Asymmetric Collaborative Bar Stabilization Tethered to Two Heterogeneous Aerial Vehicles

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Abstract—We consider a system composed of a bar tethered to two unmanned aerial vehicles (UAVs), where the cables behave as rigid links under tensile forces, and with the control objective of stabilizing the bar’s pose around a desired pose. Each UAV is equipped with a PID control law, and we verify that the bar’s motion is decomposable into three decoupled motions, namely a longitudinal, a lateral and a vertical. We then provide relations between the UAVs’ gains, which, if satisfied, allows us to decompose each of those motions into two cascaded motions; the latter relations between the UAVs’ gains are found so as to counteract the system asymmetries, such as the different cable lengths and the different UAVs’ weights. Finally, we provide conditions, based on the system’s physical parameters, that describe good and bad types of asymmetries. We present experiments that demonstrate the stabilization of the bar’s pose.

I. INTRODUCTION

Aerial vehicles provide a platform for transportation of cargos in dangerous and cluttered environments [1]. In particular, vertical take off and landing rotorcrafts, with hovering capabilities, have been used to validate different types of transportation and manipulation of objects.

Tethered transportation, when compared with manipulator-endowed transportation, is mechanically simple and inexpensive. Several control strategies for slung-load transportation, i.e., tethered transportation of a point mass load by a single UAV, are found in the literature. The swing angle of the load can be estimated, either inferred from vision or from the internal force exerted by the load on the UAV, and used in the feedback loop to avoid/dampen swing excitation [2], [3]. Trajectory planning that minimizes the loads’ swing, and exploiting differential flatness for control purposes has also been demonstrated [4]–[6].

Cooperative transportation with multiple UAVs is also found in the literature. Vision has been used to correctly place end-effectors with respect to a visual target placed on the object to be transported [7], [8], or to autonomously estimate the bar’s pose [9]. Motion planning for collision avoidance between the cargo and the UAVs with obstacles in a cluttered environment has also been studied and validated [10]–[12]. How to position a group of UAVs by specifying the desired tension on the cables and a desired pose for the tethered object is found in [13], [14]. Note that tethered transportation with multiple UAVs comes with multiple degrees of freedom, which have been explored so as to minimize the internal forces applied of the load [15], [16]. There are different grasping mechanisms in aerial transportation, such as, adhesive/gripping mechanisms at the tool-tip that stick to the grasped object [17]; a hook-based system between the end-effector and the point to grasp [12]; and magnets, electromagnets and electropermanent magnets [18] – the latter is the option we adopt.

In this manuscript, we focus on stabilization of a rod-like object tethered to two AUVs, as pictured in Fig. 1. This problem has also been considered in [9], [19], [20]. In [19], a master-slave approach for the two UAVs is put in place, with the slave UAV estimating the cable force exerted on itself. In [9], vision is used to autonomously estimate the bar’s pose. In [20], relations on the UAVs’ PID gains are provided for which stability – regarding the bar’s pose stabilization – is guaranteed.

Regarding experimental cooperative transportation, experiments have been performed where the system is taken to be symmetric [9], [9], [12], [19], [20]. In this work, we extend [20], and consider an asymmetric system, with non-identical UAVs and different cable lengths. We perform an analysis similar to that in [21]–[23], where we linearize the system, and derive conditions on the gains that guarantee exponential stability regarding the stabilization of the bar’s pose. We verify that the bar’s motion is decomposable into three decoupled motions, namely a longitudinal, a lateral and a vertical; and that if UAVs’ gains satisfy specific relations, each of those motions is in turn decomposable into two cascaded motions: e.g., the vertical motion is decomposable into the vertical linear motion and the vertical angular motion of the bar, with the latter cascaded after the former, if the vertical PID gains of the two UAVs satisfy a specific ratio. Finally, we provide conditions, based on the system’s physical parameters, that describe good and bad types of asymmetries, which may be explored to design safer experiments.

The remainder of this paper is structured as follows. In

Fig. 1: Tethered transportation of a rod-like object by two heterogeneous aerial vehicles, with different cable lengths.
For brevity, given the quantities described above, denote

\[ z := (p, n, p_1, p_2, v_1, \omega, v_2, r_1, r_2, \xi_1, \xi_2) \in \mathbb{R}^{32}, \]

\[ u := (u_1, u_2) \in \mathbb{R}^6, \]

where \( z \) and \( u \) are used next as the state and input, respectively, of the UAVs-bar system. Consider then the state space

\[ Z := \{z \in \mathbb{R}^{32} : f(z) = 0\}, \]

where the map \( f \) defined above encapsulates the constraints illustrated in Fig. 2. Specifically, the first two constraints in (3) imply that the bar’s attitude \( n \) is given by a unit vector and that the bar’s angular velocity \( \omega \) is orthogonal to that unit vector; the next four constraints in (3), imply that the distance between each contact point on the bar and the corresponding UAV is constant and equal to the corresponding cable length; and, finally, the last two constraints in (3), imply that the UAVs’ thrust vectors are also given by unit vectors.

