

Robust Estimation and Control for Heterogeneous Multi-agent Systems Based on Decentralized k -hop Prescribed Performance Observers

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Abstract—We propose decentralized k -hop Prescribed Performance State and Input Observers for heterogeneous multi-agent systems subject to bounded external disturbances. In the proposed input/state observer, each agent estimates the state and input of agents located two or more hops away by communicating only with 1-hop neighbors, while guaranteeing that the estimation errors satisfy predefined performance bounds. Theoretical analysis demonstrates that if a closed-loop controller with full state knowledge achieves the control objective and the estimation-based closed-loop system is set-Input to State Stable (set-ISS) with respect to the goal set, then the estimated states can be used to achieve the system objective with an arbitrarily small worst-case error governed by the accuracy of the states estimates. Simulation results are provided to validate the proposed approach.

I. INTRODUCTION

Heterogeneous multi-agent systems (MAS) consist of multiple autonomous agents with diverse dynamics, sensing, and computational capabilities that cooperate to achieve common objectives [1]. Unlike homogeneous MAS, heterogeneous systems combine complementary capabilities to accomplish complex missions with improved efficiency, robustness, and fault tolerance. However, this diversity increases coordination and estimation challenges, particularly under limited communication or sensing, where estimating states beyond immediate neighbors can enhance cooperative performance.

Research on distributed estimation and observer-based control for MAS has produced a variety of approaches [2]–[10]. Observer-based control schemes [2], [3] generally achieve consensus or tracking for specific system classes but are tailored to particular control objectives and lack theoretical guarantees when integrated with other controllers. Although distributed observers and consensus-based filters [5], [7], [10] enable local estimation through neighbor communication, they often assume homogeneous and disturbance-free settings, lack predefined estimation performance guarantees, and typically require each agent to reconstruct the full network state, thereby limiting scalability in large networks.

To overcome these limitations, our previous work [9] introduced a k -hop Distributed Prescribed Performance Observer

(k -hop DPPO) for homogeneous, disturbance-free MAS, enabling each agent to estimate the state of agents that are two or more hops away using only 1-hop communication. While this approach guarantees predefined estimation performance, it relies on network-dependent gain tuning and prior knowledge of input estimation error bounds, which are often difficult to determine in large-scale or complex networks and usually require centralized information. Moreover, it is formulated for homogeneous disturbance-free systems which limits its applicability to realistic heterogeneous scenarios.

Motivated by these challenges and inspired by Prescribed Performance Control (PPC) [11], we propose decentralized k -hop Prescribed Performance State and Input Observers (k -hop PPSO and k -hop PPIO) for heterogeneous MAS subject to bounded disturbances. The proposed observers enable each agent to estimate the state and input of agents located two to k hops away, while ensuring that the estimation errors satisfy predefined performance specifications set at the design stage. Unlike conventional distributed observers [7]–[10], the proposed framework is fully decentralized, relying solely on local (1-hop) communication without requiring any global network information, input bounds, or assumptions of homogeneous agent dynamics. This purely local interaction ensures scalability and facilitates deployment in large heterogeneous networks. Moreover, the prescribed performance formulation inherently guarantees robustness against bounded disturbances and model uncertainties, ensuring desired transient and steady-state behavior of the estimation errors. Beyond the observer design, we identify conditions under which the state observer can be simplified by removing the k -hop PPIO. Finally, we show that feedback controllers ensuring set-ISS stability of the closed-loop system can preserve their control objectives, with an arbitrarily small worst-case error, even when nonlocal state information is replaced by locally estimated counterparts.

The remainder of the paper is organized as follows. Section II introduces the notation, preliminaries, and problem formulation. Section III defines the disagreement vectors among the agents' estimates and derives their dynamics. Section IV presents the proposed k -hop PPSO and the conditions under which it can be simplified. Section V introduces the k -hop PPIO. Section VI describes the feedback control structure and establishes the conditions under which the k -hop estimation-based feedback controller guarantees convergence to the team objective. Section VII demonstrates the effectiveness of the proposed approach through simulation results, and Section VIII concludes the paper with final remarks and directions for future work.

This work was supported in part by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg (KAW) Foundation, Horizon Europe EIC project SymAware (101070802), the ERC LEAFHOUND Project, and the Swedish Research Council (VR).

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II. PRELIMINARIES AND PROBLEM SETTING

Notation: Denote by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{> 0}$ the sets of real, nonnegative, and positive real numbers, respectively. Let $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ be the n -dimensional Euclidean space and the set of real matrices with n rows and m columns. Let I_n and $\mathbf{1}_n$ be the identity matrix and the vector of ones of size n . For a set S , $|S|$, S^c and ∂S denote its cardinality, complement, and boundary, respectively; for $S = \{s_1, \dots, s_n\}$, let $\max_{i \in \{1, \dots, n\}} \{s_i\}$ and $\min_{i \in \{1, \dots, n\}} \{s_i\}$ be the maximum and minimum element in S . The Cartesian product of N sets $\{S_1, \dots, S_N\}$ is $\times_{i=1}^N S_i$. For symmetric $B \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ are the minimum and maximum eigenvalues of B . We use $\|B\|$ to denote the spectral norm of B , and $B \succ 0$ if B is positive definite. For $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^\top x}$. Let $\text{diag}(a_1, \dots, a_n)$ be the diagonal matrix with diagonal elements $a_1, \dots, a_n$ and let $\otimes$ be the Kronecker product. $f \in \mathcal{C}_1$ indicates that a function f is continuous differentiable. We define functions $\mathcal{K}$ and $\mathcal{KL}$ as follows: $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} : \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\}$; $\mathcal{KL} = \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} : \text{for each fixed } s, \beta(r, s) \in \mathcal{K} \text{ with respect to } r \text{ and, for each fixed } r > 0, \beta(r, s) \text{ is decreasing with respect to } s \text{ and } \lim_{s \rightarrow \infty} \beta(r, s) = 0\}$.

A. Multi-agent systems

Consider a heterogeneous MAS consisting of a set of N agents $\mathcal{V} := \{1, \dots, N\}$. Denote with $\mathbf{x}$ and $\mathbf{u}$ the global state and input and suppose each $i \in \mathcal{V}$ evolves as:

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(u_i(t)) + w_i(\mathbf{x}, t), \quad (1)$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and input of agent i , respectively, $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$ is the flow drift, $g_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ is an input function, and $w_i : \times_{i=1}^N \mathbb{R}^{n_i} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_i}$ represents external disturbances acting on i .

Let $n = \sum_{i=1}^N n_i$ and $m = \sum_{i=1}^N m_i$ be the dimensions of the global state and input. Then, $\mathbf{x} := [x_1^\top, \dots, x_N^\top]^\top \in \mathbb{R}^n$ and $\mathbf{u} := [u_1^\top, \dots, u_N^\top]^\top \in \mathbb{R}^m$.

Assumption 1: (i) $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$ is locally Lipschitz; (ii) $g_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ is measurable and essentially locally bounded; (iii) $w_i : \times_{i=1}^N \mathbb{R}^{n_i} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_i}$ is continuous and uniformly bounded in $\times_{i=1}^N \mathbb{R}^{n_i} \times \mathbb{R}_{\geq 0}$.

Assumption 2: One of the following holds: (i) g_i is bounded; (ii) $\dot{g}_i$ is bounded.

B. Communication graph

The interactions among agents are represented by an undirected graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of agents, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of communication links. An edge $(i, j) \in \mathcal{E}$ indicates that agents i and j can exchange information. A path between two agents $i, j \in \mathcal{V}$ is defined as a sequence of non-repeating edges connecting i to j . Then, a k -hop path is a path of length k connecting i and j .

