i\tilde{e}_i - \gamma T_i^{\mathrm{T}}M_i(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)e\big),\\
&= \begin{bmatrix} A_{io} & \mathbf{0} \\ A_{ir} & A_{iu}\end{bmatrix}\tilde{e}_i + \delta_i^{\sigma(t)}\begin{bmatrix} L_{io} \\ \mathbf{0}\end{bmatrix}[C_{io}\;\mathbf{0}]\,\tilde{e}_i\\
&\quad - \gamma\delta_i^{\sigma(t)}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_{n-\nu_i}\end{bmatrix} T_i^{\mathrm{T}}(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)e,\\
&= \begin{bmatrix} A_{io} + \delta_i^{\sigma(t)}L_{io}C_{io} & \mathbf{0} \\ A_{ir} & A_{iu}\end{bmatrix}\tilde{e}_i\\
&\quad - \gamma\delta_i^{\sigma(t)}\begin{bmatrix} \mathbf{0} \\ T_{iu}^{\mathrm{T}}\end{bmatrix}(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)T\tilde{e}.
\end{aligned} \tag{9}$$

where $T = \mathsf{blkdiag}\{T_1, \cdots, T_N\}$. From (9), we have

$$\dot{\tilde{e}}_{io} = (A_{io} + \delta_i^{\sigma(t)}L_{io}C_{io})\tilde{e}_{io}, \tag{10a}$$

$$\begin{aligned}\dot{\tilde{e}}_{iu} &= A_{iu}e_{iu} - \gamma\delta_i^{\sigma(t)}T_{iu}^{\mathrm{T}}(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)T_u\tilde{e}_u + A_{ir}\tilde{e}_{io}\\ &\quad - \gamma\delta_i^{\sigma(t)}T_{iu}^{\mathrm{T}}(\mathsf{L}_i^{\sigma(t)} \otimes \mathbb{I}_n)T_o\tilde{e}_o,\end{aligned} \tag{10b}$$

where $T_o = \mathsf{blkdiag}\{T_{o1}, \cdots, T_{oN}\}$, $T_u = \mathsf{blkdiag}\{T_{1u}, \cdots, T_{1N}\}$ and $\tilde{e}_o = [\tilde{e}_{1o}^{\mathrm{T}} \cdots \tilde{e}_{No}^{\mathrm{T}}]^{\mathrm{T}}$ and $\tilde{e}_u = [\tilde{e}_{1u}^{\mathrm{T}} \cdots \tilde{e}_{Nu}^{\mathrm{T}}]^{\mathrm{T}}$ are concatenated observable and unobservable errors, respectively, of appropriate dimensions. Finally, the transformed error dynamics of the distributed system is expressed as follows:

$$\dot{\tilde{e}}_o = (A_o + \delta_o^{\sigma(t)}L_oC_o)\tilde{e}_o, \tag{11a}$$

$$\begin{aligned}\dot{\tilde{e}}_u &= (A_u - \gamma\delta_u^{\sigma(t)}T_u^{\mathrm{T}}(\mathsf{L}^{\sigma(t)} \otimes \mathbb{I}_n)T_u)\tilde{e}_u + (A_r\\ &\quad - \gamma\delta_u^{\sigma(t)}T_u^{\mathrm{T}}(\mathsf{L}^{\sigma(t)} \otimes \mathbb{I}_n)T_o)\tilde{e}_o,\end{aligned} \tag{11b}$$

where $\delta_o^{\sigma(t)} = \mathsf{blkdiag}\{\delta_1^{\sigma(t)} \otimes \mathbb{R}^{\nu_1}, \cdots, \delta_N^{\sigma(t)} \otimes \mathbb{R}^{\nu_N}\}$, $\delta_u^{\sigma(t)} = \mathsf{blkdiag}\{\delta_1^{\sigma(t)} \otimes \mathbb{R}^{n-\nu_1}, \cdots, \delta_N^{\sigma(t)} \otimes \mathbb{R}^{n-\nu_N}\}$, $L_o = \mathsf{blkdiag}\{L_{1o}, \cdots, L_{No}\}$, $C_o = \mathsf{blkdiag}\{C_{1o}, \cdots, C_{No}\}$, similarly A_o, A_u and A_r are $\mathsf{blkdiag}$ matrices constructed from A_{io}, A_{iu} and A_{ir} respectively.