Given an appropriate input \( u : \mathbb{R}_{\geq 0} \to \mathbb{R}^6 \), a system’s trajectory \( z : \mathbb{R}_{\geq 0} \ni t \mapsto z(t) \in Z \) evolves according to

\[ \dot{z}(t) = Z(z(t), u(t)), \quad z(0) \in Z, \]

where the vector field \( Z : \mathbb{R}^n \ni (z, u) \mapsto Z(z, u) \in \mathbb{R}^{32} \) is given by

\[ Z(z, u) := \begin{bmatrix} Z_k(z, u) \\ Z_d(z, u) \\ Z_i(z, u) \\ Z_e(z, u) \end{bmatrix} = \begin{bmatrix} [\text{kinematics} & \text{dynamics} & \text{attitude inner loop} & \text{integrator dynamics}] \end{bmatrix}, \]

with the kinematics given by

\[ Z_k(z) := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \]

with the dynamics given by (below, \( g \) stands for the acceleration due to gravity; \( T_1, T_2 \) stand for the tensions on the cables, which are functions of the state and the input; and \( n_i \equiv \frac{n_i - (p + d_i) l_i}{l_i} \) stands for the cable \( i \in \{1, 2\} \) direction)

\[ Z_d(z, u) := \begin{bmatrix} \sum_{i \in \{1, 2\}} T_i(z, u) n_i - g e_3 \\ \sum_{i \in \{1, 2\}} T_i(z, u) S d_i n_i \\ \sum_{i \in \{1, 2\}} T_i(z, u) a_i n_i - g e_3 \\ \sum_{i \in \{1, 2\}} T_i(z, u) m_i - T_i(z, u) a_i n_i - g e_3 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{u_1} \\ \dot{u_2} \end{bmatrix}, \]

\[ \bar{u} := (\bar{u}_1, \bar{u}_2) \equiv (u_1^T, \bar{r}_i^T, u_2^T, \bar{r}_2^T, r_2^T, r_2), \]

with the attitude inner loop dynamics given by (below, \( k_i^1, k_i^2 \) stand for the positive gains of the UAVs’ attitude inner loop)

\[ Z_i(z, u) := \begin{bmatrix} \mathcal{S}(k_i^1 S(r_1, u_1)) r_1 \\ \mathcal{S}(k_i^2 S(r_2, u_2)) r_2 \end{bmatrix} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix}, \]

Sections III and IV, the model of the system and the control law are described. In Section V, we present conditions under which matrices of interest are Hurwitz. In Section VI, we linearize the closed loop vector field around the equilibrium and, under a proper similarity transformation and a proper choice of gains, we verify the longitudinal, lateral and vertical motion decomposition. Finally, in Section VII, we present illustrative experimental results.

II. NOTATION

The map \( S : \mathbb{R}^3 \ni x \mapsto S(x) \in \mathbb{R}^{3 \times 3} \) yields a skew-symmetric matrix and it satisfies \( S(a \cdot b) = a \times b \), for any \( a, b \in \mathbb{R}^3 \). \( S^2 := \{ x \in \mathbb{R}^3 : x^2 x = 1 \} \) denotes the set of unit vectors in \( \mathbb{R}^3 \). We denote \( A_1 \oplus \cdots \oplus A_n \) as the block diagonal matrix with block diagonal entries \( A_i \) to \( A_n \) (square matrices). We denote by \( e_1, \cdots, e_n \in \mathbb{R}^n \) the canonical basis vectors in \( \mathbb{R}^n \), for some \( n \in \mathbb{N} \). Given some \( n, m \in \mathbb{N} \), and a function \( f : \mathbb{R}^n \ni a \mapsto f(a) \in \mathbb{R}^m \), \( f' : \mathbb{R}^n \ni a \mapsto f'(a) \in \mathbb{R}^{m \times n} \) denotes the derivative of \( f \).

III. PROBLEM DESCRIPTION

Consider the system illustrated in Fig. 2, with two VTOL aerial vehicles, a one dimensional bar and two cables connecting the aerial vehicles to distinct contact points on the bar. Fig. 2 provides a two-dimensional picture of the real system, as shown in Fig 1, but the modeling we describe next is three dimensional. Hereafter, and for brevity, we refer to this system as UAVs-bar system. We denote by \( p_1, p_2, p \in \mathbb{R}^3 \) and by \( v_1, v_2, v \in \mathbb{R}^3 \) the UAVs’ and the bar’s center of mass positions and velocities, by \( n_i, \omega_i \in \mathbb{R}^3 \) the bar’s orientation and angular velocity; by \( r_1, r_2 \in \mathbb{R}^3 \) the UAVs’ thrust body directions; and by \( \xi_1, \xi_2 \in \mathbb{R}^3 \) the vertical integral error of each UAV. As for physical constants, we denote by \( m_1, m_2, m > 0 \) the UAVs’ and bar’s masses; by \( J > 0 \) the bar’s moment of inertia (w.r.t. the bar’s center of mass); by \( l_1, l_2 > 0 \) the cables’ lengths; and, finally, by \( d_1, d_2 \in \mathbb{R}^3 \) the contact points on the bar at which the cables are attached to. Finally, we denote by \( u_1, u_2 \in \mathbb{R}^3 \) the input forces on the UAVs-bar system: for \( i \in \{1, 2\} \),

\[ u_i := U_i r_i := u_i^T r_i, \]

is the UAV \( i \) input force, where the throttle \( U_i \) is taken as the inner product between the input \( u_i \) and the UAV’s thrust body direction (one may think of \( u_i \) as the desired value for \( u_i \)).
and finally, with the integrator dynamics given by
\[ Z_i(z, u) := \begin{bmatrix} e_i^T p_i - l_i \\ e_i^T p_i - l_i \end{bmatrix} \left( \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} \right). \]

An important relation to note is that
\[ f'(z) Z_i(z, u) = 0, \text{ for all } (z, u) \in Z \times \mathbb{R}^6, \]
which implies that a solution of (4) starting in \( Z \), remains in \( Z \) (i.e., that \( Z \) is invariant).

