Robust Connectivity Analysis for Multi-Agent Systems

Dimitris Boskos and Dimos V. Dimarogonas

***Abstract*— In this paper we provide a decentralized robust control approach, which guarantees that connectivity of a multi-agent network is maintained when certain bounded input terms are added to the control strategy. Our main motivation for this framework is to determine abstractions for multi-agent systems under coupled constraints which are further exploited for the synthesis of high level plans.**

I. Introduction

Cooperative control of multi-agent systems constitutes a highly active area of research during the last two decades. Typical objectives are the consensus problem, which is concerned with finding a protocol that achieves convergence to a common value [10], reference tracking [1] and formation control [7]. Consensus algorithms have also been extended to robust frameworks, in order to provide convergence in the presence of input disturbances [11]. A common feature in the approach to the latter problems is the design of decentralized control laws in order to achieve a global goal.

In the case of mobile robot networks with limited sensing and communication ranges, connectivity maintenance plays a fundamental role [16]. In particular, it is required to constrain the control input in such a way that the network topology remains connected during the evolution of the system. For instance, in [7] the rendezvous and formation control problems are studied while preserving connectivity, whereas in [4] swarm aggregation is achieved by means of a control scheme that guarantees both connectivity and collision avoidance.

In this framework we provide a control law for each agent comprising of a decentralized feedback component and a free input term, which ensures connectivity maintenance, for all possible free input signals up to a certain bound of magnitude. The motivation for this approach comes from distributed control and coordination of multi-agent systems with locally assigned Linear Temporal Logic (LTL) specifications. In particular, by virtue of the invariance and robust connectivity maintenance properties, it is possible to define well posed decentralized abstractions for the multi-agent system which can be exploited for motion planning. The latter problem has been studied in our recent work [2] for the single integrator dynamics case.

In this work, we design a bounded control law which results in network connectivity of the system for all future times provided that the initial relative distances of interconnected agents and the free input terms satisfy appropriate bounds. Furthermore, in the case of a spherical domain, it is shown that adding an extra repulsive vector field near the boundary of the domain can also guarantee invariance of the solutions and simultaneously maintain the robust connectivity property. The latter framework enables the construction of finite abstractions for the single integrator case. Due to the motivation for the study of the problem and to the best of our knowledge, this is the first attempt to consider robust connectivity maintenance in conjunction with invariance, by means of bounded control laws.

The rest of the paper is organized as follows. Section II introduces basic notation and preliminaries. In Section III, results on robust connectivity maintenance are provided and explicit controllers which establish this property are designed. In Section IV, the corresponding controllers are appropriately modified, in order to additionally guarantee invariance of the solution for the case of a spherical domain. We summarize the results and discuss possible extensions in Section V.

II. Preliminaries and Notation

Notation. We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{m\times n}$ we use the notation $|A| := \max\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$ for the induced Euclidean matrix norm and A^T for its transpose. For two vectors $x, y \in \mathbb{R}^n (= \mathbb{R}^{n\times 1})$ we denote their inner product by $\langle x, y\rangle := x^T y$. Given a subset S of $\mathbb{R}^n$, we denote by $\mathrm{cl}(S)$, $\mathrm{int}(S)$ and ∂S its closure, interior and boundary, respectively, where $\partial S := \mathrm{cl}(S) \setminus \mathrm{int}(S)$. For $R > 0$, we denote by $B(R)$ the closed ball with center $0 \in \mathbb{R}^n$ and radius R. Given a vector $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ we define the component operators $c_l(x) := x^l$, $l = 1, \ldots, n$. Likewise, for a vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn}$ we define the component operators $c_l(x) := (c_l(x_1), \ldots, c_l(x_N)) \in \mathbb{R}^N$, $l = 1, \ldots, n$.

Consider a multi-agent system with N agents. For each agent $i \in \{1, \ldots, N\} := \mathcal{N}$ we use the notation $\mathcal{N}_i$ for the set of its neighbors and $|\mathcal{N}_i|$ for its cardinality. We also consider an ordering of the agent's neighbors which we denote by $j_1, \ldots, j_{|\mathcal{N}_i|}$. $\mathcal{E}$ stands for the undirected network's edge set and $\{i, j\} \in \mathcal{E}$ iff $j \in \mathcal{N}_i$. The network graph $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ is connected if for each $i, j \in \mathcal{N}$ there exist $i_1, \ldots, i_l \in \mathcal{N}$ with $i_1 = i$, $i_l = j$ and $\{i_k, i_{k+1}\} \in \mathcal{E}$, for all $k = 1, \ldots, l-1$. Consider an arbitrary orientation of the network graph $\mathcal{G}$, which assigns to each edge $\{i, j\} \in \mathcal{E}$ precisely one of the ordered pairs (i, j) or (j, i). When selecting the pair (i, j) we say that i is the tail and j is the head of

