

Controlling the Relative Agent Motion in Multi-Agent Formation Stabilization

Meng Guo, Michael M. Zavlanos and Dimos V. Dimarogonas

Abstract—In this paper we propose a novel technique to control the relative motion of multiple mobile agents as they stabilize to a desired configuration. In particular, we focus on the agents’ relative velocities and the rate of change of their pairwise distances, and employ constructs from classic navigation functions (NFs) to control these quantities. Controlling agent velocities requires nontrivial extensions of the NF methodology to second-order models. Although in this work we propose a centralized framework to control the relative agent velocities, it adds a new dimension to the control of multi-agent systems with several advantages. In particular, we provide a novel approach to control the transient dynamics of a network that may facilitate the integration of continuous motion planing with discrete topology control. The result is verified theoretically and via computer simulations.

Index Terms—Cooperative control, Agents and autonomous systems.

I. INTRODUCTION

The navigation function (NF) methodology, firstly introduced in the seminal work of Rimon and Koditschek [1], has been extensively applied to multi-agent cooperative navigation due to its mathematical soundness [2], [3]. The primary control objective in these problems is to guide a team of autonomous robots to a desired configuration, while avoiding collisions both with teammates and obstacles. Different dynamic models of the agents have been analyzed including simple single integrator [4], with limited sensing abilities [2], double integrator and non-holonomic models [5]. Solutions are both centralized [3] and decentralized [4], [6] with the latter relying only on locally available information for control.

Other control schemes for multi-agent systems that do not employ NFs have also been used in the context of consensus under dynamical interaction topologies [7], cooperative search under limited communication range [8], and group coordination using nearest neighbor rules [9]. In this paper, we address the problem of cooperative, multi-agent, formation stabilization subject to relative motion constraints among neighboring agents. Specifically, we consider bounds on the relative velocities between neighboring agents and on the rate of change of their pairwise distances, which we enforce as the network converges to its final configuration. An important motivation for this work stems from integrating continuous motion dynamics with discrete network control. In fact, it is shown in our recent work [10], [11], [12] that the convergence of distributed network optimization algorithm for mobile networks depends not so much on the absolute velocity of the agents, but on the rate at which the pairwise distances between agents change. Specifically, [10] develops theoretical results that relate the distance to the optimal point to the rate at which the pairwise agent distances change.

An additional great advantage of this framework is that controlling the dynamics of the pairwise distances within a network can indirectly control its connectivity [13], which is critical to the convergence of, e.g., state agreement algorithms [14]. Moreover, controlling the

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relative motion of the agents can facilitate integration of motion planning with discrete topology control due to, e.g., flow control and routing [11]. This development can lead to hybrid multi-agent systems that can reliably relay information within the network. An alternative way to ensure a desired rate of change of the network structure is to enforce hard bounds on the individual agent velocities [15]. However, this approach is overly conservative, as it will generally require from the agents to move slow, which hinders real time implementation of this algorithm. Our approach by contrast only enforces bounds on the relative velocity or the change rate of pairwise distance between neighboring agents.

To formulate the proposed problem using navigation functions, we modify the repulsive potential in the resulting navigation function to incorporate the relative motion constraints, while keeping the goal potential as specified by the task. Due to the fact that both relative motion constraints explicitly involve the state variables of position and velocity simultaneously, second-order dynamics must be considered, which are known to be theoretically harder to deal with, compared to their first-order counterparts [16]. Under some assumptions, we show that the proposed navigation mechanism is free of local minima, while it ensures that the evolution of the network satisfies the desired transient constraints. The proposed framework reveals a new technique to tackle the multi-agent coordination problem under relative motion constraints involving velocity terms.

The main contributions of this paper are as follows: (1) two different relative motion constraints are considered to ensure the network integrity; (2) a novel way is proposed to incorporate these constraints into the NF-based cooperative controller design of multi-agent systems; and (3) two generic control schemes are proposed that are applicable to a class of cooperative formation tasks.

The rest of the paper is organized as follows: Section II describes the system model and the problem in hand. A detailed description in Section III is given about how to modify the classic navigation function to fit our needs. Stability and convergence of two different control schemes are analyzed in Section IV. Section V shows how the control scheme can be applied to several multi-agent formation tasks and the results are illustrated by computer simulations. The last section summarizes the main conclusions and indicates the further research directions.

II. PROBLEM DESCRIPTION

Since the proposed relative motion constraints involve the state variables of both velocity and position, it is necessary to consider autonomous agents satisfying second order dynamics, i.e.,

$$\begin{aligned} \dot{q}_i &= v_i \\ \dot{v}_i &= u_i, \quad i \in \mathcal{V}, \end{aligned} \quad (1)$$

where q_i, v_i stand for agent i ’s position and velocity with dimension m , i.e., $q_i, v_i \in \mathbb{R}^m$, and $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of agents. Denote further by $v, q, u \in \mathbb{R}^{Nm}$ the stack vectors composed of v_i, q_i and u_i , for $i \in \mathcal{V}$. To simplify the navigation problem, we neglect collision avoidance as we assume agents of zero volume operating within a large workspace. Furthermore, we assume that no static or moving obstacles are present in the operating workspace.