Let us provide some details on the vector field (5). The dynamics equations are obtained from the Newton-Euler’s equations of motion, considering the net force and torque on each rigid body: the bar is taken as a rigid body (with net force and torque in blue – see Fig. 2); while the UAVs are taken as point masses (with net forces in orange and green – see Fig. 2). The Newton-Euler’s equations of motion do not provide any insight into the tensions \( T_i \) and \( T_2 \) since these constitute internal forces: the tensions are found by requiring the dynamics to belong to the state space tangent set. The explicit description on the tensions is found in [24], but it is omitted here for brevity. The attitude inner loop dynamics in (6) corresponds to a first order model with attitude gain \( k_i > 0 \), that guarantees that the UAV i \( \in \{1,2\} \) thrust vector tries to align itself with a rigid body (of the input force \( u \), for a constant \( u_i \in \mathbb{R}^3 \setminus \{0\} \), a solution of \( \dot{r}_i = -k_i S(r_i) S(r_i) \dot{u} \) converges exponentially fast to \( \dot{u} = 0 \), with rate proportional to \( k_i \). Note that the model for the UAVs’ attitude inner loop in (6) is only a possible one, and there are more ways of modeling that inner loop.

Let us define the equilibrium, before explaining the integrator dynamics in (7). For any \( (\xi_1^*, \xi_2^*) \in \mathbb{R}^2 \), define
\[ z^* := (p^*, n^*, p_1^*, p_2^*, u^*, w^*, t^*, r_1^*, r_2^*, \xi_1^*, \xi_2^*) \in Z \]
\[ := (0, c_i, d_i e_i, 1, e_3, d_i e_i, 0, 0, 0, 0, 0, 0, 0, 0, e_3, \xi_1^*, \xi_2^*) \]
and \( u^* := (u_1^*, u_2^*) \in \mathbb{R}^6 \) as
\[ u^* := \left( \begin{bmatrix} m_1 + \frac{md_2}{d_1 - d_2} \\ m_2 + \frac{md_1}{d_1 - d_2} \end{bmatrix} \right) e_3. \]

Since it can be verified that \( Z(z^*, u^*) = 0_{2x2} \), it follows that \( z^* \) (under a constant input \( u^* \)) is an equilibrium of the system. Thus, the integral terms \( (\xi_1, \xi_2) \) evolving according to the integrator dynamics in (7) represent the vertical-position integral error of the UAVs. These integral errors are used in the control law, and provide robustness against disturbances and model uncertainties, as shall be verified in the experiments.

We can now formulate the problem treated in this paper.

**Problem 1:** Given the vector field \( Z \) in (5) and the equilibrium \( z^* \) in (9) (for some \( (\xi_1^*, \xi_2^*) \in \mathbb{R}^2 \)), design a control law \( w^* : Z \rightarrow \mathbb{R}^6 \) satisfying \( w^*(z^*) = u^* \) and such that \( z^* \) is an exponentially stable equilibrium of the closed loop vector field \( z \rightarrow Z(z, w^*(z)) \).

**Remark 1:** In general, we may require the bar to stabilize around any point \( p^* \in \mathbb{R}^3 \) and any attitude \( n^* \in S^2 \) with \( e_n^T n^* = 0 \). For that purpose, if suffices to place the origin of the inertial frame at \( p^* \), and align the inertial x-axis with \( n^* \): i.e., stabilizing around the pose \( (p^*, n^*) \) is reduced to stabilizing around the pose \( (0, e_1) \) as indicated in the equilibrium (9).

**Definition 1:** We say the system is symmetric if
\[ m_1 = m_2 = M, l_1 = l_2 = l, d_1 = -d_2 = d. \]

**IV. CONTROL LAW**

For each aerial vehicle – \( j \in \{1,2\} \) – consider the PID-like control law \( u^*_{pd} : Z \ni z \rightarrow w^*_{pd}(z) \in \mathbb{R}^6 \) defined as
\[ u^*_{pd}(z) := (u_{pd,1}(z), u_{pd,2}(z), u_{pd,3}(z)), \]
where
\[ u_{pd,1}(z) := -m_j (k_{pd,p} e_1^T (p_j - p^*_j) + k_{d,e} e_1^T v_j), \]
\[ u_{pd,2}(z) := -m_j (k_{pd,v} e_2^T (p_j - p^*_j) + k_{d,v} e_2^T v_j) - \cdots \]
\[ \cdots - m_j d_j l_j (k_{p,v} e_1^T n + k_{d,v} e_2^T \omega), \]
\[ u_{pd,3}(z) := -m_j (k_{pd,\psi} e_3^T (p_j - p^*_j) + k_{d,\psi} e_3^T v_j + k_{d,\psi} \xi_j), \]
where \( p^*_j, p_j^* \) are the UAVs' equilibrium positions given in (9); where, for \( i \in \{x,y,z,\psi\}, k_{pd,i} \) and \( k_{d,i} \) are positive gains related to the position and velocity feedback, respectively, of vehicle \( j \in \{1,2\} \) and the bar’s yaw attitude; and where \( k_{d,\psi} \) is a positive gain related to the integral feedback of vehicle \( j \in \{1,2\} \).