For each agent $i \in \mathcal{V}$, let $\mathcal{N}_i^{k\text{-hop}}$ denote the set of k -hop neighbors of agent i , i.e., of all nodes $N_j^i \in \mathcal{V}$ from which the shortest path to i is a p -hop path with $2 \leq p \leq k$. The $\eta_i = |\mathcal{N}_i^{k\text{-hop}}|$ elements of this set are denoted as $\{N_1^i, \dots, N_{\eta_i}^i\}$,

where $N_j^i \in \mathcal{V}$, with $j \in \{1, \dots, \eta_i\}$, indicates the global index of the j -th k -hop neighbor of i . For simplicity, we use $\mathcal{N}_i$ to indicate the set of direct (1-hop) neighbors of agent i , excluding i in the presence of self-loops.

Assumption 3: $\mathcal{G}$ is a time invariant undirected graph and each $i \in \mathcal{V}$ knows its neighborhood $\mathcal{N}_i$ and $\mathcal{N}_i^{k\text{-hop}}$.

Assumption 4: Each agent $i \in \mathcal{V}$ has access and can relay, at each time instant, the state and input of its 1-hop neighbors $j \in \mathcal{N}_i$ to $\mathcal{N}_i$.

Assumption 3 is not restrictive, as distributed neighborhood discovery algorithms have been extensively studied in the sensor network literature [12]. Furthermore, Assumption 4 is satisfied in scenarios where each agent can measure the states of its 1-hop neighbors using onboard sensors and share this information with its neighbors. As it will be explained in Remark 1, Assumption 4 can be relaxed.

C. State and input estimates

Let $\mathbf{x}^i$ and $\mathbf{g}^i$ denote the stacked state vector and stacked input-function vector associated with agents $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$:

$$\mathbf{x}^i := [x_{N_1^i}^\top, \dots, x_{N_{\eta_i}^i}^\top]^\top, \quad \mathbf{g}^i := [g_{N_1^i}^\top, \dots, g_{N_{\eta_i}^i}^\top]^\top \quad (2)$$

and let $\hat{\mathbf{x}}^i := [\hat{x}_{N_1^i}^\top, \dots, \hat{x}_{N_{\eta_i}^i}^\top]^\top$ and $\hat{\mathbf{g}}^i := [\hat{g}_{N_1^i}^\top, \dots, \hat{g}_{N_{\eta_i}^i}^\top]^\top$ be their estimates available to agent i . Denote with $\tilde{\mathbf{x}}^i$ and $\tilde{\mathbf{g}}^i$ the corresponding estimation errors:

$$\tilde{\mathbf{x}}^i := [\tilde{x}_{N_1^i}^\top, \dots, \tilde{x}_{N_{\eta_i}^i}^\top]^\top, \quad \tilde{\mathbf{g}}^i := [\tilde{g}_{N_1^i}^\top, \dots, \tilde{g}_{N_{\eta_i}^i}^\top]^\top, \quad (3)$$

where $\tilde{x}_{N_j^i}^i := \hat{x}_{N_j^i}^i - x_{N_j^i}^i$ and $\tilde{g}_{N_j^i}^i := \hat{g}_{N_j^i}^i - g_{N_j^i}^i$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$.

Define $\mathbf{x}_i := \mathbf{1}_{\eta_i} \otimes x_i$, $\mathbf{g}_i := \mathbf{1}_{\eta_i} \otimes g_i(u_i)$, and let:

$$\hat{\mathbf{x}}_i := [\hat{x}_i^{N_1^\top}, \dots, \hat{x}_i^{N_{\eta_i}^\top}]^\top, \quad \hat{\mathbf{g}}_i := [\hat{g}_i^{N_1^\top}, \dots, \hat{g}_i^{N_{\eta_i}^\top}]^\top, \quad (4)$$

be the stacked vectors of the estimates of x_i and g_i computed by the k -hop neighbors of agent i . As in (3), indicate with $\tilde{\mathbf{x}}_i := \hat{\mathbf{x}}_i - \mathbf{x}_i$ and $\tilde{\mathbf{g}}_i := \hat{\mathbf{g}}_i - \mathbf{g}_i$ the estimation errors computed by each $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, i.e.:

$$\tilde{\mathbf{x}}_i := [\tilde{x}_i^{N_1^\top}, \dots, \tilde{x}_i^{N_{\eta_i}^\top}]^\top, \quad \tilde{\mathbf{g}}_i := [\tilde{g}_i^{N_1^\top}, \dots, \tilde{g}_i^{N_{\eta_i}^\top}]^\top, \quad (5)$$

with $\tilde{x}_i^{N_j^i} := \hat{x}_i^{N_j^i} - x_i$ and $\tilde{g}_i^{N_j^i} := \hat{g}_i^{N_j^i} - g_i$.

Without loss of generality, consider $n_i = 1$ for all $i \in \mathcal{V}$ in the following; extension to higher dimensions follows by appropriate use of the Kronecker product.

D. Problem formulation

For every agent $i \in \mathcal{V}$ and for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, let $\delta_i^{N_j^i} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ be a prescribed performance function used to capture the performance bounds for the estimation errors.

Definition 1: $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ is a prescribed performance function if for all $t \in \mathbb{R}_{\geq 0}$: (i) $\rho \in \mathcal{C}^1$; (ii) $\rho(t) \leq \bar{\rho}$ for some $\bar{\rho} < \infty$ and (iii) $|\dot{\rho}(t)| \leq \bar{\rho}_d$ for some $\bar{\rho}_d < \infty$.

One conventional choice is

$$\rho(t) = (\rho(0) - \rho(\infty))e^{-lt} + \rho(\infty), \quad (6)$$

where $\rho(0)$ and $\rho(\infty)$ denote the initial and steady-state values, and $l > 0$ specifies the decay rate.

Then, the goal of this work is formulated as follows.

Problem 1: Given the heterogeneous MAS in (1) communicating over a graph $\mathcal{G}$, and prescribed performance functions $\delta_i^{N_j^i}(t)$, design a decentralized k -hop observer such that $|\tilde{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ holds for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$. Additionally, given a control objective for the MAS, derive sufficient conditions ensuring that the observer-based controllers u_i achieve the objective with arbitrarily small error while using the estimated states.

III. DISAGREEMENT DYNAMICS

For each $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, define the disagreement term $\xi_i^{N_j^i}$ on the estimate of x_i performed by N_j^i as:

$$\xi_i^{N_j^i} := \sum_{l \in (\mathcal{N}_{N_j^i} \cap \mathcal{N}_i^{k\text{-hop}})} (\hat{x}_i^{N_j^i} - \hat{x}_i^l) + |\mathcal{N}_{N_j^i} \cap \mathcal{N}_i| (\hat{x}_i^{N_j^i} - x_i), \quad (7)$$

where $\xi_i^{N_j^i}$ captures how $\hat{x}_i^{N_j^i}$ differs from: (i) the true state information x_i shared by the agents $l \in \mathcal{N}_{N_j^i} \cap \mathcal{N}_i$ and (ii) the state estimate $\hat{x}_i^l$ shared by those agents $l \in \mathcal{N}_{N_j^i} \cap \mathcal{N}_i^{k\text{-hop}}$.