The authors are with the ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden and with the KTH Centre for Autonomous Systems. boskos@kth.se, dimos@kth.se. This work was supported by the EU STREP RECONFIG: FP7-ICT-2011-9-600825, the H2020 ERC Starting Grant BUCOPHSYS and the Swedish Research Council (VR).

edge $\{i, j\}$. By considering a numbering $l = 1, \dots, M$ of the graph's edge set we define the $N \times M$ incidence matrix $D(\mathcal{G})$ corresponding to the particular orientation as follows:

$$D(\mathcal{G})_{kl} := \begin{cases} 1, & \text{if vertex } k \text{ is the head of edge } l \\ -1, & \text{if vertex } k \text{ is the tail of edge } l \\ 0, & \text{otherwise} \end{cases}$$

The graph Laplacian $L(\mathcal{G})$ is the $N \times N$ positive semidefinite symmetric matrix $L(\mathcal{G}) := D(\mathcal{G}) \times D(\mathcal{G})^T$. If we denote by $\mathbb{1}$ the vector $(1, \dots, 1) \in \mathbb{R}^N$, then $L(\mathcal{G})\mathbb{1} = D(\mathcal{G})^T \mathbb{1} = 0$. Let $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$ be the ordered eigenvalues of $L(\mathcal{G})$. Then each corresponding set of eigenvectors is orthogonal and $\lambda_2(\mathcal{G}) > 0$ iff $\mathcal{G}$ is connected.

Problem Statement. We focus on single integrator multi-agent systems with dynamics

$$\dot{x}_i = u_i, x_i \in \mathbb{R}^n, i = 1, \dots, N \tag{1}$$

We aim at designing decentralized control laws of the form

$$u_i := k_i(x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}) + v_i \tag{2}$$

which ensure that appropriate apriori bounds on the initial relative distances of interconnected agents guarantee network connectivity for all future times, for all free inputs v_i bounded by certain constant. In particular, *we assume that the agents' interaction graph is static and connected, and that the network remains connected, as long as the maximum distance between two neighboring agents does not exceed a given positive constant R*. In addition, we make the following connectivity hypothesis for the initial states of the agents.

(ICH) We assume that the agents' network is initially connected. In particular, there exists a constant $\tilde{R} \in (0, R)$ with

$$\max\{|x_i(0) - x_j(0)| : \{i, j\} \in \mathcal{E}\} \leq \tilde{R} \tag{3}$$

Potential Functions. We proceed by defining certain mappings which are exploited in order to design the control law (2) and prove that network connectivity is maintained. Let $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous function satisfying the following property.

(P) $r(\cdot)$ is increasing and $r(0) > 0$.

Also, consider the integral

$$P(\rho) = \int_0^\rho r(s) s ds, \rho \in \mathbb{R}_{\geq 0} \tag{4}$$

For each pair $(i, j) \in \mathcal{N} \times \mathcal{N}$ with $\{i, j\} \in \mathcal{E}$ we define the potential function $V_{ij} : \mathbb{R}^{Nn} \to \mathbb{R}_{\geq 0}$ as $V_{ij}(x) = P(|x_i - x_j|), \forall x = (x_1, \dots, x_N) \in \mathbb{R}^{Nn}$. Notice that $V_{ij}(\cdot) = V_{ji}(\cdot)$. Furthermore, it can be shown that $V_{ij}(\cdot)$ is continuously differentiable and that

$$\frac{\partial}{\partial x_i} V_{ij}(x) = r(|x_i - x_j|)(x_i - x_j)^T \tag{5}$$

where $\frac{\partial}{\partial x_i}$ stands for the derivative with respect to the x_i-coordinates.

III. Connectivity Analysis

In the following proposition we provide a control law (2) and an upper bound on the magnitude of the input terms $v_i(\cdot)$ which guarantee connectivity of the multi-agent network.

Proposition 3.1: For the multi agent system (1), assume that (ICH) is fulfilled and define the control law

$$u_i = -\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i \tag{6}$$

for certain continuous $r(\cdot)$ satisfying Property (P). Also, consider a constant $\delta > 0$ and define

$$K := \frac{2\sqrt{N(N-1)}|D(\mathcal{G})^T|}{\lambda_2(\mathcal{G})^2} \tag{7}$$

where $D(\mathcal{G})$ is the incidence matrix of the systems' graph and $\lambda_2(\mathcal{G})$ the second eigenvalue of the graph Laplacian. We assume that the positive constant δ, the maximum initial distance $\tilde{R}$ and the function $r(\cdot)$ satisfy the restrictions