Remark 4. (On switches in (11)) The transformed error dynamics in (11) distinctly separates the nature of switches in (11a) and (11b). While (11a) is dependent on switching observer gains through the term $\delta_o^{\sigma(t)}L_oC_o$, (11b) is dependent on switching network $\mathcal{G}^\sigma$. With switching observer gains, the switched system in (11a) toggles between various configurations plausibly involving unstable modes and requires switching signals that guarantee stability. On the other hand, the dynamics with switching networks in (11b) is widely studied under joint network connectivity assumptions such as Assumption 3.

5. MAIN RESULTS

Now we present the main result of our paper that addresses the stability of the transformed error system in (11) thus guaranteeing asymptotic omniscience.

Theorem 3. (Stabilization of $(\tilde{e}_o, \tilde{e}_u)$) Let Assumptions 1, 2, and 3 hold. If a periodic switching signal $\sigma(t)$ in (3) satisfies the following conditions:

(1) $\sigma(t) \in \mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$ stabilizes (11a),
(2) $\sigma(t)$ with period $\mathcal{T}_s$ stabilizes

$$\dot{\tilde{e}}_u = (A_u - \gamma\delta_u^{\sigma(t)}T_u^{\mathrm{T}}(\mathsf{L}^{\sigma(t)} \otimes \mathbb{I}_n)T_u)\tilde{e}_u, \text{ and} \tag{12}$$

(3) scalars $\alpha_c, \forall c \in \mathcal{M}$, are such that

$$\tau_{\min}/\mathcal{T}_s \le \alpha_c \le \tau_{\max}/\mathcal{T}_s, \tag{13}$$

then there exists a sufficiently large $\gamma > 2|A|/\lambda_2(\mathsf{L})$ such that the transformed error dynamics in (11) under $\sigma(t)$ is asymptotically stable where $\lambda_2(\mathsf{L})$ denotes the second largest eigenvalue of L.

Proof. Stability of the transformed observer error system described by the dynamics in (11) can be parsed into the following components:

(1) Asymptotic stability of the switched system in (11a);
(2) Exponential stability of the switched system in (12);
(3) Compatibility of $\sigma(t)$ for (11a) and (12);
(4) Asymptotic stability of the system in (11b).

► *Analysis of the switched system in* (11a)*:*
Here, we exploit Theorem 2 by Xiang and Xiao (2014) (partially recapped in Lemma 1 in this work). Let $\bar{A}_o^\sigma = A_o + \delta_o^{\sigma(t)}L_oC_o$, then the switched system in (11a) is

$$\dot{\tilde{e}}_o = \bar{A}_o^\sigma\tilde{e}_o. \tag{14}$$

For system $\dot{\tilde{e}}_o = \bar{A}_o^i\tilde{e}_o$, $\forall i \in \mathcal{M}$, consider the following discretized Lyapunov function as per Xiang and Xiao (2014)

$$\bar{V}_i(t) = \begin{cases} e_o^{\mathrm{T}}P_i^{(q)}(\pi)e_o, & t \in [t_k + qh, t_k + (q+1)h), \\ e_o^{\mathrm{T}}P_{i,D}e_o, & t \in [t_k + \tau_{\min}, t_{k+1}), \end{cases} \tag{15}$$

where linear interpolation $P_i^{(q)}(\pi) = (1-\pi)P_{i,q} + \pi P_{i,q+1} = P_i(t)$, $q \in \{0, \cdots, D-1\}$, $\pi = (t - t_k - qh)/h$, $h = \tau_{\min}/D$ and $P_{i,q} \succ 0, \forall q \in \{0, \cdots, D-1\}$, are matrices to-be-designed. To elaborate, the Lyapunov function $\bar{V}_i(t)$ in (15) is a time-varying function over the interval $[t_k, t_{k+1}]$ whose design is split in the following fashion: i) over the interval $[t_k, t_k+\tau_{\min})$ that is divided into D segments each of length h, $P_i^{(q)}(\pi)$ denotes the time-varying matrix function for each segment (indexed by $\forall q \in \{0, \cdots, D-1\}$), and ii) over the interval $[t_k + \tau_{\min}, t_{k+1})$, $P_i(t)$ adopts a constant matrix $P_{i,D}$. The design of these matrices is described by the following result adopted from Xiang and Xiao (2014).