A. Disagreement vector and problem reformulation

By stacking $\xi_i^{N_j^i}$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, and using the state estimation error definition in (5), the **disagreement vector** defined as $\xi_i := [\xi_i^{N_1^i}, \dots, \xi_i^{N_{n_i}^i}]^\top$ can be expressed as:

$$\xi_i = (L_i^{\text{kc}} + H_i^{\text{kc}}) \tilde{x}_i = M_i^{\text{kc}} \tilde{x}_i, \quad (8)$$

where L_i^{kc} is the Laplacian matrix of the sub-graph $\mathcal{G}_i := (\mathcal{N}_i^{k\text{-hop}}, \mathcal{E}_i)$ induced by the k -hop neighbors of agent i , with $\mathcal{E}_i := \{(p, q) \in \mathcal{E} : \{p, q\} \subseteq \mathcal{N}_i^{k\text{-hop}}\}$; $H_i^{\text{kc}} := \text{diag}(|\mathcal{N}_{N_1^i} \cap \mathcal{N}_i|, \dots, |\mathcal{N}_{N_{n_i}^i} \cap \mathcal{N}_i|) \in \mathbb{R}^{n_i \times n_i}$ and $M_i^{\text{kc}} := L_i^{\text{kc}} + H_i^{\text{kc}}$.

Lemma 1 ([8]): If $\mathcal{G}$ is connected, then $M_i^{\text{kc}} \succ 0$ for all $i \in \mathcal{V}$ with $\mathcal{N}_i^{k\text{-hop}} \neq \emptyset$.

Remark 1: If Assumption 4 fails, (7) is not locally computable for those N_j^i with $|\mathcal{N}_{N_j^i} \cap \mathcal{N}_i| \neq 0$, as they lack access to x_i without 2-hop information flow. Yet, local computability can be restored, while preserving $M_i^{\text{kc}} \succ 0$, by augmenting the k -hop neighborhood with 1-hop neighbors. Then, L_i^{kc} remains the Laplacian matrix of the subgraph induced by the k -hop neighbors of agent i , and $H_i^{\text{kc}} := \text{diag}(h_i^{N_1^i}, \dots, h_i^{N_{n_i}^i}) \in \mathbb{R}^{n_i \times n_i}$, where $h_i^{N_j^i} = 1$ if $N_j^i \in \mathcal{N}_i$.

Lemma 2: Let $\rho(t) := [\rho_1(t), \dots, \rho_m(t)]^\top$, where each $\rho_i(t)$ is a prescribed performance function. Then, $\|\rho(t)\|$ is also a prescribed performance function as per Definition 1.

Proof: (i) $\rho_i \in C^1$ for all $i \in \{1, \dots, m\}$ by definition, hence $\rho \in C^1$. The Euclidean norm is smooth on $\mathbb{R}^{n_i} \setminus \{0\}$. Furthermore, $\rho(t) \neq 0$ for all $t \geq 0$. Thus, $\|\rho\| \in C^1$. (ii) From Definition 1, $0 < \rho_i(t) \leq \bar{\rho}_i$ for some $\bar{\rho}_i < \infty$. Hence, $0 < \|\rho(t)\| \leq \bar{\rho}$, with $\bar{\rho} = \sqrt{\sum_{i=1}^m \bar{\rho}_i^2}$. (iii) Differentiating the Euclidean norm gives $\frac{d\|\rho(t)\|}{dt} = \frac{\rho^\top(t) \dot{\rho}(t)}{\|\rho(t)\|}$. Since $\|\rho(t)\| > 0$, $0 < \rho_i(t) < \infty$, and $|\dot{\rho}_i(t)| < \infty$ for all $i \in \{1, \dots, m\}$, it follows that $\frac{d\|\rho(t)\|}{dt}$ is upper bounded. ■

Given the validity of Lemma 1, M_i^{kc} is always invertible and $\tilde{x}_i = M_i^{\text{kc}-1} \xi_i$. Thus, $\|\tilde{x}_i\| \leq \|M_i^{\text{kc}-1}\| \|\xi_i\|$, and $\|\tilde{x}_i(t)\| \leq \lambda_{\min}^{-1}(M_i^{\text{kc}}) \|\xi_i(t)\|$ holds. Since $|\tilde{x}_i^{N_j^i}(t)| \leq \|\tilde{x}_i(t)\|$, to satisfy Problem 1 and ensure $|\tilde{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, it suffices to impose $\|\xi_i(t)\| < \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, n_i\}} \{\delta_i^{N_j^i}(t)\}$ by constraining $|\xi_i^{N_j^i}| < \rho_i^{N_j^i}(t)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, where each $\rho_i^{N_j^i}(t)$ is a prescribed performance function selected such that $\rho_i(t) := [\rho_i^{N_1^i}(t), \dots, \rho_i^{N_{n_i}^i}(t)]^\top$ satisfies $\|\rho_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, n_i\}} \{\delta_i^{N_j^i}(t)\}$. As a result, Problem 1 can be partially reformulated as:

Problem 2: Given the heterogeneous MAS in (1) communicating over a graph $\mathcal{G}$, and prescribed performance functions $\delta_i^{N_j^i}(t)$, design a decentralized k -hop observer such that $|\xi_i^{N_j^i}| < \rho_i^{N_j^i}(t)$ holds for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, where all $\rho_i^{N_j^i}$ are prescribed performance functions satisfying $\|\rho_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, n_i\}} \{\delta_i^{N_j^i}(t)\}$.

Remark 2: Since all $\rho_i^{N_j^i}(t)$ and $\delta_i^{N_j^i}(t)$ are design choices, we can impose desired behavior on every $\tilde{x}_i^{N_j^i}(t)$.

For $n_i > 1$, similar reasoning applies componentwise to $\tilde{x}_i^{N_j^i}(t)$, allowing desired performance to be imposed by constraining every component $\xi_{i,l}^{N_j^i}$, $l \in \{1, \dots, n_i\}$, of $\xi_i^{N_j^i}$.

B. Prescribed Performance Observer

Inspired by the PPC literature [11], we design a k -hop Prescribed Performance Observer (PPO) to impose

$$-\rho_i^{N_j^i}(t) < \xi_i^{N_j^i}(t) < \rho_i^{N_j^i}(t) \quad (9)$$

for all $t \in \mathbb{R}_{\geq 0}$, $i \in \mathcal{V}$, and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, where $\rho_i^{N_j^i}(t)$ is a prescribed performance function, defined as in (6), satisfying $\rho_i^{N_j^i}(0) \geq \rho_i^{N_j^i}(\infty) > 0$ and $\rho_i^{N_j^i}(0) > |\xi_i^{N_j^i}(0)|$. Given the initial condition $\rho_i^{N_j^i}(0)$, $\rho_i^{N_j^i}(\infty) = \lim_{t \rightarrow \infty} \rho_i^{N_j^i}(t)$ represents the maximum allowable value at steady state.