We call agent i and j neighbors if they exchange information with each other and denote $(i, j) \in E$, where $E \subset \mathcal{V} \times \mathcal{V}$ is the edge set of the communication topology [17] $G = (\mathcal{V}, E)$. $\mathcal{N}_i \subset \mathcal{V}$ denotes the neighboring set of agent i so that $i \in \mathcal{N}_j$ if $(i, j) \in E$. In this paper, we only consider static and undirected graphs that satisfy $(i, j) \in E$ for all $t > 0$ if $(i, j) \in E$ at $t = 0$, and $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$, respectively. This implies that E is constant and pre-defined at the system startup. Furthermore, we assume that G is connected [17], namely there exists a path from any node i to another node j .

In what follows, we employ the two different relative motion constraints below:

(C.1) Constraints on relative agent velocities:

$$\|v_i - v_j\| < \varepsilon_1, \forall (i, j) \in E, \quad (2)$$

where $\varepsilon_1 > 0$ and $\|\cdot\| \triangleq \|\cdot\|_2$ denotes the Euclidean norm. These constraints impose upper bounds on the relative velocities between neighboring agents.

(C.2) Constraints on the rate of change of the pairwise relative agent distances:

$$|(q_i - q_j)^T(v_i - v_j)| < \varepsilon_2, \forall (i, j) \in E, \quad (3)$$

where $\varepsilon_2 > 0$. To obtain (3) note that the squared relative distance between $(i, j) \in E$ is given by $\|q_i - q_j\|^2$ and its changing rate is $\frac{\partial}{\partial t}\|q_i - q_j\|^2 = 2(q_i - q_j)^T(v_i - v_j)$.

The constraint (C.1) can be thought of as an alternative way to control the relative distances. Zero relative velocity means that the relative distance stays the same and small relative velocities means that the relative distance changes slowly. The constraint (C.2) directly controls the change rate of relative distances.

Controlling the relative motion of the agents in multi-agent systems can allow to indirectly control the connectivity of the network, which can have a significant impact in, e.g., convergence of state agreement algorithms. Moreover, controlling the relative agent motion can allow to control the rate of change of the network structure, which may facilitate integration of motion planning with iterative optimization algorithms, such as communication control, that depend on static or slowly varying networks for convergence. On the other hand, we intend to design a generic control scheme that serves various formation objectives, while satisfying constraints on the agents' relative motion. Such *objectives* can be:

(O.1) consensus, captured by the condition $q_i = q_j, \forall (i, j) \in E$, or $q_i = q_j = q_0$, where $q_0 \in \mathbb{R}^m$ is the pre-defined consensus point, or

(O.2) formation stabilization, captured by the condition $q_i - q_j = c_{ij}, \forall (i, j) \in E$, where $c_{ij} \in \mathbb{R}^m$ is the relative position between neighbors i and j , or the condition $q_i = c_i, \forall i$, where $c_i \in \mathbb{R}^m$ is the absolute destination for agent i .

In the sequel, we develop and study the stability properties of navigation functions that minimize generic objectives including consensus and formation stabilization, as discussed above, while respecting the relative motion constraints (2) and (3). It is worth mentioning that even with the centralized approaches, it is not trivial to tackle the coordination of multi-agent system under relative motion constraints.

III. CLASSIC NAVIGATION FUNCTIONS

In this part, we first briefly discuss the notion of a navigation function and then describe the method of modify the classic navigation function in order to take into account the relative motion constraints. The navigation function firstly proposed by Rimon and Koditschek in [1] is given by:

$$\phi \triangleq \frac{\gamma}{(\gamma^k + \beta)^{\frac{1}{k}}}, \quad (4)$$

where ϕ represents the potential, γ the attractive potential from the goal and β the repulsive potential from the sphere obstacles in the workspace. Note that k is the critical tuning parameter that guarantee its correctness, namely there exists a lower bound of k such that ϕ is a valid navigation function [1]. Besides its provable mathematical correctness, another strength of (4) is that it provides a straightforward motion planning algorithm. By simply following the negated gradient $-\nabla_q \phi$, it is guaranteed that $\gamma \rightarrow 0$ when $t \rightarrow \infty$ and $\beta > 0$ holds for all $t \geq 0$. That is to say, a collision free path is

guaranteed from almost any initial position (except a set of measure zero) to any goal position in a valid workspace [1].