Lemma 4. Consider the switched system in (14). If there exists a set of matrices $P_{i,q} \succ \mathbb{I}$, $\forall q \in \{0, \cdots, D-1\}$,

$\forall i \in \mathcal{M}$, such that for a fixed $1 > \mu > 0$, $\tau_{\max}$, $\forall q \in \{0, \cdots, D-1\}$, $\forall i, j \in \mathcal{M}$,

$$(\bar{A}_o^i)^{\mathrm{T}} P_{i,q} + P_{i,q}^{\mathrm{T}}(\bar{A}_o^i) + \Psi_i^{(q)} - \eta \mathbb{I} \prec \mathbf{0}, \tag{16a}$$
$$(\bar{A}_o^i)^{\mathrm{T}} P_{i,q+1} + P_{i,q+1}^{\mathrm{T}}(\bar{A}_o^i) + \Psi_i^{(q)} - \eta \mathbb{I} \prec \mathbf{0}, \tag{16b}$$
$$(\bar{A}_o^i)^{\mathrm{T}} P_{i,D} + P_{i,D}^{\mathrm{T}}(\bar{A}_o^i) - \eta \mathbb{I} \prec \mathbf{0}, \tag{16c}$$
$$P_{j,0} - \mu P_{i,L} \preceq \mathbf{0}, \qquad i \neq j \tag{16d}$$
$$\ln \mu + \eta \tau_{\max} < 0 \tag{16e}$$

where $\Psi_i^{(q)} = D(P_{i,q+1} - P_{i,q})/\tau_{\min}$. Then, the system in (14) is asymptotically stable, i.e., $\tilde{e}_o \to \mathbf{0}$, under any switching law $\sigma(t) \in \mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$ where $\tau_{\min} \in [\tau_{\min}^*, \tau_{\max}]$

$$\tau_{\min}^* = \min_{\tau_{\min} < \tau_{\max}} \{\tau_{\min} \mid \text{conditions in (16) hold}\}. \tag{17}$$

Proof. Refer to the article by Xiang and Xiao (2014).

Here, it is worth noting that the system in (14) can be stabilized, by carefully tuning parameter μ, for any given $\tau_{\max}$ and corresponding $\tau_{\min}^*$ as per (17).

This addresses item (1) in Theorem 3.

► *Analysis of the unforced system in* (12)*:*
For the switching signal in (3), under Assumptions 2 and 3, we have $\sum_{c=1}^{M} \alpha_c \mathsf{L}^c = \mathsf{L}$. This implies that any convex combination of matrices

$$\sum_{c=1}^{M} \alpha_c (A_u - \gamma \delta_u^c T_u^{\mathrm{T}} (\mathsf{L}^c \otimes \mathbb{I}_n) T_u) = A_u - \gamma T_u^{\mathrm{T}} (\mathsf{L} \otimes \mathbb{I}_n) T_u \tag{18}$$

results in a stable (i.e., Hurwitz) matrix for sufficiently large $\gamma > 2|A|/\lambda_2(\mathsf{L})$ (details shown by Wang and Guay (2024)). Therefore, under Lemma 2, there exists $\mathcal{T}_{ub} > 0$ and a periodic signal $\sigma(t)$ with period $\mathcal{T}_s \in (0, \mathcal{T}_{ub}]$, such that the switched system in (12) is exponentially stable.

This addresses item (2) in Theorem 3.