Let $e_i^{N_j^i}(t) \in (-1, 1)$ denote the normalization of $\xi_i^{N_j^i}(t)$ with respect to $\rho_i^{N_j^i}(t)$, i.e., $e_i^{N_j^i}(t) := \rho_i^{N_j^i}(t)^{-1} \xi_i^{N_j^i}(t)$, and let $T : (-1, 1) \rightarrow \mathbb{R}$ be a strictly increasing transformation satisfying $T(0) = 0$. For all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, define the transformed normalized disagreement as:

$$\epsilon_i^{N_j^i}(t) := T(e_i^{N_j^i}) = T(\rho_i^{N_j^i}(t)^{-1} \xi_i^{N_j^i}(t)). \quad (10)$$

In this work, we select $T(e) = \ln(\frac{1+e}{1-e})$, which has a strictly positive derivative $J_T(e) = \frac{2}{1-e^2}$. Then, by defining $\epsilon_i := [\epsilon_i^{N_1^i}, \dots, \epsilon_i^{N_{n_i}^i}]^\top$, its dynamics result into:

$$\dot{\epsilon}_i = J_i P_i^{-1} (\dot{\xi}_i - \dot{P}_i \epsilon_i), \quad (11)$$

where $J_i := \text{diag}(J_T(e_i^{N_1^i}), \dots, J_T(e_i^{N_{n_i}^i}))$, $P_i := \text{diag}(\rho_i^{N_1^i}, \dots, \rho_i^{N_{n_i}^i})$, $\dot{P}_i := \text{diag}(\dot{\rho}_i^{N_1^i}, \dots, \dot{\rho}_i^{N_{n_i}^i})$, $e_i :=$

$$P_i^{-1}\xi_i \text{ and } \dot{\xi}_i := [\dot{\xi}_i^{N_i^1}, \dots, \dot{\xi}_i^{N_i^{\eta_i}}]^\top.$$

Remark 3: From (10), if ϵ_i remains bounded for all $t \in \mathbb{R}_{\geq 0}$, then $e_i^{N_j^i} \in (-1, 1)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$. Hence, $\xi_i^{N_j^i}$ evolves according to (9) for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$.

IV. k -HOP PRESCRIBED PERFORMANCE STATE OBSERVER

In this section, a decentralized k -hop Prescribed Performance State Observer (k -hop PPSO) is introduced to solve Problem 2. To this end, assume $\hat{x}_i^{N_j^i}$ is updated as:

$$\dot{\hat{x}}_i^{N_j^i} = f_i(\hat{x}_i^{N_j^i}) + \hat{g}_i^{N_j^i} - \rho_i^{N_j^i}(t)^{-1} J_T(e_i^{N_j^i}) \epsilon_i^{N_j^i}(t), \quad (12)$$

where $\hat{g}_i^{N_j^i}$ is the estimate of $g_i(u_i(t))$ computed by N_j^i , and $\rho_i^{N_j^i}(t)$, $J_T(e_i^{N_j^i})$ and $\epsilon_i^{N_j^i}$ are defined as in Section III-B.

Remark 4: Note that $\epsilon_i^{N_j^i}(t)$, and consequently $\hat{x}_i^{N_j^i}(t)$, is computed exclusively based on information received from the neighbors of agent N_j^i . Hence, provided that each $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ possesses knowledge of the structure of f_i , the proposed observer operates in a fully decentralized manner.

By stacking $\dot{\hat{x}}_i^{N_j^i}$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, the dynamics of $\hat{x}_i$, defined as in (4), becomes:

$$\dot{\hat{x}}_i = f_i(\hat{x}_i) + \hat{g}_i - P_i^{-1} J_i \epsilon_i, \quad (13)$$

where $f_i(\hat{x}_i) := [f_i(\hat{x}_i^{N_i^1}), \dots, f_i(\hat{x}_i^{N_i^{\eta_i}})]^\top$, $\hat{g}_i := [\hat{g}_i^{N_i^1}, \dots, \hat{g}_i^{N_i^{\eta_i}}]^\top$ and P_i , J_i , and ϵ_i are defined as in (11).

Assume each agent runs a convergent input observer guaranteeing $\|\tilde{g}_i(t)\| \leq \delta_i^{\tilde{g}}$ to hold, with $\delta_i^{\tilde{g}} < \infty$. Then:

Theorem 1: Consider a heterogeneous MAS (1) with connected graph $\mathcal{G}$ and decentralized state observers (12). For all $i \in \mathcal{V}$, assume that $\|\tilde{g}_i(t)\|$ is upper bounded by some $\delta_i^{\tilde{g}} \in \mathbb{R}_{\geq 0}$. Then, for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $i \in \mathcal{V}$, $|\tilde{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ is satisfied provided that $|\xi_i^{N_j^i}(0)| < \rho_i^{N_j^i}(0)$ holds, and $\rho_i^{N_j^i}(t)$ is designed so that $\|\rho_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, \eta_i\}} \{\delta_i^{N_j^i}(t)\}$.

Proof: Consider agent $i \in \mathcal{V}$. Under Assumption 3, $\mathcal{G}$ is a time invariant graph. Thus, M_i^{kc} is constant, $\dot{\xi}_i = M_i^{\text{kc}} \dot{\hat{x}}_i$ from (8), and (11) can be rewritten as:

$$\dot{\epsilon}_i = J_i P_i^{-1} (M_i^{\text{kc}} \dot{\hat{x}}_i - \dot{P}_i e_i). \quad (14)$$

From (1), (5) and (13), $\dot{\hat{x}}_i = f_i(\hat{x}_i) - f_i(x_i) + \tilde{g}_i - P_i^{-1} J_i \epsilon_i - w_i$ holds with $f_i(x_i) := 1_{\eta_i} \otimes f_i(x_i)$ and $w_i := 1_{\eta_i} \otimes w_i(x, t)$. Consider $V = \frac{1}{2} \epsilon_i^\top \epsilon_i$. By replacing (14) and $\dot{\hat{x}}_i$, $\dot{V} = \epsilon_i^\top \dot{\epsilon}_i$ results into:

$$\dot{V} = -\epsilon_i^\top J_i P_i^{-1} M_i^{\text{kc}} P_i^{-1} J_i \epsilon_i + \epsilon_i^\top J_i P_i^{-1} \left\{ M_i^{\text{kc}} [\tilde{g}_i + f_i(\hat{x}_i) - f_i(x_i) - w_i] - \dot{P}_i e_i \right\}. \quad (15)$$

Since $M_i^{\text{kc}} \succ 0$ from Lemma 1, $-\epsilon_i^\top J_i P_i^{-1} M_i^{\text{kc}} P_i^{-1} J_i \epsilon_i \leq -\lambda_{\min}(M_i^{\text{kc}}) \alpha_J \alpha_\rho \epsilon_i^\top \epsilon_i$ holds with $\alpha_J = \min_{N_j^i \in \mathcal{N}_i^{k\text{-hop}}} \left\{ \min_{e_i^{N_j^i} \in (-1, 1)} J_T(e_i^{N_j^i})^2 \right\} = 4$ and $\alpha_\rho = (\max_{N_j^i \in \mathcal{N}_i^{k\text{-hop}}} \{\max_{t \in \mathbb{R}_{\geq 0}} \rho_i^{N_j^i}(t)^2\})^{-1}$. From

Definition 1, there exists $\bar{\rho}_i^{N_j^i} < \infty$ such that $\rho_i^{N_j^i}(t) \leq \bar{\rho}_i^{N_j^i}$.