**Remark 2:** The real control law is subject to saturations [24], which are of practical importance. Since these saturations do not interfere with the analysis we perform in the next sections, we omit them here for brevity.

Let us provide some insight into the control law (12) (and recall that we wish to align the bar with the inertial \( x \)-direction, i.e., \( n^* = e_1 \)). The control law along the \( x \)-direction \( (u_{pd,1}) \) is composed of two terms, one proportional and one derivative that try to bring the UAV to its desired \( x \) position: this control law will only influence the \( x \) linear motion of the bar, and the \( x \) linear motion between the UAVs. The control law along the \( y \)-direction \( (u_{pd,2}) \) is composed of four terms: one proportional and one derivative that try to bring the UAV to its desired \( y \) position; and one proportional and one derivative that try to bring the bar to its desired \( y \) angular position: this control law will only influence the \( y \) linear motion of the bar, as well as the \( y \) angular motion of the bar (the yaw motion). Finally, the control law along the \( z \)-direction \( (u_{pd,3}) \) is composed of three terms: a proportional, a derivative and an integral that try to bring the UAV to its desired \( z \) position: this control law will only influence the \( z \) linear motion of the bar, as well as the \( z \) angular motion of the bar (the pitch motion). Given the equilibrium input defined in (10), the complete control law is then defined as
\[ Z \ni z \rightarrow w^*(z) := u^* |_{m=m_1} + (w^*_{pd}(z), u_{pd,3}(z)) \in \mathbb{R}^6, \]
where \( \hat{m} \) is the mass of the bar as known by the controller: e.g., if the the bar’s weight is known, then \( \hat{m} := m \), and if the bar’s weight is unknown, then \( \hat{m} := 0 \). It then follows that for (see Problem 1 and (10))
\[ (\xi_1^*, \xi_2^*) := \left( \begin{bmatrix} \frac{m-n}{m_1-d_1} \\ \frac{m-n}{m_1-d_1} \end{bmatrix}, \frac{(m-n)d_2}{m_1(d_2-d_1)} \right) \Rightarrow w^*(z^*) = u^* \]
where we emphasize that if the bar’s mass is known, then the equilibrium integral errors vanish, i.e., \( (\xi_1^*, \xi_2^*) = (0,0) \); however, even if the bar’s weight is known, having an
integral action is still of practical importance as it provides robustness against other types of model uncertainties. In the next sections, we study the stability of the equilibrium \( z^* \) (with \( (\xi^*_i, \xi^*_i) \) as in (13)) of the closed loop vector field
\[
z \mapsto Z^c(z) := Z(z, u^c(z)). \tag{14}
\]

V. ROUTH’S CRITERION

In Section VI, we linearize the closed loop vector field \( Z^c \) in (14) around the equilibrium \( z^* \) in (9), and we verify that the Jacobian is similar to a block triangular matrix, whose block diagonal entries are in controllable form. This section provides tools for the analysis of the eigenvalues of those matrices in controllable form. Denote then, for any \( a \in \mathbb{R}^n \), \( C_n(a) := \begin{bmatrix} e_2 & \cdots & e_n & -a \end{bmatrix}^T \in \mathbb{R}^{n \times n} \), as a matrix in controllable form. It follows from the Routh’s criterion that
\[
C_3((a_0, a_1, a_2)) \text{ is Hurwitz } \iff a_0, a_1, a_2 > 0 \land a_0 < a_1 a_2, \tag{15}
\]
which we make use of later on. In what follows, denote \( q \in \mathbb{R}, f := (f_3, f_4) \in (\mathbb{R}^r)^2, k := (k_1, k_2) \in (\mathbb{R}^r)^2 \), where, in later sections, \( f \) and \( q \) provide physical constants of interest, and \( k \) provides the controller gains (in particular a proportional and a derivative gain). There are two matrices (in controllable form) that appear several times in Section VI, and therefore we introduce them here. Specifically, we define \( \Gamma_3 \) and \( \Gamma_5 \) as
\[
\Gamma_3(f, k) := C_3((f_3(k_1 + f_3), f_3(k_2 + f_4), f_3)), \tag{16}
\]
\[
\Gamma_5(q, f, k) := C_5(e), \tag{17}
\]
\[
e \equiv f_3(k_1, k_2, k_3, k_4 + f_3) + f_3(1 + q), k_4 + \frac{f_5}{f_3}(1 + q), 1
\]
Since we are interested in determining the stability of an equilibrium, it proves useful to determine when a matrix is Hurwitz. That is the case iff all the elements in the first column of the Routh’s table are positive (or negative) [25]. It follows from the Routh’s criterion that (16) and (17) are Hurwitz if and only if
\[
q > 0 \text{ and } f_3 > k_3/k_4, \tag{18}
\]