$$\delta \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R} \tag{8}$$

with K as given in (7) and

$$M P(\tilde{R}) \leq P(R) \tag{9}$$

where $P(\cdot)$ is given in (4), and $M = |\mathcal{E}|$ is the cardinality of the system's graph edge set. Then, the system remains connected for all positive times, provided that the input terms $v_i(\cdot)$, $i = 1, \dots, N$ satisfy

$$|v_i(t)| \leq \delta, \forall t \geq 0 \tag{10}$$

Proof: For the proof we follow parts of the analysis in [7] (see also [9, Section 7.2]). Consider the energy function

$$V := \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} V_{ij} \tag{11}$$

where the mappings V_{ij}, $\{i, j\} \in \mathcal{E}$ are given in Section II. Then it follows from (5) that

$$\frac{\partial}{\partial x_i} V(x) = \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j)^T \tag{12}$$

Also, in accordance with [9, Section 7.2] we have for $l = 1, \dots, n$ that

$$c_l \left(\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) \right) = L_w(x) c_l(x) \tag{13}$$

The weighted Laplacian matrix $L_w(x)$ is given as

$$L_w(x) = D(\mathcal{G}) W(x) D(\mathcal{G})^T \tag{14}$$

where $D(\mathcal{G})$ is the incidence matrix of the communication graph (see Notation) and

$$\begin{aligned} W(x) :=& \mathrm{diag}\{w_1(x), \dots, w_M(x)\} \\ :=& \mathrm{diag}\{r(|x_i - x_j|), \{i, j\} \in \mathcal{E}\} \end{aligned} \tag{15}$$

(recall that $M = |\mathcal{E}|$). Then, by evaluating the time derivative of V along the trajectories of (1)-(6) and taking into account (11), (12) and (13) we get

$$\begin{aligned}\dot{V} &= \sum_{l=1}^{n} c_l \left(\frac{\partial}{\partial x} V(x)\right) c_l(\dot{x}) \\ &= -\sum_{l=1}^{n} c_l(x)^T L_w(x)(L_w(x)c_l(x) - c_l(v)) \\ &\leq -\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right|\end{aligned} \tag{16}$$

We want to provide appropriate bounds for the right hand side of (16) which can guarantee that the sign of $\dot{V}$ is negative whenever the maximum distance between two agents exceeds the bound $\tilde{R}$ on the maximum initial distance as given in (3). First, we provide certain useful inequalities between the eigenvalues of the weighted Laplacian $L_w(x)$ and the Laplacian matrix of the graph $L(\mathcal{G})$. Notice, that due to (15), for each $i = 1, \ldots, M$ we have $w_i(x) = r(|x_k - x_\ell|)$ for certain $\{k, \ell\} \in \mathcal{E}$ and hence, by virtue of Property (P), it holds

$$0 < r(0) \leq w_i(x) \leq \max_{\{k,\ell\} \in \mathcal{E}} r(|x_k - x_\ell|) \tag{17}$$

From (17), it follows that $L_w(x)$ has precisely the same properties with those provided for $L(\mathcal{G})$ in the Notation subsection. Furthermore, it holds

$$\lambda_2(x) \geq \lambda_2(\mathcal{G}) r(0) \tag{18}$$

where $0 = \lambda_1(x) < \lambda_2(x) \leq \cdots \leq \lambda_N(x)$ and $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$ are the eigenvalues of $L_w(x)$ and the Laplacian matrix of the graph $L(\mathcal{G})$, respectively. Indeed, in order to show (18), notice that $L_w(x) = D(\mathcal{G})\mathrm{diag}\{w_1(x), \ldots, w_M(x)\}D(\mathcal{G})^T = D(\mathcal{G})\mathrm{diag}\{r(0), \ldots, r(0)\}D(\mathcal{G})^T + D(\mathcal{G})\mathrm{diag}\{w_1(x) - r(0), \ldots, w_M(x) - r(0)\}D(\mathcal{G})^T = r(0)L(\mathcal{G}) + B$, where (17) implies that $B := D(\mathcal{G})\mathrm{diag}\{w_1(x) - r(0), \ldots, w_M(x) - r(0)\}D(\mathcal{G})^T$ is positive semidefinite. Hence, it holds $L_w(x) \succeq r(0)L(\mathcal{G})$, with $\succeq$ being the partial order on the set of symmetric $N \times N$ matrices and thus, we deduce from Corollary 7.7.4(c) in [5, page 495] that (18) is fulfilled.