Thus, $\alpha_\rho \geq (\max_{N_j^i \in \mathcal{N}_i^{k\text{-hop}}} \{(\bar{\rho}_i^{N_j^i})^2\})^{-1}$. By summing and subtracting $\zeta \|P_i^{-1} J_i \epsilon_i\|^2$ for some $0 < \zeta < \lambda_{\min}(M_i^{\text{kc}})$, (15) can be upper bounded as $\dot{V} \leq -(\lambda_{\min}(M_i^{\text{kc}}) - \zeta) \alpha_J \alpha_\rho \|\epsilon_i\|^2 + \epsilon_i^\top J_i P_i^{-1} b(t) - \zeta \|P_i^{-1} J_i \epsilon_i\|^2$, where $b(t) = M_i^{\text{kc}} [\tilde{g}_i + f_i(\hat{x}_i) - f_i(x_i) - w_i] - \dot{P}_i e_i$. Since $\epsilon_i^\top J_i P_i^{-1} b(t) - \zeta \|P_i^{-1} J_i \epsilon_i\|^2$ resemble terms of $\|\sqrt{\zeta} P_i^{-1} J_i \epsilon_i - \frac{1}{2\sqrt{\zeta}} b(t)\|^2$, $\dot{V} \leq -(\lambda_{\min}(M_i^{\text{kc}}) - \zeta) \alpha_J \alpha_\rho \|\epsilon_i\|^2 + \frac{1}{4\zeta} b^\top(t) b(t)$ holds and $\dot{V}$ can be written as

$$\dot{V} \leq -\kappa V + b(t), \quad (16)$$

with $\kappa := 2(\lambda_{\min}(M_i^{\text{kc}}) - \zeta) \alpha_J \alpha_\rho$ and $b(t) := \frac{1}{4\zeta} \{\lambda_{\max}(M_i^{\text{kc}}) [\|f_i(\hat{x}_i) - f_i(x_i)\| + \|w_i\| + \|\tilde{g}_i\|] + \|\dot{P}_i e_i\|\}^2$.

Let's check whether $b(t)$ admits an upper bound $\bar{b}(t)$. Define with $\tilde{\mathcal{X}}_i(t) := \{\tilde{x}_i \in \mathbb{R}^{\eta_i} \mid -1_{\eta_i} < e_i = P_i^{-1} \xi_i < 1_{\eta_i}\}$ the set containing the $\tilde{x}_i$ for which $\xi_i^{N_j^i}(t)$ satisfy (9) for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$. As introduced in Section III-A, $|\tilde{x}_i^{N_j^i}(t)| \leq \|\tilde{x}_i(t)\| \leq \lambda_{\min}^{-1}(M_i^{\text{kc}}) \|\xi_i(t)\|$ is valid by construction. Moreover, $\|\xi_i(t)\| < \|P_i 1_{\eta_i}\|$ holds in $\tilde{\mathcal{X}}_i(t)$, with $\|P_i 1_{\eta_i}\|$ bounded as a direct result of Definition 1. Thus, since $\|\tilde{x}_i(t)\| < \lambda_{\min}^{-1}(M_i^{\text{kc}}) \|P_i 1_{\eta_i}\|$ is valid in $\tilde{\mathcal{X}}_i(t)$, $\tilde{\mathcal{X}}_i(t)$ is a bounded open set. Being f a Lipschitz continuous function, $\|f_i(\hat{x}_i) - f_i(x_i)\|$ is bounded in $\tilde{\mathcal{X}}_i(t)$. Moreover, since $\dot{P}_i e_i$ is a column vector with $\dot{\rho}_i^{N_j^i} e_i^{N_j^i}$ as entries, and $|\dot{\rho}_i^{N_j^i}(t) e_i^{N_j^i}| < |\dot{\rho}_i^{N_j^i}(t)| < \bar{\rho}_{d,i}^{N_j^i}$ holds with $\bar{\rho}_{d,i}^{N_j^i} < \infty$ from Definition 1, also $\|\dot{P}_i e_i\|$ is bounded. Then, since $\|\tilde{g}_i\|$ and $\|w_i\|$ are bounded by assumption, an upper bound $\bar{b}(t) < \infty$ on $b(t)$ exists for all $\tilde{x}_i \in \tilde{\mathcal{X}}_i(t)$.

Inspired by [13, Thm. 22], to prove the invariance of $\tilde{\mathcal{X}}_i(t)$, we introduce $S(e_i) = 1 - e^{-V(e_i)}$. Note that S satisfies: (i) $S(e_i) \in [0, 1)$ for all $e_i \in \mathcal{D}$ with $\mathcal{D} = \times_{j=1}^{\eta_i} (-1, 1)$, and (ii) $S(e_i) \rightarrow 1$ as $e_i \rightarrow \partial \mathcal{D}$. Therefore, studying the boundedness of $\epsilon_i(e_i)$ through the one of V reduces to proving that $S(e_i) < 1$ holds for all t .

By replacing (16) and $V(e_i) = -\ln(1 - S(e_i))$ in $\dot{S}(e_i) = \dot{V}(e_i)(1 - S(e_i))$, $\dot{S}(e_i) \leq -\kappa(1 - S(e_i)) \left(-\frac{1}{\kappa} b(t) - \ln(1 - S(e_i)) \right)$ is obtained. Since $\kappa > 0$ and $1 - S(e_i) > 0$, verifying $\dot{S}(e_i) \leq 0$ reduces to checking when $-\frac{1}{\kappa} b(t) - \ln(1 - S(e_i)) \geq 0$. Note that $-\frac{1}{\kappa} b(t) - \ln(1 - S(e_i)) \geq 0$ for all $e_i \in \Omega_e^c$, where $\Omega_e := \{e_i \in \mathcal{D} \mid S(e_i) < 1 - e^{-\frac{b(t)}{\kappa}}\}$, and that $-\frac{1}{\kappa} b(t) - \ln(1 - S(e_i)) = 0$ holds for $e_i \in \partial \Omega_e$. Thus, $\dot{S}(e_i) \leq 0$ for $e_i \in \Omega_e^c$, with $\dot{S}(e_i) = 0$ iff $e_i \in \partial \Omega_e$.

Since $|\xi_i^{N_j^i}(0)| < \rho_i^{N_j^i}(0)$ by assumption, $e_i^{N_j^i}(0) \in (-1, 1)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, and $S(e_i(0)) < 1$ holds. Then, since $e^{-\frac{b(t)}{\kappa}} \geq e^{-\frac{b(t)}{\kappa}} > 0$ by definition, $S(e_i) < 1$ is preserved for all $t \in \mathbb{R}_{\geq 0}$ regardless of whether $e_i(0) \in \Omega_e$ or $e_i(0) \in \Omega_e^c$. From $S(e_i) < 1$, boundedness of $V(e_i)$ follows. Thus, ϵ_i is bounded, and (9) holds. If $\rho_i^{N_j^i}(t)$ is designed so that $\|\rho_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, \eta_i\}} \{\delta_i^{N_j^i}(t)\}$, $|\tilde{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ is guaranteed by construction for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ as explained in Section III-A. ■

Theorem 1 assumes the existence of a convergent input observer ensuring $\|\tilde{g}_i(t)\| \leq \delta_i^{\tilde{g}} < \infty$. To relax this assumption, a k -hop Prescribed Performance Input Observer will be proposed in Section V.

Remark 5: Under Assumption 2-(i), prescribed performance can be guaranteed following Theorem 1 while treating g_i as a bounded disturbance. Thus, the input observer can be omitted, and (12) reduces to $\hat{x}_i = f_i(\hat{x}_i) - P_i^{-1} J_i \epsilon_i$. In this framework, g_i can be extended to $g_i(x_i, u_i)$.

Under further assumptions on $g_i(u_i)$, (12) can be modified to avoid the need of $f_i(\hat{x}_i^{N_j^i})$.

Theorem 2: Consider a heterogeneous MAS (1) with connected graph $\mathcal{G}$ and decentralized state observers $\hat{x}_i^{N_j^i} = \hat{g}_i^{N_j^i} - \rho_i^{N_j^i}(t)^{-1} J_T(e_i^{N_j^i}) \epsilon_i^{N_j^i}(t)$ for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$. For all $i \in \mathcal{V}$, assume $\|\tilde{g}_i(t)\|$ is upper bounded by some $\delta_i^{\tilde{g}} \in \mathbb{R}_{\geq 0}$ and that $g_i(u_i)$ is designed such that (1) evolves in a bounded set $\mathbb{X}_i \subset \mathbb{R}^{n_i}$. Then, for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and all $i \in \mathcal{V}$, $|\tilde{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ is satisfied provided that $|\xi_i^{N_j^i}(0)| < \rho_i^{N_j^i}(0)$ holds and $\rho_i^{N_j^i}(t)$ is designed so that $\|\rho_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, \eta_i\}} \{\delta_i^{N_j^i}(t)\}$.