In particular, for an agent satisfying single integrator model $\dot{q} = u$, convergence of the closed loop system under the control law $u \triangleq -\nabla_q \phi$ could be verified by considering the Lyapunov function candidate $V \triangleq \phi$. Since $\dot{V} = (\nabla_q \phi)^T \dot{q} = -\|\nabla_q \phi\|^2 \leq 0$, it has been shown in [1] that $\nabla_q \phi = 0$ only if $q = q_d$ except a set of measure zero points. Furthermore, similar arguments also hold for double integrator models as in (1). In this case the control law is given by $u \triangleq -\nabla_q \phi - v$. Consider the Lyapunov function candidate $V \triangleq \phi + \frac{1}{2} v^T v$, where $\frac{1}{2} v^T v$ represents the kinetic energy of the system [16], [3]. Its time derivative along the solution of the closed-loop system is $\dot{V} = (\nabla_q \phi)^T v + v^T u = -\|v\|^2 \leq 0$. $\dot{V} = 0$ holds when $v = 0$, which implies $u = \dot{v} = 0$ and further $\nabla_q \phi = 0$, with $\nabla_q \phi = 0$ only if $q = q_d$ except for a set of measure zero points. Moreover, it is of great importance to point out why $\beta > 0$ is ensured during the process. Since we have shown that the Lyapunov function candidate is monotonically decreasing before the agent reaches the goal position, $V(t) \leq V(0) < 1, \forall t \geq 0$. Due to the fact that $V(t) \geq \phi \geq 1$ when $\beta \leq 0$, it is guaranteed that $\beta > 0, \forall t \geq 0$.

Inspired by the reasoning above, we incorporate the relative motion constraints (2) and (3) into two different repulsive potential functions β_1 and β_2 , respectively. In particular, constraint (C.1), namely (2) is equivalent to $\varepsilon_1^2 - (v_i - v_j)^T(v_i - v_j) > 0$, which can be captured by the repulsive potential $\beta_1 \triangleq \beta_1(v) = \prod_{(i,j) \in E} (\varepsilon_1^2 - (v_i - v_j)^T(v_i - v_j))$. Similarly, constraint (C.2), namely (3) is equivalent to $\varepsilon_2^2 - ((q_i - q_j)^T(v_i - v_j))^2 > 0$, which can be captured by the repulsive potential $\beta_2 \triangleq \beta_2(q, v) = \prod_{(i,j) \in E} (\varepsilon_2^2 - ((v_i - v_j)^T(q_i - q_j))^2)$. We want to keep both of the repulsive potentials positive, like the collision avoidance mechanism. In what follows, we assume a general form of the goal potential function γ that satisfies the following conditions: (a) $\gamma \geq 0$; (b) $\gamma = \gamma(q)$; and (c) $\gamma(q) = 0$ and $\nabla_q \gamma = 0$ if and only if $q \in q_d$, where $q_d \in \mathbb{R}^{Nm}$ is the set of desired formations. Specific choices for γ that meet the requirements in (a), (b), and (c), but also model objectives (O.1) and (O.2) are discussed in Section V.

IV. CONTROLLER DESIGN

In this section, we propose two different controller designs to ensure the satisfaction of the relative motion constraints (2) and (3), respectively. Our designs can accommodate a variety of formation objectives and their correctness is shown using Lyapunov stability.

A. Relative Velocity Constraints

In this part, we mainly consider the multi-agent formation control problem subject to relative velocity constraints (2), namely $\|v_i - v_j\| \leq \varepsilon_1, \forall (i, j) \in E$, where $\varepsilon_1 > 0$ and E is the edge set. Let $\beta_1 \triangleq \prod_{(i,j) \in E} \beta_{ij}$ denote the repulsive potential, where $\beta_{ij} \triangleq \varepsilon_1^2 - (v_i - v_j)^T(v_i - v_j)$. The function β_1 in matrix form is equivalent to

$$\beta_1 \triangleq \prod_{(i,j) \in E} \beta_{ij} = \prod_{(i,j) \in E} (\varepsilon_1^2 - (v^T B_{ij} v)), \quad (5)$$

where $B_{ij} \in \mathbb{R}^{Nm \times Nm}$ is defined as $B_{ij} = E_{ij} \otimes I_m$, with \otimes denoting the Kronecker product [18] and $E_{ij} \in \mathbb{R}^{N \times N}$ having the following structure:

$$E_{ij} \triangleq \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Specifically, E_{ij} is a symmetric matrix, with the (i, i) and (j, j) entries being 1 and the (i, j) and (j, i) entries being -1 , while the rest being zero. The resulting potential function ϕ_1 in this case is given by

$$\phi_1 \triangleq \frac{\gamma}{(\gamma^k + \beta_1)^{\frac{1}{k}}}, \quad (6)$$

where $k > 0$, $\gamma \triangleq \gamma(q)$ is defined in Section III and β_1 is given by (5). Furthermore, the gradients of ϕ_1 with respect to the state variables q, v are given by

$$\nabla_q \phi_1 = \frac{\beta_1 \nabla_q \gamma}{(\gamma^k + \beta_1)^{\frac{1}{k} + 1}}, \quad (7)$$

$$\nabla_v \phi_1 = \frac{-\frac{1}{k} \gamma \nabla_v \beta_1}{(\gamma^k + \beta_1)^{\frac{1}{k} + 1}}, \quad (8)$$

where we use the fact that $\nabla_q \beta_1 = 0$ as β_1 is a function of v and $\nabla_v \gamma = 0$ as γ is a function of q . For brevity, set $h_1 \triangleq \frac{1}{(\gamma^k + \beta_1)^{\frac{1}{k} + 1}} > 0$. Then we have

$$\begin{aligned} \nabla_q \phi_1 &= \beta_1 h_1 \nabla_q \gamma, \\ \nabla_v \phi_1 &= -\frac{1}{k} h_1 \gamma \nabla_v \beta_1. \end{aligned}$$

The gradient of the modified repulsive potential β_1 with respect to v is given by:

$$\nabla_v \beta_1 = - \left(2 \sum_{(i,j) \in E} \bar{\beta}_{ij} B_{ij} \right) v = -A_1 v,$$

where $A_1 \triangleq 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} B_{ij}$ and $\bar{\beta}_{ij} \triangleq \prod_{(i,k) \in E, k \neq j} \beta_{ij}$ is the omit product.