VI. LINEARIZATION

Before linearizing the closed loop vector field \( Z^c \) in (14) around the equilibrium \( z^* \) in (9), let us provide a vector field that serves only the purpose of analysis. Recall the map \( f \) in (3) containing the constraints that define the state space. Consider then, for any \( z \in \mathbb{R}^3 \) and for some \( \lambda > 0 \),
\[
\hat{Z}(z) = -f^c(z)^T (f^c(z)^T f^c(z))^{-1} (f^c(z)^T Z^c(z) + \lambda f(z)), \tag{19}
\]
where, it follows from (2) and (8), that for any \( z \in \mathbb{Z}, \hat{Z}(z) = 0_{32} \). Consider then the new vector field
\[
\hat{Z}^c(z) := Z^c(z) + \hat{Z}(z), \tag{20}
\]
where we emphasize that \( \hat{Z}^c(z) = Z^c(z) \) for any \( z \in \mathbb{Z} \). The sole purpose of the vector field \( \hat{Z} \) in (19) is to permit the analysis we conduct next.

Linearization of the closed loop vector field \( \hat{Z}^c \) in (20) around \( z^* \) in (9) yields the Jacobian
\[
A := D\hat{Z}(z^*) \in \mathbb{R}^{32 \times 32}, \tag{21}
\]
which is not a diagonal matrix, and thus determining whether it is Hurwitz is not straightforward. For that purpose, we provide a similarity matrix, i.e., \( P \in \mathbb{R}^{32 \times 32} \), such that \( PAP^{-1} \) is a block triangular matrix, and where each block diagonal matrix is in controllable form (allowing us to invoke the results from Section V). Consider then
\[
P := \begin{bmatrix} P_z & P_{y, \psi} & P_{\nu, \theta} & P_v & P_{\nu, \theta} \end{bmatrix} \in \mathbb{R}^{32 \times 32} \tag{22}
\]
where (below \( A \) is the Jacobian in (21), and \( e_1, \cdots, e_{32} \) are the canonical basis vectors in \( \mathbb{R}^{32} \))
\[
P_z := \begin{bmatrix} \nu & A \nu & A^2 \nu \end{bmatrix} \bigg|_{z = 0, \nu = 0} \in \mathbb{R}^{32 \times 3},
P_{y, \psi} := \begin{bmatrix} \nu & A \nu & A^2 \nu \end{bmatrix} \bigg|_{y = 0, \psi = 0} \in \mathbb{R}^{32 \times 3},
P_{\nu, \theta} := \begin{bmatrix} e_1 & A e_1 & A^2 e_1 \end{bmatrix} \bigg|_{\nu = 0, \theta = 0} \in \mathbb{R}^{32 \times 5},
P_v := \begin{bmatrix} e_3 & A e_2 & A^2 e_2 \end{bmatrix} \bigg|_{\nu = 0, \psi = 0} \in \mathbb{R}^{32 \times 5},
\]
and, finally, where
\[
P_{\nu, \theta} := (f^c(z^*))^T \in \mathbb{R}^{32 \times 8}.
\]

Remark 3: Recall the state decomposition in (1), and that \( \dot{z} = Az \), for the linearized motion around the equilibrium. Then (for brevity, denote \( p = (x, y, z) \) and \( n = (\phi, \psi, \theta) \))
\[
\begin{bmatrix} P^T_z z \\
P^T_{y, \psi} z \\
P^T_{\nu, \theta} z \\
P^T_v z \end{bmatrix} = \begin{bmatrix} (x(0), x(1), x(2), x(3)) \\
(\phi(0), \phi(1), \phi(2), \phi(3)) \\
(\psi(0), \psi(1), \psi(2), \psi(3)) \\
(\nu(0), \nu(1), \nu(2), \nu(3)) \end{bmatrix}
\]
and (the equalities below can only be verified under an appropriate coordinate transformation)
\[
\begin{bmatrix} P^T_z z \\
P^T_{y, \psi} z \\
P^T_{\nu, \theta} z \end{bmatrix} = \begin{bmatrix} (x(0), x(1)) \\
(\phi(0), \phi(1)) \\
(\psi(0), \psi(1)) \end{bmatrix}
\]
That is, \( P_z \) is associated with the \( x \)-linear motion of the bar (fifth order system) and \( P_{y, \psi} \) is associated with the \( x \)-linear motion between the UAVs (third order system); \( P_{\nu, \theta} \) is associated with the \( y \)-linear motion of the bar (fifth order system) and \( P_v \) is associated with the \( y \)-angular motion of bar (fifth order system). And finally, \( P_v \) is associated with the \( z \)-linear motion of the bar (third order system) and \( P_{\nu, \theta} \) is associated with the \( z \)-angular motion of bar (third order system). (The sum of the integral errors is then associated with the \( z \)-linear position of the bar, and the difference is associated with the \( z \)-angular position of the bar.)