In the sequel we introduce some additional notation. Let H be the subspace $H := \{x \in \mathbb{R}^{Nn} : x_1 = x_2 = \cdots = x_N\}$. For a vector $x \in \mathbb{R}^{Nn}$ we denote by $\bar{x}$ its projection to the subspace H, and $x^\perp$ its orthogonal complement with respect to that subspace, namely $x^\perp := x - \bar{x}$. By taking into account that for all $y \in H$ we have $D(\mathcal{G})^T c_l(y) = 0$ and hence, due to (14), that $c_l(y) \in \ker(L_w(x))$, it follows that for every vector $x \in \mathbb{R}^{Nn}$ with $x = \bar{x} + x^\perp$ it holds

$$L_w(x)c_l(x) = L_w(x)c_l(x^\perp) \tag{19}$$

We also denote by $\Delta x \in \mathbb{R}^{Mn}$ the stack column vector of the vectors $x_i - x_j, \{i, j\} \in \mathcal{E}$ with the edges ordered as in the case of the incidence matrix. It is thus straightforward to check that for all $x \in \mathbb{R}^{Nn}$

$$D(\mathcal{G})^T c_l(x) = c_l(\Delta x), \forall l = 1, \ldots, n \tag{20}$$

and furthermore, due to (17), that

$$|W(x)| \leq r(|\Delta x|_\infty) \tag{21}$$

where $|\Delta x|_\infty := \max\{|\Delta x_i|, i = 1, \ldots, M\}$. Before proceeding we state the following elementary facts, whose proofs can be found in the Appendix of [3]. In particular, for the vectors $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathbb{R}^{Nn}$ the following properties hold.

Fact I: $|L_w(x)c_l(x^\perp)| \geq \lambda_2(x)|c_l(x^\perp)|, \forall l = 1, \ldots, n.$

Fact II: $\sum_{l=1}^{n} |c_l(x)||c_l(y)| \leq |x||y|.$

Fact III: $|x^\perp| \geq \frac{1}{\sqrt{2(N-1)}}|\Delta x|.$

Fact IV: $\sqrt{2}|x^\perp| \geq |\Delta x|_\infty.$

We are now in position to bound the derivative of the energy function V and exploit the result in order to prove the desired connectivity maintenance property. We break the subsequent proof in two main steps.

Step 1: Bound estimation for the rhs of (16)**.**

Bound for the first term in (16)**.** By taking into account (19), it follows that

$$\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) = \sum_{l=1}^{n} \left|L_w(x)c_l(x^\perp)\right|^2 \tag{22}$$

and by exploiting Fact I and (18), we get

$$\begin{aligned}\sum_{l=1}^{n} \left|L_w(x)c_l(x^\perp)\right|^2 &\geq \sum_{l=1}^{n} \lambda_2(x)^2 |c_l(x^\perp)|^2 \\ &\geq \sum_{l=1}^{n} [\lambda_2(\mathcal{G})r(0)]^2 |c_l(x^\perp)|^2 = [\lambda_2(\mathcal{G})r(0)]^2 |x^\perp|^2\end{aligned} \tag{23}$$

Thus, it follows from (22) and (23) that

$$\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) \geq [\lambda_2(\mathcal{G})r(0)]^2 |x^\perp|^2 \tag{24}$$

Bound for the second term in (16)**.** For this term, we have from (14) and (20) that

$$\begin{aligned}&\left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right| \\ &\leq \sum_{l=1}^{n} |c_l(x)^T D(\mathcal{G}) W(x) D(\mathcal{G})^T c_l(v)| \\ &\leq \sum_{l=1}^{n} |c_l(\Delta x)||W(x)||D(\mathcal{G})^T||c_l(v)|\end{aligned} \tag{25}$$

By taking into account (21), we obtain

$$\begin{aligned}&\sum_{l=1}^{n} |c_l(\Delta x)||W(x)||D(\mathcal{G})^T||c_l(v)| \\ &\qquad \leq \sum_{l=1}^{n} |c_l(\Delta x)| r(|\Delta x|_\infty)|D(\mathcal{G})^T||c_l(v)|\end{aligned} \tag{26}$$

Also, by exploiting Fact II, we get that

$$\begin{aligned}&\sum_{l=1}^{n} |c_l(\Delta x)| r(|\Delta x|_\infty)|D(\mathcal{G})^T||c_l(v)| \\ &\qquad \leq r(|\Delta x|_\infty)|D(\mathcal{G})^T||\Delta x||v| \\ &\qquad \leq r(|\Delta x|_\infty)|D(\mathcal{G})^T||\Delta x|\sqrt{N}|v|_\infty\end{aligned} \tag{27}$$

where $|v|_\infty := \max\{|v_i|, i = 1, \ldots, N\}$. Hence, it follows from (25)-(27) that

$$\left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right| \leq \sqrt{N}|D(\mathcal{G})^T||\Delta x| r(|\Delta x|_\infty)|v|_\infty \tag{28}$$