Lemma 1. A_1 is positive semidefinite when $\beta_{ij} > 0$ in (5), $\forall (i, j) \in E$. Moreover, $(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})$ is positive definite for any $\rho > 0$.

Proof. A_1 is symmetric and therefore so is $(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})$. Let $x \in \mathbb{R}^{Nm}$ be any nonzero vector. The quadratic term $x^T (\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm}) x$ can be computed as follows:

$$\begin{aligned} & x^T \left(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm} \right) x \\ &= \frac{2}{k} \gamma h_1 \left(\sum_{(i,j) \in E} \bar{\beta}_{ij} x^T B_{ij} x \right) + \rho \|x\|^2, \end{aligned}$$

where $B_{ij} = E_{ij} \otimes I_m$. The term $x^T B_{ij} x = (x_i - x_j)^2 \geq 0$, $\forall (i, j) \in E$. The equality holds when $x_i = x_j$, $\forall (i, j) \in E$, implying $x \in \text{span}\{\mathbf{1}\}^T$ because the underlying communication topology G is connected, where $\mathbf{1}$ is the row vector with all ones. The terms γ, h_1 and $\bar{\beta}_{ij}$ are all positive as $\beta > 0$ is assumed. Thus A_1 is positive semidefinite, and is actually a standard Laplacian matrix [19] of the underlying communication graph with nonnegative edge weights with $(v_i - v_j)^T (v_i - v_j) \geq 0$, $\forall (i, j) \in E$.

Moreover $\|x\|^2 \geq 0$ and the equality holds only when $x = 0$. Thus $x^T (\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm}) x > 0$, $\forall x \neq 0$. This means that $\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm}$ is positive definite and therefore invertible. \square

Theorem 2. Assume that $k, \rho > 0$, $v = 0$ initially, and the communication topology is undirected and connected. System (1) is globally stabilized to the invariant set $\mathbf{S}_1 = \{(q, v) | q \in q_d, v = 0\}$ by following the control law:

$$u \triangleq - \left(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm} \right)^{-1} (\beta_1 h_1 \nabla_q \gamma + \rho v). \quad (9)$$

Moreover, the relative velocity constraints (2) are satisfied for all $t \geq 0$.

Proof. Consider the following Lyapunov candidate

$$V_1 \triangleq \phi_1 + \frac{1}{2} \rho v^T v,$$

where $\rho > 0$ is a control parameter that depends on the control preference. The time derivative of V_1 along the solution of system (1) under control law (9) is given by

$$\begin{aligned} \dot{V}_1 &= (\nabla_q \phi_1)^T v + (\nabla_v \phi_1)^T u + \rho v^T u \\ &= \beta_1 h_1 (\nabla_q \gamma)^T v + v^T \left(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm} \right) u, \end{aligned} \quad (10)$$

where we have used equations (7), (8). Then \dot{V}_1 is given by

$$\dot{V}_1 = -\rho \|v\|^2 \leq 0,$$

which means that V_1 remains decreasing as long as $\|v\| \neq 0$. At $t = 0$, we assume that $v = 0$, i.e., zero initial velocity. Thus $\beta_{ij} = \varepsilon_1^2 - (v^T B_{ij} v) = \varepsilon_1^2 > 0$, $\forall (i, j) \in E$ and $\beta_1^{t=0} = \varepsilon_1^{2|E|} > 0$. The Lyapunov function at $t = 0$ is evaluated as $V_1^{t=0} = \phi_1^{t=0} + \frac{1}{2} \rho v^T v = \phi_1^{t=0} = \frac{\gamma}{(\gamma^k + \beta_1^{t=0})^{\frac{1}{k}}} < 1$. Thus $0 \leq \phi_1 \leq V_1 < V_1^{t=0} < 1$, $\forall t > 0$.

The fact that $\beta_1 > 0$ is maintained during the whole process can be proved by contradiction. In fact, if $\beta_1 = 0$ at certain time instants, then $\phi_1 = 1$, which violates the condition that $\phi_1 < 1$, $\forall t > 0$. Similar arguments can be applied to show that $\beta_{ij} > 0$, $\forall (i, j) \in E$ is ensured. Since β_{ij} , $(i, j) \in E$ are independent continuous variables and initialized as positive numbers, they need to approach zero before becoming negative. If one of β_{ij} becomes zero, then $\beta_1 = 0$, which is in contradiction to the fact that $\beta_1 > 0$, $\forall t > 0$. Thus we can draw the conclusion that the constraint 2 is fulfilled, namely $(v_i - v_j)^T (v_i - v_j) < \varepsilon_2^2$, $\forall (i, j) \in E$ and $\forall t > 0$. It is worth mentioning that $V_1 < 1$ implies $\|v\| \leq \sqrt{\frac{2}{\rho}}$, namely the upper bound is inversely proportional to $\sqrt{\rho}$.