Proof: Consider $V = \frac{1}{2} \epsilon_i^\top \epsilon_i$. Following similar steps to those in Theorem 1, $\dot{V}$ can be upper bounded as $\dot{V} \leq -\kappa V + \mathbf{b}(t)$, with $\mathbf{b}(t) = \frac{1}{4\zeta} \{\lambda_{\max}(M_i^{\text{kc}}) [\|\mathbf{f}_i(\mathbf{x}_i)\| + \|\mathbf{w}_i\| + \|\tilde{g}_i\|] + \|\dot{P}_i e_i\|^2\}$. Then, since f_i is Lipschitz, and $g_i(u_i)$ ensures the agent dynamics (1) to evolve in $\mathbb{X}_i \subset \mathbb{R}^{n_i}$, $\|\mathbf{f}_i(\mathbf{x}_i)\|$ is bounded, and so is $\mathbf{b}(t)$. Thus, validity of Theorem 2 follows by introducing $S(e_i)$ as in Theorem 1. ■

V. k -HOP PRESCRIBED PERFORMANCE INPUT OBSERVER

While the results of the previous sections hold for any k -hop input observer, e.g. the one in [8], here we propose a decentralized k -hop Prescribed Performance Input Observer (k -hop PPIO) that estimates $g_i(u_i(t))$ while guaranteeing $|\tilde{g}_i^{N_j^i}(t)| < \theta_i^{N_j^i}(t)$ for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, where $\theta_i^{N_j^i}(t)$ is a prescribed performance function as in (6).

A. Input disagreement dynamics

Following the k -hop PPSO design in Sections III-IV, define the input disagreement term on the estimate of g_i performed by N_j^i as:

$$\mu_i^{N_j^i} := \sum_{l \in (\mathcal{N}_{N_j^i} \cap \mathcal{N}_i^{k\text{-hop}})} (\hat{g}_i^{N_j^i} - \hat{g}_l^l) + |\mathcal{N}_{N_j^i} \cap \mathcal{N}_i| (\hat{g}_i^{N_j^i} - g_i). \quad (17)$$

By stacking $\mu_i^{N_j^i}$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, a relationship similar to (8) holds for $\boldsymbol{\mu}_i := [\mu_i^{N_1^i}, \dots, \mu_i^{N_{\eta_i}^i}]^\top$, i.e.:

$$\boldsymbol{\mu}_i = M_i^{\text{kc}} \tilde{\mathbf{g}}_i. \quad (18)$$

Then, similarly to Section III-A, to guarantee $|\tilde{g}_i^{N_j^i}(t)| < \theta_i^{N_j^i}(t)$ for all $i \in \mathcal{V}$ and $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, it suffices to enforce $|\mu_i^{N_j^i}| < \omega_i^{N_j^i}(t)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, where each $\omega_i^{N_j^i}(t)$ is a prescribed performance function designed so that

$$\|\omega_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, \eta_i\}} \{\theta_i^{N_j^i}(t)\} \text{ holds with } \omega_i(t) := [\omega_i^{N_1^i}(t), \dots, \omega_i^{N_{\eta_i}^i}(t)]^\top.$$

Denote with $q_i^{N_j^i}(t) \in (-1, 1)$ the normalization of $\mu_i^{N_j^i}(t)$ with respect to $\omega_i^{N_j^i}$, i.e., $q_i^{N_j^i}(t) := \omega_i^{N_j^i}(t)^{-1} \mu_i^{N_j^i}(t)$, and let $\nu_i^{N_j^i}(t) := T(q_i^{N_j^i}) = T(\omega_i^{N_j^i}(t)^{-1} \mu_i^{N_j^i}(t))$ denote the transformed input disagreement. By defining the transformed input disagreement vector $\boldsymbol{\nu}_i := [\nu_i^{N_1^i}, \dots, \nu_i^{N_{\eta_i}^i}]^\top$,

$$\dot{\boldsymbol{\nu}}_i = \mathbf{J}_i^g \boldsymbol{\Omega}_i^{-1} (\boldsymbol{\mu}_i - \boldsymbol{\Omega}_i \mathbf{q}_i) \quad (19)$$

holds with $\mathbf{J}_i^g := \text{diag}(J_T(q_i^{N_1^i}), \dots, J_T(q_i^{N_{\eta_i}^i}))$, $\boldsymbol{\Omega}_i := \text{diag}(\omega_i^{N_1^i}, \dots, \omega_i^{N_{\eta_i}^i})$, $\dot{\boldsymbol{\Omega}}_i := \text{diag}(\dot{\omega}_i^{N_1^i}, \dots, \dot{\omega}_i^{N_{\eta_i}^i})$, $\mathbf{q}_i := \boldsymbol{\Omega}_i^{-1} \boldsymbol{\mu}_i$ and $\dot{\boldsymbol{\mu}}_i := [\dot{\mu}_i^{N_1^i}, \dots, \dot{\mu}_i^{N_{\eta_i}^i}]^\top$.

Remark 6: Bounded $\nu_i(t)$ implies $q_i^{N_j^i}(t) \in (-1, 1)$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $t \geq 0$, hence $|\mu_i^{N_j^i}(t)| < \omega_i^{N_j^i}(t)$.

B. k -hop Prescribe Performance Input Observer design

Let each agent $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ update its estimate $\hat{g}_i^{N_j^i}$ as:

$$\dot{\hat{g}}_i^{N_j^i} = -\omega_i^{N_j^i}(t)^{-1} J_T(q_i^{N_j^i}) \nu_i^{N_j^i}(t). \quad (20)$$

By stacking $\dot{\hat{g}}_i^{N_j^i}$ for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, the dynamics of $\hat{\mathbf{g}}_i$, defined as in (4), becomes:

$$\dot{\hat{\mathbf{g}}}_i = -\boldsymbol{\Omega}_i^{-1} \mathbf{J}_i^g \boldsymbol{\nu}_i, \quad (21)$$

where $\boldsymbol{\Omega}_i$, $\mathbf{J}_i^g$, and $\boldsymbol{\nu}_i$ are defined as in (19).

Theorem 3: Consider a heterogeneous MAS (1) with connected graph $\mathcal{G}$ and decentralized input observers (20). Under Assumption 2-(ii), $|\tilde{g}_i^{N_j^i}(t)| < \theta_i^{N_j^i}(t)$ is satisfied for all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $i \in \mathcal{V}$, provided that $|\mu_i^{N_j^i}(0)| < \omega_i^{N_j^i}(0)$ holds and $\omega_i^{N_j^i}(t)$ is designed so that $\|\omega_i(t)\| \leq \lambda_{\min}(M_i^{\text{kc}}) \min_{j \in \{1, \dots, \eta_i\}} \{\theta_i^{N_j^i}(t)\}$.