Given the state matrix \( A \) in (21) and the similarity matrix \( P \) in (22), it then follows that
\[
\begin{bmatrix} A_{x, \phi} & A_{x, \psi} & A_{\nu, \theta} & A_{y, \psi} & A_{\nu, \theta} \end{bmatrix} \begin{bmatrix} 0_{32 \\ 24} \\
\nu \times 32 \times 32 \end{bmatrix} \begin{bmatrix} \lambda I_{32 \times 32} \end{bmatrix} \begin{bmatrix} \nu \times 32 \times 32 \end{bmatrix}
\]
where (23) is a block triangular matrix, with the first block as a block diagonal matrix with three blocks (note that the \( \lambda \) in (23) is that chosen in (19)). Thus \( \text{eig}(A) = \{ -\lambda \} \cup \text{eig}(A_{x, \phi}) \cup \text{eig}(A_{x, \psi}) \cup \text{eig}(A_{\nu, \theta}) \), and, therefore, determining whether the Jacobian \( A \) in (21) is Hurwitz amounts to checking whether each of the three blocks in (23) is Hurwitz. Let us look at each of these matrices separately, corresponding to three decoupled motions: longitudinal, lateral and vertical.
A. Longitudinal motion

Recall Remark 3, and note that $P_x$ and $P_δ$ are associated to $A_{x,δ} ∈ R^{8×8}$ in (23). As such, $A_{x,δ}$ is associated with the longitudinal motion, namely the $x$ motion of the bar, and the $x$ motion difference between the two UAVs.

In what follows denote

$$F_x ≡ F_x(k^1_{p,x}, k^2_{p,x}, k^1_{d,x}, k^2_{d,x}, k^1_δ, k^2_δ) ∈ R^3,$$

where $F_x$ is some function of the gains shown above. Note then that $A_{x,δ}$ has a specific structure, namely (below $*$ denotes a vector in $R^8$)

$$A_{x,δ} = \left[ \begin{array}{cc} A_x & 0_{5×3} \\ *_{3×5} & A_δ \end{array} \right] ∈ R^{8×8}. \quad (24)$$

Notice that $A_{x,δ}$ can be rendered block triangular, if one chooses the gains such that $F_x$ in (24) vanishes. That is accomplished if, for $i ∈ \{1,2\}$,

$$k^i_{p,x} = k_{p,x} + \frac{d_1 l_1}{d_2 l_2} f_p Δ_x,$$

$$k^i_{d,x} = k_{d,x} + \frac{d_1 l_1}{d_2 l_2} f_p Δ_x,$$

$$k^i = k_{r,i} \quad \text{and} \quad \Delta_x = \frac{m(d_1 l_1 m_1 + d_2 l_2 m_2)}{m_2 m_1(d_1 l_1 + d_2 l_2)},$$

for some positive $k_{p,x}, k_{d,x}$, and $k_{r,i}$, and where $f_p$ is that in (27). That is, the proportional and derivative gains of each vehicle must be the same up to some difference that is proportional to the asymmetry of the system – quantized by $Δ_i$. If the vehicles’ gains are chosen as above, then

$$A_{x,δ} = \left[ \begin{array}{cc} A_{x} & 0_{5×3} \\ *_{3×5} & A_{δ} \end{array} \right] ∈ R^{8×8}. \quad (26)$$

where (recall $Γ_5$ in (17))

$$A_{x} = Γ_5(q, f, k) \mid f_p = k_{p,x}, k = (k_{p,x}, k_{d,x}),$$

$$f_p = \frac{g(l_1 + l_2)}{2l_1 l_2} > 0, \quad q = \frac{m(d_1 l_1 m_1 + d_2 l_2 m_2)}{m_2 m_1(d_1 l_1 + d_2 l_2)} > 0,$$

and where (recall $Γ_3$ in (16))

$$A_{δ} = Γ_3(f, k) \mid f_p = k_{p,x}, k = (k_{p,x}, k_{d,x}), \quad g_m(d_1 l_1 m_1 + d_2 l_2 m_2) = \frac{l_1 l_2 m_1 m_2 d_1^2 l_1 + d_1^2 l_2 + d_2^2 l_1 + d_2^2 l_2}{l_1 l_2 m_1 m_2 d_1^2 l_1 + d_1^2 l_2 + d_2^2 l_1 + d_2^2 l_2}.$$ It follows from (18) that $A_{x}$ and $A_{δ}$ above are Hurwitz, provided that

$$k_{r,i} > k_{p,x}/k_{d,x},$$

provided that the attitude gain is big enough. This constraint can be comprehended intuitively: fast tracking along the longitudinal direction requires a fast attitude inner loop.

Remark 4: If the system is symmetric (see (24)), then

$$A_{x} = Γ_5(q, f, k) \mid q = \frac{g_m(f_p, f_p)}{2}(f_p, f_p) = (q, k_r, k = (k_{p,x}, k_{d,x})).$$

That is, the (linearized) $x$ motion of the bar is exactly that of container in a container-crane system, with a cable of length $l$, being pulled by a crane of mass $2M$, and with a motor constant $k_r$ [24].