By Lemma 1 the matrix $(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})$ is positive definite for any $\rho > 0$ and $\beta_{ij} > 0$, $\forall (i, j) \in E$. Consequently, $(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})^{-1}$ always exists, which verifies the feasibility of the proposed control scheme. By LaSalle's Invariance principle [20], the system converges to the invariant set $\mathbf{S}_1 = \{(q, v) | v = 0\}$. Within this invariant set \mathbf{S}_1 , $v = 0$ implies $u = \dot{v} = 0$, namely $-(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})^{-1} (\beta_1 h_1 \nabla_q \gamma + \rho v) = 0$. Since $(\frac{1}{k} \gamma h_1 A_1 + \rho I_{Nm})$ is positive definite, it means that $\beta_1 h_1 \nabla_q \gamma + \rho v = 0$, which implies $\nabla_q \gamma = 0$. Since $\nabla_q \gamma = 0$ only if $q \in q_d$ from Section III, this leads to the conclusion that $\mathbf{S}_1 = \{(q, v) | q \in q_d, v = 0\}$, i.e., all agents keep still at the desired formation. This completes the proof. \square

B. Constraints on the Change Rate of Relative Distances

In the second part, we mainly consider the formation control problem with constraints (3) on the rate of change of the relative distance, namely $|(v_i - v_j)^T (q_i - q_j)| \leq \varepsilon_2$, $\forall (i, j) \in E$, where $\varepsilon_2 > 0$ and E is the edge set. Let $\beta_2 \triangleq \prod_{(i,j) \in E} \beta_{ij}$ where $\beta_{ij} \triangleq \prod_{(i,j) \in E} (\varepsilon_2^2 - |(v_i - v_j)^T (q_i - q_j)|^2)$. The function β_2 denotes the repulsive potential, which can be written in matrix form as

$$\beta_2 \triangleq \prod_{(i,j) \in E} \beta_{ij} = \prod_{(i,j) \in E} (\varepsilon_2^2 - (v^T B_{ij} q)^2), \quad (11)$$

where q, v and B_{ij} are defined in the same way as in Section IV-A. In particular, the potential function ϕ_2 in this case is defined as

$$\phi_2 \triangleq \frac{\gamma}{(\gamma^k + \beta_2)^{\frac{1}{k}}}, \quad (12)$$

where $k > 0$, γ is defined in Section III and β_2 is given by (11). Furthermore, in this case the gradients of ϕ_2 with respect to q, v are

given by

$$\begin{aligned}\nabla_q \phi_2 &= \frac{\beta_2 \nabla_q \gamma - \frac{1}{k} \gamma \nabla_q \beta_2}{(\gamma^k + \beta_2)^{\frac{1}{k} + 1}}, \\ \nabla_v \phi_2 &= \frac{-\frac{1}{k} \gamma \nabla_v \beta_2}{(\gamma^k + \beta_2)^{\frac{1}{k} + 1}}.\end{aligned}$$

Since β_2 is a function of both q and v , the gradients of β_2 with respect to v and q are computed as

$$\begin{aligned}\nabla_q \beta_2 &= \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (-2(v^T B_{ij} q)) \cdot B_{ij} v \\ &= - \left(2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q) \cdot B_{ij} \right) v = -A_2 v, \\ \nabla_v \beta_2 &= \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (-2(v^T B_{ij} q)) \cdot B_{ij} q \\ &= -2 \left(\sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q) \cdot B_{ij} \right) q = -A_2 q,\end{aligned}$$

where $\bar{\beta}_{ij} \triangleq 2 \prod_{(i,k) \in E, k \neq j} \beta_{ij}$ is the omit product and $A_2 \triangleq \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q) \cdot B_{ij}$. Note here, that unlike A_1 defined in (8), A_2 is neither a standard Laplacian matrix nor positive semidefinite, because the edge weights $v^T B_{ij} q = (v_i - v_j)^T (q_i - q_j)$ are not guaranteed to be positive. Consequently, the previous approach does not apply in this case. However by simple calculations, we can verify that

$$q^T A_2 v = 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q)^2 \geq 0.$$

This observation gives rise to the following transformation that plays an important role in the stability analysis

$$\begin{aligned}A_2 q &= 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q) (B_{ij} q) \\ &= 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (B_{ij} q) (q^T B_{ij} v) \\ &= \left(2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (B_{ij} q q^T B_{ij}) \right) v \triangleq M v,\end{aligned} \quad (13)$$

where the second equality follows from the fact that $v^T B_{ij} q$ is a scalar. Namely, $A_2 q = M v$ and $M = M(q, v) = 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (B_{ij} q q^T B_{ij})$. For brevity, set $h_2 \triangleq \frac{1}{(\gamma^k + \beta_2)^{\frac{1}{k} + 1}} > 0$, and then

$$\begin{aligned}\nabla_q \phi_2 &= \beta_2 h_2 \nabla_q \gamma + \frac{1}{k} h_2 \gamma A_2 v, \\ \nabla_v \phi_2 &= \frac{1}{k} h_2 \gamma A_2 q = \frac{1}{k} h_2 \gamma M v.\end{aligned}$$

Lemma 3. M is symmetric and positive semidefinite when $\beta_{ij} > 0$ in (11), $\forall (i, j) \in E$.