Thus, we get from (16), (24) and (28) that

$$\dot{V} \leq -[\lambda_2(\mathcal{G})r(0)]^2|x^\perp|^2 + \sqrt{N}|D(\mathcal{G})^T||\Delta x| r(|\Delta x|_\infty)|v|_\infty$$

and by exploiting Facts III and IV, that

$$\begin{aligned}\dot{V} &\leq -[\lambda_2(\mathcal{G})r(0)]^2 \frac{1}{\sqrt{2(N-1)}}|\Delta x| \frac{1}{\sqrt{2}}|\Delta x|_\infty \\ &\quad + \sqrt{N}|D(\mathcal{G})^T||\Delta x| r(|\Delta x|_\infty)|v|_\infty \\ &= |\Delta x| \left(-\frac{1}{2\sqrt{N-1}}[\lambda_2(\mathcal{G})r(0)]^2|\Delta x|_\infty \right. \\ &\quad \left. + \sqrt{N}|D(\mathcal{G})^T| r(|\Delta x|_\infty)|v|_\infty \right)\end{aligned}$$

By using the notation $|\Delta x|_\infty := s$, in order to guarantee that the above rhs is negative for $s \geq \tilde{R}$, it should hold $\frac{2\sqrt{N(N-1)}|D(\mathcal{G})^T|}{\lambda_2(\mathcal{G})^2}|v|_\infty \leq r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R}$, or equivalently

$$|v|_\infty \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R} \tag{29}$$

with K as given in (7). Hence, we have shown that for v satisfying (29) the following implication holds

$$|\Delta x|_\infty \geq \tilde{R} \Rightarrow \dot{V} \leq 0 \tag{30}$$

Step 2: Proof of connectivity. By assuming that conditions (10), (8) and (9) in the statement of the proposition are fulfilled and recalling that according to (ICH) (3) holds, we can show that the system will remain connected for all future times. Indeed, let $x(\cdot)$ be the solution of the closed loop system (1)-(6) with initial condition satisfying (3), defined on the maximal right interval $[0, T_{\max})$. We claim that the system remains connected on $[0, T_{\max})$, namely, that $\max\{|x_i(t) - x_j(t)| : \{i, j\} \in \mathcal{E}\} \leq R$ for all $t \in [0, T_{\max})$, which by boundedness of the dynamics on the set $\mathcal{F} := \{x \in \mathbb{R}^{Nn} : |x_i - x_j| \leq R, \forall \{i, j\} \in \mathcal{E}\}$ implies that $T_{\max} = \infty$. We proceed with the proof of connectivity. First, notice that due to (3) and (9), it holds

$$V(x(0)) \leq \frac{1}{2}\sum_{i=1}^{N}\sum_{j \in \mathcal{N}_i} P(\tilde{R}) = \frac{M}{2}P(\tilde{R}) \leq \frac{1}{2}P(R) \tag{31}$$

In order to prove our claim, it suffices to show that

$$V(x(t)) \leq \frac{1}{2}P(R), \forall t \in [0, T_{\max}) \tag{32}$$

because if $|x_i(t) - x_j(t)| > R$ for certain $t \in [0, T_{\max})$ and $\{i, j\} \in \mathcal{E}$, then $V(x(t)) \geq \frac{1}{2}P(|x_i(t) - x_j(t)|) > \frac{1}{2}P(R)$. We prove (32) by contradiction. Indeed, suppose on the contrary that there exists $T \in (0, T_{\max})$ (due to (31)) such that

$$V(x(T)) > \frac{1}{2}P(R) \tag{33}$$

and define

$$\tau := \min\{t \in [0, T] : V(x(\bar{t})) > \tfrac{1}{2}P(R), \forall \bar{t} \in (t, T]\} \tag{34}$$

which due to (33) and continuity of $V(x(\cdot))$ is well defined. Then it follows from (31) and (34) that

$$V(x(\tau)) = \frac{1}{2}P(R), V(x(t)) > \frac{1}{2}P(R), \forall t \in (\tau, T] \tag{35}$$

hence, there exists $\bar{\tau} \in (\tau, T)$ such that

$$\dot{V}(x(\bar{\tau})) = \frac{V(x(T)) - V(x(\tau))}{T - \tau} > 0 \tag{36}$$

On the other hand, due to (35), it holds

$$V(x(\bar{\tau})) > \frac{1}{2}P(R) \tag{37}$$

which implies that there exists $\{i, j\} \in \mathcal{E}$ with

$$|x_i(\bar{\tau}) - x_j(\bar{\tau})| > \tilde{R} \tag{38}$$

Indeed, if (38) does not hold, then we can show as in (31) that $V(x(\bar{\tau})) \leq \frac{1}{2}P(R)$ which contradicts (37). Notice that by virtue of (10) and (8), (29) is fulfilled. Hence, we get from (38) that $|\Delta x(\bar{\tau})|_\infty > \tilde{R}$ and thus from (30) it follows that $\dot{V}(x(\bar{\tau})) \leq 0$, which contradicts (36). We conclude that (32) holds and the proof is complete. ∎

In the following corollary, we apply the result of Proposition 3.1 in order to provide two explicit feedback laws of the form (6), a linear and a nonlinear one and compare their performance in the subsequent remark.