Remark 5: If one wishes both gains $k^1_{p,x}$ and $k^1_{p,x}$, in (25), to be positive, then one must impose that $k_{p,x} > \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}$, for some positive $\overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}$, and which are both Hurwitz, provided that

$$k_{r} > k_{p,y}/k_{d,y},$$

Fig. 3: Good and bad asymmetries (a good asymmetry only requires the gains $k_{p,x}, k_{d,x}$ to be positive, and a bad asymmetry requires the proportional gain to be strictly positive): it is better for the heavier UAV to be attached to the shorter cable.

$$\overline{q} \begin{pmatrix} l_1 = 2l & \overline{q} \begin{pmatrix} l_1 = l & \overline{q} \begin{pmatrix} l_1 = 2l \overline{q} \begin{pmatrix} l_1 = l & \overline{q} \begin{pmatrix} l_1 = 2l \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

bad asymmetry for longitudinal motion

good asymmetry for longitudinal motion

B. Lateral motion

Recall Remark 3, and note that $P_y$ and $P_ψ$ are associated to $A_{y,ψ} ∈ R^{10×10}$ in (23). As such, $A_{y,ψ}$ is associated with the lateral motion, namely the $ψ$ linear motion of the bar, and the $y$ angular motion of the bar (yaw motion). In what follows denote

$$F_y ≡ F_y(k^1_{p,y}, k^2_{p,y}, k^1_{d,y}, k^2_{d,y}, k^1_ψ, k^2_ψ, k^1_δ, k^2_δ) ∈ R^8,$$

where $F_y$ is some function of the gains shown above. Note then that $A_{y,ψ}$ has a specific structure, namely

$$A_{y,ψ} = \left[ \begin{array}{cc} A_y & \overline{0}_{5×5} \\ \overline{e} e^{TF} & A_ψ \end{array} \right] ∈ R^{8×8}. \quad (28)$$

Notice that $A_{y,ψ}$ can be rendered block triangular, if one chooses the gains such that $F_y$ in (28) vanishes (no choice of gains makes $F_y$ vanish). Similarly to as in Subsection VI-A, $F_y$ vanishes under an appropriate choice of gains, which we omit here for brevity (details are found [24]). We only state here that

$$A_y = Γ_5(q, f, k) \mid q = \frac{g_m(f_p, f_p)}{2}(f_p, f_p) = (q, k_r, k = (k_{p,y}, k_{d,y})), \quad A_ψ = Γ_5(q, f, k) \mid q = \frac{g_m(f_p, f_p)}{2}(f_p, f_p) = (q, k_r, k = (k_{p,y}, k_{d,y})).$$

for some positive $\overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}, \overline{q}$, and which are both Hurwitz, provided that

$$k_{r} > k_{p,y}/k_{d,y},$$

\( \overline{q} \begin{pmatrix} l_1 = 2l & \overline{q} \begin{pmatrix} l_1 = l & \overline{q} \begin{pmatrix} l_1 = 2l \overline{q} \begin{pmatrix} l_1 = l & \overline{q} \begin{pmatrix} l_1 = 2l \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \)
i.e., provided that the attitude gain is big enough. Note that similar remarks to Remarks 4, 5, and 6 can be made at this point regarding the lateral motion.

C. Vertical motion

Remark 7: In this section, and for brevity, we assume \( d_i = -d_2 =: d \) (for some \( d \)) when presenting the results. The results without these assumptions are found in [24].

Recall Remark 3, and note that \( P_i \) and \( P_0 \) are associated to \( A_{i,\theta} \in \mathbb{R}^{6 \times 6} \) in (23). As such, \( A_{i,\theta} \) is associated with the vertical motion, namely the \( z \)-linear motion of the bar, and the \( z \) angular motion of the bar (pitch motion).

In what follows denote

\[
F_i \equiv F_i(k_{p,i}^z, k_{d,i}^z, k_{d,i}^z, k_{i,i}^z, k_{i,i}^z, k_{i,i}^z) \in \mathbb{R}^3,
\]

where \( F_i \) is some function of the gains shown above. Note then that \( A_{i,\theta} \) has a specific structure, namely

\[
A_{i,\theta} = \begin{bmatrix} A_i & e_3 \bar{F}_i^T \\ e_3 & F_i & A_0 \end{bmatrix} \in \mathbb{R}^{(3+3) \times (3+3)}.
\]

Notice that \( A_{i,\theta} \) can be rendered block triangular, if one chooses the gains such that either \( F_i \) or \( \bar{F}_i \) in (29) vanish. We choose to cancel \( F_i \), implying that we decouple the \( z \)-linear motion, from the \( z \)-angular motion. That is accomplished if \((i \in \{1, 2\}) \)

\[
\begin{align*}
k_{p,i}^z &= k_{d,i}^2 \\
k_{d,i}^z &= k_{i,i}^z = m_2(J + 2d^2m_1) \\
k_{i,i}^z &= m_1(J + 2d^2m_2).
\end{align*}
\]

That is, the proportional, derivative and integral gains of each vehicle must respect a ratio, which is exactly 1 under symmetry conditions (see (11)). In order to satisfy the conditions above, let, for \( h \in \{p, i, d\} \),

\[
\begin{align*}
k_{h,i}^z &= \frac{2(J + 2d^2m_i)m_x}{4d^2m_i m_x + J(m_1 + m_2)} k_{h,z} \\
k_{h,i}^z &= \frac{2(J + 2d^2m_i)m_x}{4d^2m_i m_x + J(m_1 + m_2)} k_{h,z},
\end{align*}
\]

for some positive \( k_{p,i}^z, k_{d,i}^z \), and \( k_{i,i}^z \). If the vehicles’ gains are chosen as in (30), then

\[
A_{i,\theta} = \begin{bmatrix} A_i & 0_{3 \times 3} \\ 0_{3 \times 3} & A_0 \end{bmatrix} \in \mathbb{R}^{(3+3) \times (3+3)}
\]