► *Designing $\sigma(t)$ for simultaneous stabilization:*
As seen above, stabilizing the two switched systems in (14) and (12) resulted in different criteria on the switching signal $\sigma(t)$. In order to ensure simultaneous stabilization of these two subsystems, for a given periodic switching signal in (3) with period $\mathcal{T}_s$ that stabilizes (12), we require the following to hold

$$\tau_{\min} \leq \alpha_c \mathcal{T}_s \leq \tau_{\max}, \quad \forall c \in \mathcal{M}, \tag{19}$$

where $\tau_{\min}$ is determined through (17) for any given $\tau_{\max} \geq \max_c \alpha_c \mathcal{T}_s, \forall c \in \mathcal{M}$. A simple solution for $\alpha_c, c \in \mathcal{M}$, that satisfies conditions in (19) is as follows: let $\alpha_c = 1/M, \forall c \in \mathcal{M}$, then $\tau_{\min} \leq \mathcal{T}_s/M \leq \tau_{\max}$. If $\tau_{\max} = \mathcal{T}_s/M$, then there exists some $\sigma(t)$, with $\tau_{\min}^* \leq \tau_{\min} \leq \tau_{\max}$ satisfying (17), that stabilizes both subsystems (14) and (12) simultaneously. More general solutions for scalars $\alpha_c, c \in \mathcal{M}$, satisfying (13) can be obtained as deviations of the simple solution described here.

This addresses item (3) in Theorem 3.

► *Analysis of the perturbed system in* (11b)*:*
Finally, note that the system in (11b) can be viewed as a perturbed system of (12) with vanishing inputs, i.e., $(A_r - \gamma T_u^{\mathrm{T}} \mathsf{L}^{\sigma(t)} T_o)\tilde{e}_o$. The system in (11b) is input-to-state stable since (11b) is exponentially stable, as noted in Lemma 4.6 by Khalil (2002). This implies that $|\tilde{e}_u| \leq \theta(|\tilde{e}_u(0)|, t - t_0) + \psi(\sup_{t \geq t_0} |w(t)|)$ where $|w(t)| = |(A_r - \gamma T_u^{\mathrm{T}} \mathsf{L}^{\sigma(t)} T_o)||\tilde{e}_o|$, $\theta(\cdot, \cdot)$ and $\psi(\cdot)$ are class $\mathcal{KL}$ function and class $\mathcal{K}$ function, respectively. Here, note that $|(A_r - \gamma T_u^{\mathrm{T}} \mathsf{L}^{\sigma(t)} T_o)|$ is bounded from both above and below. Now, consider an arbitrary constant $\varepsilon > 0$. Since $\psi(\cdot) \in \mathcal{K}$, for some $\epsilon > 0$, $\psi(\epsilon) < \varepsilon$, and there exists some $t > \bar{t}$ such that $|\tilde{e}_o(t)| \leq \epsilon$. Similarly, since $\theta(\cdot, \cdot) \in \mathcal{KL}$, $\theta(|\tilde{e}_u(0)|, t - t_0) < \varepsilon$ for some $t > \hat{t}$. Combining the two arguments, for some $t > \max\{\bar{t}, \hat{t}\}$, $|\tilde{e}_u(t)| \leq 2\varepsilon$. This implies that $\tilde{e}_u \to \mathbf{0}$ asymptotically.

This concludes the proof. ■

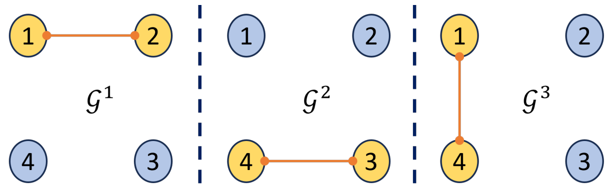


Fig. 1. Observer nodes switch between modes as shown in $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}^3$. Nodes in yellow are active and those in blue are passive.

Remark 5. (On stability analysis of (11)) It is fair to expect that the conditions on switching signal $\sigma(t)$ for stabilization of the two subsystems in (11) might be different (as described in items (1) and (2) in Theorem 3). Consequently, item (3) in Theorem 3 combines the two criteria and addresses this concern through conditions on α_c in (13) to ensure simultaneous stabilization of both $\tilde{e}_o$ and $\tilde{e}_u$ subsystems. Furthermore, the feasibility of (13) is discussed in the proof of Theorem 3.