Corollary 3.2: For the multi agent system (1), assume that (ICH) is fulfilled and consider the control law (2) as given by (6). By imposing the additional requirement $r(0) = r(\tilde{R}) = 1$ and defining $\delta := \frac{\tilde{R}}{K}$, with $\tilde{R}$ and K as given in (3) and (7), respectively, the system remains connected for all positive times, provided that the function $r(\cdot)$ and the constant $\tilde{R}$ are selected as in the following two cases (L) and (NL) (providing a linear and a nonlinear feedback, respectively).

Case (L). We select $r(s) := 1$, $s \geq 0$ and $\tilde{R} \leq \frac{1}{\sqrt{M}}R$ (recall that $M = |\mathcal{E}|$).

Case (NL). We select $r(s) := \begin{cases} 1, & s \in [0, \tilde{R}] \\ \frac{s}{\tilde{R}}, & s \in (\tilde{R}, R] \\ \frac{R}{\tilde{R}}, & s \in (R, \infty) \end{cases}$ and $\tilde{R} \leq \left(\frac{2}{3M-1}\right)^{\frac{1}{3}} R$.

Proof: The proof is rather straightforward and therefore omitted. However it can be found in [3]. ∎

Remark 3.3: At this point we derive the advantage of using the nonlinear controller over the linear one by comparing the ratio of the maximal initial relative distance that maintains connectivity for these two cases. In both cases we have the same bound on the free input terms and the same feedback law up to some distance between neighboring agents, which allows us to compare their performance under the criterion of maximizing the largest initial distance between two interconnected agents. In particular, this ratio, which depends on the number of edges in the systems' graph, is given by $Rat(M) := \frac{1}{\sqrt{M}} / \left(\frac{2}{3M-1}\right)^{\frac{1}{3}}$. It is then rather straightforward to show that $Rat(\cdot)$ is a strictly decreasing function of M with values less than 1 for $M \geq 1$.

IV. Invariance Analysis

In what follows, we assume that the agents' initial states belong to a given domain $D \subset \mathbb{R}^n$. In order to simplify the subsequent analysis, we assume that $D = \mathrm{int}(B(\mathfrak{R}))$, namely the interior of the ball with center $0 \in \mathbb{R}^n$ and radius $\mathfrak{R} > 0$. We aim at designing an appropriate modification of the feedback law (6) which guarantees that the trajectories of the agents remain in D for all future times.

For each $\varepsilon \in (0, \mathfrak{R})$, let N_ε be the region with distance ε from the boundary of D towards the interior of D, namely

$$N_\varepsilon := \{x \in \mathbb{R}^n : \mathfrak{R} - \varepsilon \le |x| < \mathfrak{R}\} \tag{39}$$

and

$$D_\varepsilon := D \setminus N_\varepsilon \tag{40}$$

We proceed by defining a repulsive from the boundary of D vector field, which when added to the dynamics of each agent in (6), will ensure the desired invariance of the closed loop system and simultaneously guarantee the same robust connectivity result established above. Let $h : [0,1] \to [0,1]$ be a Lipschitz continuous function that satisfies

$$h(0) = 0;\ h(1) = 1;\ h(\cdot) \text{ strictly increasing} \tag{41}$$

We define the vector field $g : D \to \mathbb{R}^n$ as

$$g(x) := \begin{cases} -c\delta h\left(\frac{\varepsilon + |x| - \mathfrak{R}}{\varepsilon}\right)\frac{x}{|x|}, & \text{if } x \in N_\varepsilon \\ 0, & \text{if } x \in D_\varepsilon \end{cases} \tag{42}$$

with $h(\cdot)$ as given above and appropriate positive constants c, δ which serve as design parameters. Then, it follows from (41), (42) and the Lipschitz property for $h(\cdot)$ that the vector field $g(\cdot)$ is Lipschitz continuous on D.

Having defined the mappings for the extra term in the dynamics of the modified controller which will guarantee the desired invariance property, we now state our main result.

Proposition 4.1: For the multi-agent system (1), assume that $D = \mathrm{int}(B(\mathfrak{R}))$, for certain $\mathfrak{R} > 0$ and that (ICH) is fulfilled. Furthermore, let $\varepsilon \in (0, \mathfrak{R})$, N_ε and D_ε as defined by (39) and (40), respectively and assume that the initial states of all agents lie in D_ε. Then, there exists a control law (2) (with free inputs v_i) which guarantees both connectivity and invariance of D for the solution of the system for all future times and is defined as

$$u_i = g(x_i) - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i \tag{43}$$

with $g(\cdot)$ given in (42) and certain $r(\cdot)$ satisfying Property (P). We choose the same positive constant δ in both (10) and (42) and select the constant c in (42) greater that 1. Then the connectivity-invariance result is valid provided that the parameters δ, $\tilde{R}$ and the function $r(\cdot)$ satisfy the restrictions (8), (9) and the input terms $v_i(\cdot)$, $i = 1, \ldots, N$ satisfy (10).