Proof. Clearly, M is symmetric. Let $x \in \mathbb{R}^{Nm}$ be any nonzero vector. Then the quadratic term $x^T M x$ can be computed as:

$$\begin{aligned}x^T M x &= x^T \cdot \left(2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (B_{ij} q q^T B_{ij}) \right) \cdot x \\ &= 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot \left((q^T B_{ij} x)^T (q^T B_{ij} x) \right) \\ &= 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (q^T B_{ij} x)^2.\end{aligned}$$

Since $\bar{\beta}_{ij} \triangleq \prod_{(i,k) \in E, k \neq j} \beta_{ij} > 0$ and $(q^T B_{ij} x)^2 \geq 0$, $x^T M x \geq 0$. The equality holds only when $q \in \text{span}\{\mathbf{1}\}^T$ or $x \in \text{span}\{\mathbf{1}\}^T$ as the underlying communication topology is connected. This completes the proof. \square

Theorem 4. Assume that $k, \rho > 0$, $v = 0$ initially, and the communication topology is undirected and connected. System (1) is globally stabilized to the invariant set $\mathbf{S}_2 = \{(q, v) | q \in q_d, v = 0\}$ by using the control law:

$$u \triangleq - \left(\frac{1}{k} h_2 \gamma M + \rho I_{Nm} \right)^{-1} (\nabla_q \phi_2 + \rho v). \quad (14)$$

Moreover, the constraints on the rate of change of pairwise distances (3) are satisfied for $t \geq 0$.

Proof. Consider the Lyapunov candidate

$$V_2 \triangleq \phi_2 + \frac{1}{2} \rho v^T v,$$

where $\rho > 0$ is a control parameter as before. Its time derivative along the solution of system (1) under control law (14) is given by

$$\begin{aligned}\dot{V}_2 &= (\nabla_q \phi_2)^T v + (\nabla_v \phi_2)^T u + \rho v^T u \\ &= (\nabla_q \phi_2)^T v + (\nabla_v \phi_2 + \rho v)^T u \\ &= (\nabla_q \phi_2)^T v + v^T \left(\frac{1}{k} h_2 \gamma M + \rho I_{Nm} \right)^T u.\end{aligned} \quad (15)$$

Through combining (14) and (15), \dot{V}_2 becomes

$$\dot{V}_2 = -\rho \|v\|^2 \leq 0,$$

which means that V_2 keeps decreasing as long as $\|v\| \neq 0$. The statement that $\beta_2 > 0$ holds for $t > 0$ can be verified by applying similar arguments as in Theorem 2. Namely, at $t = 0$, we assume that $v = 0$, i.e., zero initial velocity. Thus $\beta_{ij} = \varepsilon_2^2 - (q^T B_{ij} v)^2 = \varepsilon_2^2 > 0$, $\forall (i, j) \in E$ and $\beta_2^{t=0} = \varepsilon_2^{2|E|} > 0$. The Lyapunov function at $t = 0$ is evaluated as $V_2^{t=0} = \phi_2^{t=0} + \frac{1}{2} \rho v^T v = \phi_2^{t=0} = \frac{\gamma}{(\gamma^k + \beta_2^{t=0})^{\frac{1}{k}}} < 1$. Since we have shown that V_2 keeps decreasing until the system reaches \mathbf{S}_2 , it implies $V_2 < V_2^{t=0} < 1$. Thus $0 \leq \phi_2 \leq V_2 < 1$, $\forall t > 0$. Then by contradiction, if $\beta_2 = 0$ at certain time instants, $\phi_2 = 1$, which violates the condition that $\phi_2 < 1$, $\forall t > 0$. Furthermore, since β_{ij} are independent continuous variables, initialized as positive numbers, they need to approach zero before becoming negative. If one of β_{ij} becomes zero, then $\beta_2 = 0$, which contradicts the observation that $\beta_2 > 0$, $\forall t > 0$. Thus we can conclude that the constraint 3 is satisfied, namely $|(q_i - q_j)^T (v_i - v_j)| < \varepsilon_2$, $\forall (i, j) \in E$ and $\forall t > 0$. Similarly as in Theorem 2 we can derive that $\|v\| \leq \sqrt{\frac{2}{\rho}}$ as $V_2 < 1$, namely the upper bound is inversely proportional to $\sqrt{\rho}$.