Proof: The proof follows similar arguments to those of Theorem 1. Consider $V = \frac{1}{2} \boldsymbol{\nu}_i^\top \boldsymbol{\nu}_i$. From (18), (19), (21) and the definition of $\tilde{\mathbf{g}}_i$ in (5), $\dot{V}$ becomes $\dot{V} = \boldsymbol{\nu}_i^\top \mathbf{J}_i^g \boldsymbol{\Omega}_i^{-1} \{M_i^{\text{kc}} [-\dot{\tilde{\mathbf{g}}}_i - \boldsymbol{\Omega}_i^{-1} \mathbf{J}_i^g \boldsymbol{\nu}_i] - \dot{\boldsymbol{\Omega}}_i \mathbf{q}_i\}$. By adding and subtracting $\zeta \|\boldsymbol{\Omega}_i^{-1} \mathbf{J}_i^g \boldsymbol{\nu}_i\|^2$ for some $0 < \zeta < \lambda_{\min}(M_i^{\text{kc}})$, $\dot{V} \leq -\kappa V + \mathbf{b}(t)$ holds with $\kappa := 2(\lambda_{\min}(M_i^{\text{kc}}) - \zeta) \alpha_J \alpha_\omega$, $\mathbf{b}(t) := \frac{1}{4\zeta} \{\lambda_{\max}(M_i^{\text{kc}}) \|\tilde{\mathbf{g}}_i\| + \|\dot{\boldsymbol{\Omega}}_i \mathbf{q}_i\|^2\}$, $\alpha_J = 4$ and $\alpha_\omega = (\max_{N_j^i \in \mathcal{N}_i^{k\text{-hop}}} \{(\bar{\omega}_i^{N_j^i})^2\})^{-1}$, where $\omega_i^{N_j^i}(t) \leq \bar{\omega}_i^{N_j^i}$ holds according to Definition 1. Since $\|\tilde{\mathbf{g}}_i\|$ is bounded by Assumption 2-(ii), and $\|\dot{\boldsymbol{\Omega}}_i \mathbf{q}_i\|$ is bounded for similar reason as $\|\dot{P}_i e_i\|$ in the proof of Theorem 1, $\mathbf{b}(t)$ is bounded. By introducing $S(\mathbf{q}_i) = 1 - e^{-V(\mathbf{q}_i)}$, the proof of Theorem 3 follows the one of Theorem 1. Thus, due to space limitations, the remaining part is omitted here. ■

Theorem 3 ensures $|\tilde{g}_i^{N_j^i}(t)| < \theta_i^{N_j^i}(t)$, which guarantees the boundedness of $\|\tilde{\mathbf{g}}_i(t)\|$ required by Theorem 1.

Remark 7: Note that, as in Theorem 1, estimation convergence is guaranteed regardless of the upper bound on $\dot{g}_i$.

VI. k -HOP PPSO-BASED CONTROLLER

The k -hop PPSO allows each agent $i \in \mathcal{V}$ to estimate the state of all $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, enabling closed-loop control laws that use local estimates to accomplish the team objective.

Consider the vectorized form of the dynamics in (1), i.e., $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{u}(t)) + \mathbf{w}(\mathbf{x}, t)$, where $\mathbf{x}(t) := [x_1(t), \dots, x_N(t)]^\top$, $\mathbf{f}(\mathbf{x}(t)) := [f_1(x_1), \dots, f_N(x_N)]^\top$, $\mathbf{g}(\mathbf{u}(t)) := [g_1(u_1), \dots, g_N(u_N)]^\top$, $\mathbf{w}(\mathbf{x}, t) := [w_1(\mathbf{x}, t), \dots, w_N(\mathbf{x}, t)]^\top$, and $\mathbf{u}(t) := [u_1(t), \dots, u_N(t)]^\top$ represents a nonlinear control input of the form:

$$\mathbf{u} = \psi(\mathbf{x}) := [\psi_1(\bar{\mathbf{x}}_1, \mathbf{x}^1), \dots, \psi_N(\bar{\mathbf{x}}_N, \mathbf{x}^N)]^\top, \quad (22)$$

where, for each $i \in \mathcal{V}$, $\mathbf{x}^i$ is defined as in (2), and $\bar{\mathbf{x}}_i$ contains the state information of agent i and of all $j \in \mathcal{N}_i$.

Since $\mathbf{x}^i$ is not locally available, the controller in (22) is implemented using $\hat{\mathbf{x}}^i$ for all $i \in \mathcal{V}$, i.e.:

$$\mathbf{u} = \psi(\bar{\mathbf{x}}, \hat{\mathbf{x}}) := [\psi_1(\bar{\mathbf{x}}_1, \hat{\mathbf{x}}^1), \dots, \psi_N(\bar{\mathbf{x}}_N, \hat{\mathbf{x}}^N)]^\top. \quad (23)$$

Noting that $\bar{\mathbf{x}}_i$ consists of components of $\mathbf{x}$ and that $\hat{\mathbf{x}}^i = \mathbf{x}^i + \tilde{\mathbf{x}}^i$ holds by definition, by introducing $\tilde{\mathbf{x}} := [\tilde{\mathbf{x}}^1, \dots, \tilde{\mathbf{x}}^N]^\top$, $\mathbf{u}$ can be written as $\mathbf{u} = \psi(\mathbf{x}, \mathbf{x} + \tilde{\mathbf{x}})$.

Definition 2 ([14]): A system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u, t)$, with $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, is set-Input to State Stable (set-ISS) with respect to u if, for each initial condition $\mathbf{x}(0)$ and any locally essentially bounded input u satisfying $\sup_{t \geq 0} \|u(t)\| < \infty$, $\|\mathbf{x}(t)\|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} \{\|\mathbf{x} - a\|\}$ satisfies $\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(0)\|_{\mathcal{A}}, t) + \gamma(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)$ for all $t \in \mathbb{R}_{\geq 0}$, where β and γ are $\mathcal{KL}$ and $\mathcal{K}$ functions, respectively.

Assumption 5: $\mathbf{u} = \psi(\mathbf{x})$ in (22) guarantees convergence of the MAS to a set $\mathcal{A}$, representing the team objective, in the presence of $\mathbf{w}(\mathbf{x}, t)$ satisfying Assumption 1.

Assumption 5 is not restrictive: since $\mathbf{w}(\mathbf{x}, t)$ is uniformly bounded by Assumption 1, (22) can be designed robustly using its upper bound, thereby guaranteeing convergence to $\mathcal{A}$ despite $\mathbf{w}$. Thus, $\mathbf{w}$ can be considered part of the nominal dynamics, and the system under (23) can be studied by analyzing $\dot{\mathbf{x}} = \Phi(\mathbf{x}, \tilde{\mathbf{x}}, t) := \mathbf{f}(\mathbf{x}) + \mathbf{g}(\psi(\mathbf{x}, \mathbf{x} + \tilde{\mathbf{x}})) + \mathbf{w}(\mathbf{x}, t)$, where $\tilde{\mathbf{x}}$ is treated as an input disturbance affecting the nominal unforced system.

Let $\delta_{\tilde{\mathbf{x}}}^i(t) := \|\delta_{\tilde{\mathbf{x}}}^1(t)^\top, \dots, \delta_{\tilde{\mathbf{x}}}^N(t)^\top\|$, with $\delta_{\tilde{\mathbf{x}}}^i(t) := [\delta_{N_1^i}^i(t), \dots, \delta_{N_{n_i}^i}^i(t)]^\top$, be the norm of the prescribed performance functions associated with $\tilde{\mathbf{x}}$. Moreover, denote with $\bar{\delta}_{\tilde{\mathbf{x}}}$ the desired upper bound on $\|\tilde{\mathbf{x}}\|$.

Assumption 6: There exists a finite time $t_x > 0$ for which $\delta_{\tilde{\mathbf{x}}}^i(t) \leq \bar{\delta}_{\tilde{\mathbf{x}}}$ holds for all $t \geq t_x$.