where (recall \( C_3 \) in (15))

\[
A_i = C_3 \left( \gamma_i(k_{p,i}^z, k_{d,i}^z, k_{i,i}^z) \right), \quad A_0 = C_3 \left( \gamma_0(k_{p,i}^z, k_{d,i}^z, k_{i,i}^z) \right),
\]

and where

\[
\begin{align*}
\gamma_i &= \left( 4m_i m_2 J + 2d^2m_1(m_1 + m_2) + 2d^2m_2(m_1 + m_2) \right) \left( 4m_i m_2 J + 2d^2m_1(m_1 + m_2) \right)^{-1} \\
\gamma_0 &= \left( 4d^2m_i m_2 J + 2d^2m_2(m_1 + m_2) \right) \left( 4d^2m_i m_2 J + 2d^2m_2(m_1 + m_2) \right)^{-1}
\end{align*}
\]

It follows from (15) that \( A_i \) and \( A_0 \) are Hurwitz (and therefore also \( A_{i,\theta} \)), provided that

\[
k_{i,i}^z \leq \min(\gamma_i, \gamma_0) k_{p,i}^z k_{d,i}^z,
\]

i.e., provided that the integral gain is small enough. Note that, for a regular PID, it is required that \( k_{i,i}^z < k_{p,i}^z k_{d,i}^z \), while the constraint above is more restrictive, since \( \gamma_0 < 1 \). Moreover, notice that \( \gamma_0 \) vanishes when \( d \) vanishes (the distance of the contact points to the bar’s center of mass): as such, it is advisable to have a big \( d \) (big compared with \( \sqrt{\frac{J(m_1 + m_2)}{m_1 m_2}} \)), because \( \gamma_0 \) is closer to 1 (and thus the bound on the integral gain is less restrictive). This also agrees with intuition, which suggests that controlling the bar’s attitude when the contact points are too close to the bar’s center of mass is difficult.

Remark 8: The attitude gains of the vehicles do not play a role in the linearized vertical motion.

VII. EXPERIMENTAL RESULTS

A video of the experiment that is described in the sequel is found at https://youtu.be/rgweowQ8fAE, whose results are visualized in Fig. 4. For the experiment, two hexacopters were used, namely one ASTEC-Neo weighting 2.22 kg, and one Tarot FY680 weighting 3.53 kg. The bar was made out of a core of aluminum, surrounded with PVC pipe, and with two metal plates at the contact points with the cables: in total, it weighted 1.48 kg, with the individual plates weighting 0.6 kg each (the bar’s weight corresponds to 60% of the ASTEC-Neo and to 40% of the Tarot FY680). The bar has a length of 2m, with the contact points between the bar and the cables at the extremities of the bar, and thus \( d_i = -d_2 = 1 \) m; the cables are attached to the bar’s contact points by means of permanent magnets. The ASTEC-Neo is tethered to the bar by a 1.45 cm cable, and the Tarot FY680 by a 1.2 cm cable. The commands for controlling the hexacopter were processed on a ground station, developed in a ROS environment, and sent to the on-board autopilot, which allowed for remotely controlling the aerial vehicles. The ASTEC-Neo is equipped with a proprietary flight controller, which we communicate with by publishing a message of the type mav_msgs/RollPitchYawrateThrust; while the Tarot FY680 is equipped with an open source flight controller (namely a PixHawk), which we communicate with by publishing a message of the type mavros_msgs/OverRideRCIn. The hexacopter’s and the bar’s poses and twists were estimated by 12 cameras from a Qualisys motion capture system.

In the beginning of the experiment the bar is required to stabilize around \( z^* \) (see (9)) where \( \psi^* = (0.4, -0.5, 0.4)^{\top} \) and \( n^* = e_2 \) (see Remark 1), i.e, the bar is required to hover at 0.4m and required to be aligned with the \( y \)-axis. In Fig. 4(b), the bar attitude is parameterized with a pitch and yaw angle (i.e., \( n = (\cos(\theta) \cos(\psi), \cos(\theta), \sin(\theta)) \), and, as can be seen in Fig. 4(b) the bar is initially aligned with the \( y \)-axis (\( \psi^* = 90^\circ \)). At around 55 sec, the bar is required to translate 0.5m in the \( x \)-direction, and at around 60 sec, the bar is required to align itself with the \( x \)-axis (\( n^* = e_1 \), \( \psi = 0^\circ \), which can be seen in Figs. 4(a) and 4(a). At around 80 sec, the bar is required to move in the \( y \)-direction, while keeping the same orientation, which can again be seen in Figs. 4(a) and 4(a). During the same experiment, we also tested robustness against impulse disturbances, which illustrate the size of the basin of attraction of the equilibrium. First, at around 100s, we disturbed the Tarot FY680 in the \( y \)-direction, as can be seen in Fig. 4(d); and, at around 110s, we disturbed the ASTEC-Neo in the \( y \)-direction, as can be seen in Fig. 4(c). In both cases, the system returns to its
equilibrium point.

In Figs. 4(e) and 4(f), the control inputs are shown, and in Fig. 4(g) the integral terms for both UAVs are shown. The equilibrium integral term is inversely proportional to the vehicle’s weight, which explains why the integral term for the ASTEC-Neo is smaller than that for the Tarot FY680.

REFERENCES