Due to Lemma 3, M is positive semidefinite when $\beta_{ij} > 0$, $\forall (i, j) \in E$. Hence $\frac{1}{k} h_2 \gamma M + \rho I_{Nm}$ is positive definite and invertible for any positive ρ and $\beta_2 > 0$. Thus $\left(\frac{1}{k} h_2 \gamma M + \rho I_{Nm} \right)^{-1}$ always exists and this validates the proposed controller. By LaSalle's Invariance Principle, the closed-loop system converges to the invariant set $\mathbf{S}_2 = \{(q, v) | v = 0\}$. Moreover, within the invariant set \mathbf{S}_2 , $v = 0$ implies $A_2 = 2 \sum_{(i,j) \in E} \bar{\beta}_{ij} \cdot (v^T B_{ij} q) \cdot B_{ij} = 0$, i.e., $\nabla_v \phi_2 = 0$. On the other hand, $v = 0$ implies $u = \dot{v} = 0$, which in turn by (14) indicates $\nabla_q \phi_2 = 0$ and $\nabla_q \gamma = 0$. Consequently, the invariant set \mathbf{S}_2 is equivalent to $\{(q, v) | \nabla_q \gamma = 0, v = 0\} = \{(q, v) | q \in q_d, v = 0\}$, i.e., agents stay still at the desired formation. \square

At last, we would like to point out that it is not trivial to extend the existing technique to take into account collision avoidance among the group or with static obstacles in the workspace. Actually, including another term containing position variables in the β function will

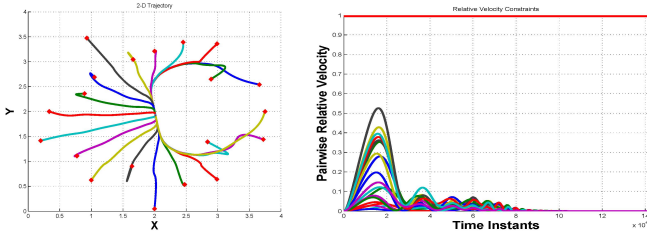


Fig. 1. Twenty agents perform con- Fig. 2. The norm of relative velocity sensus under the relative velocity con- between neighboring agents stays be- straint. The final consensus point is low the bound $\varepsilon_1 = 1$. $[2, 2]^T$.

change the invariant set of the closed-loop system and may introduce undesired local minimal. In particular, equation (7) would be altered by adding the gradient of β_1 with respect to q , namely $\nabla_q \beta_1$. As a result, $\nabla_q \phi_1 = 0$ dose not imply $\nabla_q \gamma = 0$, which is the desired equilibrium. As stated before, this challenging issue is a topic of future research.

V. APPLICATION TO FORMATION CONTROL

In this section we will apply the control law (9) and (14) to different formation objectives. Results from Theorem 2 and 4 are valid for generic goal potential function satisfying $\gamma = \gamma(q) \geq 0$, $\gamma(q) = 0$ and $\nabla_q \gamma = 0$ if and only if $q \in q_d$, where $q_d \in \mathbb{R}^{Nm}$ is the set of desired formations.

(1) Consensus

Consensus is one of the most fundamental formation stabilization problems that aims at aligning all agent positions at the same location [9], [7]. Even though many distributed control protocols have been proposed and studied, the same problem for second-order agents under relative motion constraints has not received significant consideration.

The goal potential is given by $\gamma_c \triangleq q^T L_m q$, similar to the Lyapunov function proposed in [19], where $L_m \in \mathbb{R}^{Nm \times Nm}$ and $L_m = L \otimes I_m$, where L is the standard Laplacian matrix [17] for the static underlying communication graph. Clearly, $\gamma_c = 0$ if q belongs to the desired consensus set, i.e., $q \in \text{span}\{\mathbf{1}\}^T$. Also $\nabla_q \gamma_c = L_m q = 0$ if $q \in \text{span}\{\mathbf{1}\}^T$. The corresponding control scheme under relative velocity constraints is given by substituting the gradient $\nabla_q \gamma_c = L_m q$ in (9). Similarly, the corresponding control scheme under constraints on the rate of change of the relative distance is given by substituting the gradient $\nabla_q \gamma_c = L_m q$ in (14).

We simulate a multi-agent system of twenty agents in 2-D configuration space in favor of better visualization. All agents, satisfying double integrator dynamics (1), start from twenty random positions within the circle with center $[2, 2]^T$ and radius 2. The underlying communication graph is a static line graph, namely agent i is connected with agent $i + 1$, $\forall i = 1, 2, \dots, 9$. The simulation step-size is set to 0.001s. Fig. 1 shows the full trajectory when the limit $\varepsilon_1 = 1$, $\rho = 0.1$ and $k = 0.5$, under the constraints (2). Fig. 2 illustrates the evolution of relative velocities $|(v_i - v_{i+1})^T (v_i - v_{i+1})|$, $\forall i = 1, 2, \dots, 9$, corresponding to nine communication edges in the network along with time. All start from zero and eventually converge to zero, while staying below the constraint given in (2) during all time. On the other hand, Fig. 3 shows the agent trajectories under the constraints (3) when $\varepsilon_2 = 1$, ρ, k remain the same. Fig. 4 illustrates the evolution of the rate of change of relative distance $|(q_i - q_{i+1})^T (v_i - v_{i+1})|$ among neighboring agents.