6. ILLUSTRATIVE EXAMPLE

Consider a set of 4 observer nodes that are employed to estimate the state of system given by:

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} x, \tag{20}$$

and the outputs corresponding to each observer node are as follows:

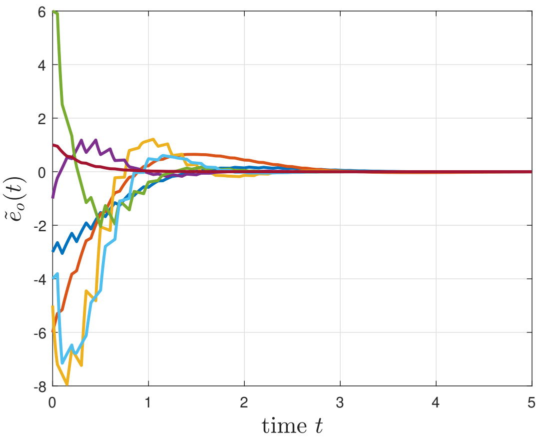
$$y^{\mathrm{T}} = [y_1 \ y_2 \ y_3 \ y_4]^{\mathrm{T}} = \mathbb{I}_4 x. \tag{21}$$

Note that the state matrix A has eigenvalues with positive real parts. The transformation matrices $T_i, \forall i \in \mathcal{V}$, for the 4 observer nodes are as follows:

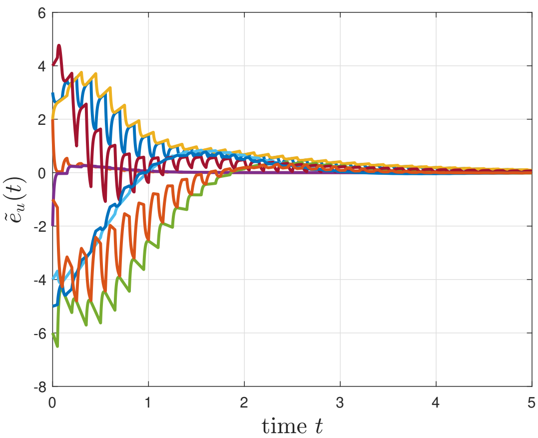
$$T_1 = T_2 = \mathbb{I}_4, \ T_3 = \begin{bmatrix} \mathbf{0} & \mathbb{I}_2 \\ \mathbb{I}_2 & \mathbf{0} \end{bmatrix}, \ T_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

Fig. 1 captures the set of active and passive nodes in each disconnected graphs $\mathcal{G}^1$, $\mathcal{G}^2$, and $\mathcal{G}^3$, whose union graph is connected and static. Note also that no pair of connected (i.e., active) observer nodes in Fig. 1 offer observability. The switching signal $\sigma(t)$, defined in (3), has a period $\mathcal{T}_s = 0.15$ and $\alpha_c = 1/3, \forall c \in \{1, 2, 3\}$, and satisfies conditions in Theorem 3 with $\tau_{\min} = \tau_{\max} = 0.05$ (see Algorithm 1 by Xiang and Xiao (2014) for details on computation of $\tau_{\min}$). In (5), $\gamma = 40$ and L_{io} in (6) are chosen as follows:

$$L_{1o} = \begin{bmatrix} -5.5 \\ -4 \end{bmatrix}, \ L_{2o} = \begin{bmatrix} -65 \\ -16 \end{bmatrix}, \ L_{3o} = \begin{bmatrix} -15 \\ -18 \end{bmatrix}, \ L_{4o} = -4$$



(a) Trajectory of $\tilde{e}_o \in \mathbb{R}^7$



(b) Trajectory of $\tilde{e}_u \in \mathbb{R}^9$

Fig. 2. Trajectories of transformed observer error $\tilde{e} \in \mathbb{R}^{16}$.

Finally, figures 2a and 2b demonstrate asymptotic stabilization of transformed observer errors to the origin.

7. CONCLUSION

In this paper, we addressed the problem of distributed state estimation where observer nodes feature two modes: active and passive. Specifically, we considered a Luenberger-based switching dynamics for state estimation at each observer node in the network and designed a switching signal that schedules mode switching in the observer nodes, and by extension, switching in the underlying communication network. We exploit Kalman decomposition and transform the problem of omniscience to the problem of stabilization of two distinct switched subsystems in transformed observer errors. As a result, we establish conditions on the periodic switching signals that guarantee asymptotic omniscience of the designed mode-dependent plug-and-play observer dynamics.

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