Proof: We break the proof in two steps. In the first step, we show that as long as the invariance assumption is satisfied, namely, the solution of the closed loop system (1)-(43) is defined and remains in D, network connectivity is maintained. In the second step, we show that for all times where the solution is defined, it remains inside a compact subset of D, which implies that the solution is defined and remains in D for all future times, thus providing the desired invariance property.

Step 1: Proof of network connectivity. The proof of this step is based on an appropriate modification of the corresponding proof of Proposition 3.1. In particular, we exploit the energy function V as given by (11) and show that when $|\Delta x|_\infty \ge \tilde{R}$, namely, when the maximum distance between two agents exceeds $\tilde{R}$ then its derivative along the solutions of the closed loop system is negative. Thus by using the same arguments with those in proof of Proposition 3.1 we can deduce that the system remains connected. Indeed, by evaluating the derivative of V along the solutions of (1)-(43) we obtain

$$\begin{aligned} \dot{V} \le & \sum_{i=1}^{N} \frac{\partial}{\partial x_i} V(x) g(x_i) - \sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) \\ & + \left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right| \end{aligned} \tag{44}$$

By taking into account (16) and using precisely the same arguments with those in proof of Steps 1 and 2 of Proposition 3.1 it suffices to show that the first term of inequality (44), which by virtue of (12) is equal to $\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)\langle (x_i - x_j), g(x_i) \rangle$, is nonpositive for all $x \in D$. Given the partition D_ε, N_ε of D, we consider for each agent $i \in \mathcal{N}$ the partition $\mathcal{N}_i^{D_\varepsilon}$, $\mathcal{N}_i^{N_\varepsilon}$ of its neighbors' set, corresponding to its neighbors that belong to D_ε and N_ε, respectively. Also, we denote by $\mathcal{E}^{N_\varepsilon}$ the set of edges $\{i, j\}$ with both $x_i, x_j \in N_\varepsilon$. Then, by taking into account that due to (42), $g(x_i) = 0$ for $x_i \in D_\varepsilon$, it follows that

$$\begin{aligned} & \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)\langle (x_i - x_j), g(x_i) \rangle \\ = & \sum_{\{i \in \mathcal{N} : x_i \in N_\varepsilon\}} \sum_{j \in \mathcal{N}_i^{D_\varepsilon}} r(|x_i - x_j|)\langle (x_i - x_j), g(x_i) \rangle \\ + & \sum_{\{i,j\} \in \mathcal{E}^{N_\varepsilon}} r(|x_i - x_j|)[\langle (x_i - x_j), g(x_i) \rangle \\ & \qquad + \langle (x_j - x_i), g(x_j) \rangle] \end{aligned} \tag{45}$$

In order to prove that both terms in (45) are less than or equal to zero and hence derive our desired result on the sign of $\dot{V}$, we exploit the following facts.

Fact V. Consider the vectors $\alpha, \beta, \gamma \in \mathbb{R}^n$ with the following properties:

$$|\alpha| = 1,\ |\beta| = 1 \tag{46}$$

$$\langle \alpha, \gamma \rangle \ge 0,\ \langle \beta, \gamma \rangle \le 0 \tag{47}$$

Then for every quadruple $\lambda_\alpha, \lambda_\beta, \mu_\alpha, \mu_\beta \in \mathbb{R}_{\ge 0}$ satisfying

$$\lambda_\alpha \ge \lambda_\beta, \mu_\alpha \ge \mu_\beta \tag{48}$$

it holds

$$\langle (\mu_\alpha \alpha - \mu_\beta \beta), \tilde{\delta} \rangle \ge 0 \tag{49}$$

where

$$\tilde{\delta} := \lambda_\alpha \alpha + \gamma - \lambda_\beta \beta \tag{50}$$

The proof of Fact V can be found in the Appendix of [3].

Fact VI. For any $x, \tilde{x} \in N_\varepsilon$ with $x = \lambda\tilde{x}$, $\lambda > 0$ and $y \in \mathrm{cl}(D_\varepsilon)$ it holds $\langle(\tilde{x} - y), x\rangle \geq 0$.

The proof of Fact VI is based on the elementary properties $y \in \mathrm{cl}(D_\varepsilon) \Rightarrow |y| \leq \mathfrak{R} - \varepsilon$ and $\tilde{x} \in N_\varepsilon \Rightarrow \mathfrak{R} - \varepsilon \leq |\tilde{x}|$. Hence we have that $\langle(\tilde{x} - y), x\rangle \geq |x||\tilde{x}| - |x||y| \geq 0$.