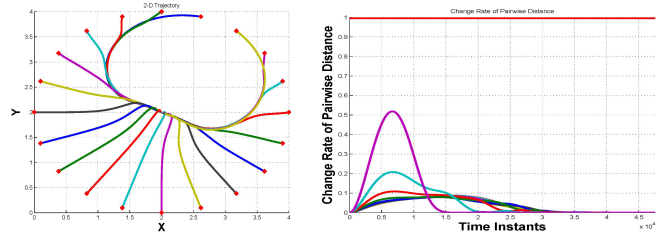


Fig. 3. Consensus from another initial Fig. 4. The change rate of pairwise state under constraint (3). The consen- sus point is $[2, 2]^T$.

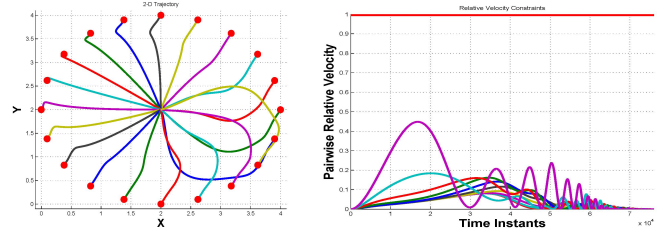


Fig. 5. Twenty agents start from Fig. 6. Under control law (9), the $[2, 2]^T$ and form a circle with radius norm of pairwise relative velocity stays 2 under constraint (2). below the bound $\varepsilon_1 = 1$.

(2) Relative Formation

The relative formation [21] is another cooperative task we take into account for multi-agent systems. In other words, the desired formation is specified by the set $q_d = \{q \mid q_i - q_j = c_{ij}, \forall (i, j) \in E\}$, where $c_{ij} \in \mathbb{R}^m$ is the desired relative position of neighboring agents. Correspondingly, the goal potential is defined as $\gamma_f(q) = \sum_{(i,j) \in E} \|q_i - q_j - c_{ij}\|^2$. Then its gradient with respect to q is computed by

$$\nabla_{q_i} \gamma_f = \sum_{j \in \mathcal{N}_i} (q_i - q_j - c_{ij}), \quad i \in \mathcal{V}. \quad (16)$$

It is easy to verify that $\gamma_f = 0$ when $q \in q_d$ and also $\nabla_q \gamma_f = 0$ when $q \in q_d$. The corresponding control scheme under relative velocity constraints is obtained by combining (9) and (16). The corresponding control scheme under constraints on the changing rate of relative distance is given by combining (14) and (16). The above control schemes are validated by computer simulations over the same twenty-agent system.

All agents start from the origin initially and the desired formation is the circle with center $[2, 2]^T$ and radius 2. This is to mimic the inverse consensus or dispersion [22]. The communication topology is now a static spanning tree the root of which is chosen randomly. In particular, c_{ij} in (16) is defined by $c_{ij} = 2 \cdot [\cos(\theta_0 \cdot i), \sin(\theta_0 \cdot i)]^T - 2 \cdot [\cos(\theta_0 \cdot j), \sin(\theta_0 \cdot j)]^T$, where $(i, j) \in E$ and $\theta_0 = \frac{\pi}{5}$. Fig. 5 shows the trajectory when the limit $\varepsilon_1 = 1$, $\rho = 0.1$ and $k = 1$, under the constraint (2). The evolution of relative velocities among neighboring agents $|(v_i - v_j)^T (v_i - v_j)|$ are shown in Fig. 6, with respect to time. All start from zero and eventually converge to zero, while staying below the constraint given in (2) during all time. On the other hand, Fig. 7 and 8 illustrate the agent trajectories under the constraint (3) and the evolution of the change rate of relative distances among neighboring agents. Red circles are used to denote the final positions.

We did not encounter any computational issues regarding the matrix inverse operation, which relies on the `inv` function in MATLAB. It is worth mentioning that the simulation time step needs to be kept small.

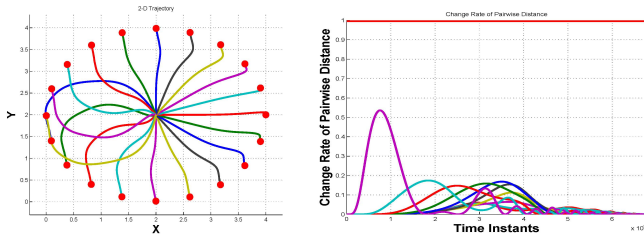


Fig. 7. Twenty agents form a circle under constraint (3). Fig. 8. Under control law (14), the norm of the change rate of pairwise relative distances is below $\varepsilon_2 = 1$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper we propose a novel method to control the relative agent motion in multi-agent systems as they stabilize to a desired configuration. We focused on the agents' relative velocities and the rate of change of their pairwise distances, and we employed constructs from classic navigation functions (NFs) to impose bounds on these quantities. The proposed controllers were analyzed theoretically in terms of their ability to stabilize the system at the desired configuration while respecting relative motion constraints, and verified by various computer simulations. The contribution of our proposed approach lies not only in its theoretical merit, but also in its potential impact in providing a powerful technique to control the connectivity of mobile networks and facilitate integration of path planning with network control. Our future work will involve extensions of this framework to more challenging settings, involving e.g. collision and obstacle avoidance. Emphasis will also be given to distributed implementations.

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