We are now in position to show that both terms in the right hand side of (45) are nonpositive, which according to our previous discussion establishes the desired connectivity maintenance result.

Proof of the fact that the first term in (45) **is nonpositive.** For each i, j in the first term in (45) we get by applying Fact VI with $x, \tilde{x} = x_i \in N_\varepsilon$ and $y = x_j \in D_\varepsilon$ that $r(|x_i - x_j|)\langle(x_i - x_j), g(x_i)\rangle = r(|x_i - x_j|)\frac{-c\delta h\left(\frac{\varepsilon + |x_i| - \mathfrak{R}}{\varepsilon}\right)}{|x_i|}\langle(x_i - x_j), x_i\rangle \leq 0$ and hence, that the first term is nonpositive.

Proof of the fact that the second term in (45) **is nonpositive.** We exploit Fact V in order to prove that for each $\{i, j\} \in \mathcal{E}^{N_\varepsilon}$ the quantity

$$\langle(x_i - x_j), g(x_i)\rangle + \langle(x_j - x_i), g(x_j)\rangle \tag{51}$$

in the second term of (45) is nonpositive as well. Notice that both $x_i, x_j \in N_\varepsilon$ and without loss of generality we may assume that

$$|x_i| \geq |x_j| \tag{52}$$

namely, that x_i is farther from the boundary of D_ε than x_j. Then by setting

$$\alpha := \frac{x_i}{|x_i|};\ \beta := \frac{x_j}{|x_j|};\ \gamma := \tilde{x}_i - \tilde{x}_j \tag{53}$$

with

$$\tilde{x}_i := x_i - (|x_i| + \varepsilon - \mathfrak{R})\frac{x_i}{|x_i|} \tag{54}$$

$$\tilde{x}_j := x_j - (|x_j| + \varepsilon - \mathfrak{R})\frac{x_j}{|x_j|} \tag{55}$$

$$\lambda_\alpha := |x_i| + \varepsilon - \mathfrak{R};\ \lambda_\beta := |x_j| + \varepsilon - \mathfrak{R} \tag{56}$$

$$\mu_\alpha := c\delta h\left(|x_i| + \varepsilon - \mathfrak{R}\right);\ \mu_\beta := c\delta h\left(|x_j| + \varepsilon - \mathfrak{R}\right) \tag{57}$$

it follows from (53) that $|\alpha| = |\beta| = 1$ and from (41), (52), (56) and (57) that $\lambda_\alpha \geq \lambda_\beta \geq 0, \mu_\alpha \geq \mu_\beta \geq 0$. Furthermore, we get from (54) and (55) that $|\tilde{x}_i| = |\tilde{x}_j| = \mathfrak{R} - \varepsilon \Rightarrow \tilde{x}_i, \tilde{x}_j \in \partial D_\varepsilon$. Thus it follows from (53) and application of Fact VI with $x = x_i$, $\tilde{x} = \tilde{x}_i$ and $y = \tilde{x}_j$ that $\langle\alpha, \gamma\rangle \geq 0$ and similarly that $\langle\beta, \gamma\rangle \leq 0$.

It follows that all requirements of Fact V are fulfilled. Furthermore, by taking into account (53)-(56), we get that $\tilde{\delta} = \lambda_\alpha\alpha + \gamma - \lambda_\beta\beta = x_i - x_j$. Thus we establish by virtue of (42), (49), (50), (53), (57) and the latter that $\langle(\mu_\alpha\alpha - \mu_\beta\beta), \tilde{\delta}\rangle = -\langle(g(x_i) - g(x_j)), (x_i - x_j)\rangle \geq 0 \iff \langle(x_i - x_j), g(x_i)\rangle + \langle(x_j - x_i), g(x_j)\rangle \leq 0$, as desired.

Step 2: Proof of forward invariance of D with respect to the solution of (1)-(43)**.** Due to space limitations, the proof of this step is omitted. However, it can be found in [3]. ■

V. Conclusions

We have provided a distributed control scheme which guarantees connectivity of a multi-agent network governed by single integrator dynamics. The corresponding control law is robust with respect to additional free input terms which can further be exploited for motion planning. For the case of a spherical domain, adding a repulsive vector field near the boundary ensures that the agents remain inside the domain for all future times. The latter framework is motivated by the fact that it allows us to abstract the behaviour of the system through a finite transition system and exploit formal method tools for high level planning.

Further research directions include the generalization of the invariance result of Section IV for the case where the domain is convex and has smooth boundary and the improvement of the bound on the free input terms, by allowing the bound to be state dependent.

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