Since $\delta_{N_j^i}^i(t)$ are design choices, they can be designed to satisfy Assumption 6, making it non-restrictive in practice.

Theorem 4: Consider a heterogeneous MAS system (1) with connected graph $\mathcal{G}$ and distributed observers (12). Suppose each agent implements the control law in (23). Then, under Assumptions 2 and 6, the MAS converges to $\mathcal{A}_e := \{\mathbf{x} : \|\mathbf{x}\|_{\mathcal{A}} < \gamma(\bar{\delta}_{\tilde{\mathbf{x}}})\}$ if $\Phi(\mathbf{x}, \tilde{\mathbf{x}}, t)$ is set-ISS with respect to $\mathcal{A}$ and Assumption 5 holds.

Proof: Under Assumption 2, Theorems 1 and 3 guarantee $|\hat{x}_i^{N_j^i}(t)| < \delta_i^{N_j^i}(t)$ for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$, and $t \in \mathbb{R}_{\geq 0}$. Thus, under Assumption 6, there exists t_x such

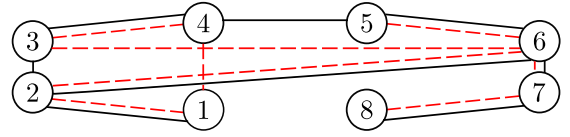


Fig. 1: Graphs $\mathcal{G}_C$ and $\mathcal{G}_T$, respectively in solid and dashed lines.

that $\|\tilde{\mathbf{x}}(t)\| < \bar{\delta}_{\tilde{\mathbf{x}}}$ for all $t \geq t_x$. Since $\Phi(\mathbf{x}, \tilde{\mathbf{x}}, t)$ is set-ISS, and from $\mathbf{x}(t_x)$ the system evolves satisfying $\dot{\mathbf{x}} = \Phi(\mathbf{x}, \tilde{\mathbf{x}}, t)$ with $\|\tilde{\mathbf{x}}\| < \bar{\delta}_{\tilde{\mathbf{x}}}$, $\|\mathbf{x}(t)\|_{\mathcal{A}} < \beta(\|\mathbf{x}(t_x)\|_{\mathcal{A}}, t - t_x) + \gamma(\bar{\delta}_{\tilde{\mathbf{x}}}) \forall t \geq t_x$. Thus, since $\beta(\|\mathbf{x}(t_x)\|_{\mathcal{A}}, t - t_x)$ is a $\mathcal{KL}$ function, $\mathbf{x}$ approaches $\mathcal{A}_e = \{\mathbf{x} : \|\mathbf{x}\|_{\mathcal{A}} < \gamma(\bar{\delta}_{\tilde{\mathbf{x}}})\}$ as $t \rightarrow \infty$. ■

Remark 8: While Assumption 2-(i) is independent of the components ψ_i , $i \in \mathcal{V}$, in (23), careful design of ψ_i is required for Assumption 2-(ii) to hold.

Theorem 4 shows that, under Assumptions 2, 5 and 6, the estimated states can be used in the decentralized controllers u_i to achieve the system objective with a worst-case error governed by $\gamma(\bar{\delta}_{\tilde{\mathbf{x}}})$. Thus, since $\bar{\delta}_{\tilde{\mathbf{x}}}$ is a design choice, the desired degree of accuracy can be imposed at design stage.

VII. SIMULATIONS

Consider a MAS of $N = 8$ agents communicating and collaborating according to the connected graphs $\mathcal{G}_C = (\mathcal{V}, \mathcal{E}_C)$ and $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$ in Fig. 1. Denote with $\mathcal{N}_i^C$ and $\mathcal{N}_i^T$ the i -th agent's neighbors in $\mathcal{G}_C$ and $\mathcal{G}_T$. Suppose $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + u_i$, where $\mathbf{x}_i = [x_{i,1}, x_{i,2}]^\top$ denotes the state of agent $i \in \mathcal{V}$, $\mathbf{f}(\mathbf{x}_i) = [\tanh(0.5x_{i,1} + 0.5x_{i,2}) \sin(0.5x_{i,1} - 0.5x_{i,2})]^\top$ is a Lipschitz continuous function with Lipschitz constant $l_f = 1$, and $u_i = k_c[\sum_{j \in (\mathcal{N}_i^C \cap \mathcal{N}_i^T)} \tanh(x_j - x_i) + \sum_{j \in \mathcal{N}_i^T \setminus \mathcal{N}_i^C} \tanh(\hat{x}_j^i - x_i)]$, with design parameter k_c , is a bounded input designed to reach consensus by exploiting only the edges of $\mathcal{G}_T$. When applied to vectors, $\tanh()$ has to be intended component-wise. The choice of homogeneous dynamics and a simple control objective is done to simplify the verification of the assumptions required to guarantee observer convergence. In particular, $\hat{x}_j^i$ in $\sum_{j \in \mathcal{N}_i^T \setminus \mathcal{N}_i^C} \tanh(\hat{x}_j^i - x_i)$ is introduced to cope with the lack of local information available to agent i about the state of $j \in \mathcal{N}_i^T \setminus \mathcal{N}_i^C$. This necessity stems from enforcing consensus using only the edges of $\mathcal{G}_T$, a condition imposed to assess the proposed observers, which are applied here with $k = 3$. Note that u_i is bounded by definition. Thus, since Assumption 2-(i) holds, the k -hop PPSO is simplified as per Remark 5. Then, since $\hat{u}_i$ is bounded in the case of convergent k -hop PPSO, Theorem 3 applies and the k -hop PPIO converges. To validate the complete framework, we avoid canceling $\mathbf{f}(\mathbf{x}_i)$ with u_i . To analyze stability under ideal input and characterize the set-ISS property of the closed-loop system, define the disagreement projection operator as $\Pi := I_{2N} - (1_N 1_N^\top \otimes I_2)/N$ [1]. Accordingly, let $\mathbf{e}_c := [e_{c,1}, \dots, e_{c,N}]^\top = \Pi \mathbf{x}$ be the consensus disagreement vector, $\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i(t)$, and $\bar{\mathbf{f}}(t) := \frac{1}{N} \sum_{i=1}^N \mathbf{f}(\mathbf{x}_i(t))$. By rewriting u_i as $u_i = u_i^{\text{ideal}} + u_i^{\text{error}}$, where $u_i^{\text{ideal}} := k_c \sum_{j \in \mathcal{N}_i^T} \tanh(x_j - x_i)$ and $u_i^{\text{error}} := k_c \sum_{j \in \mathcal{N}_i^T \setminus \mathcal{N}_i^C} [\tanh(\hat{x}_j^i + x_j - x_i) - \tanh(x_j - x_i)]$, and by computing $\dot{\mathbf{e}}_c$, $V = \frac{1}{2} \mathbf{e}_c^\top \mathbf{e}_c$ can be used to prove, following

standard Lyapunov-based analysis for consensus, that the MAS achieves consensus under the ideal input, and that the closed-loop system with true input is set-ISS with respect to the consensus manifold [14]. As a result, Theorem 4 holds and the proposed controller is expected to drive the MAS toward consensus. Note that, since $\hat{x}(t) = \frac{1}{N} \sum_{i=1}^N \hat{x}_i(t)$, and $\hat{f}(t) \neq 0$ holds in general, the MAS under u_i^{ideal} achieves consensus around a time-varying mean. Thus, similar result is also expected under the true decentralized inputs u_i .

For simulation purposes, a sampling time $dt = 10^{-5}$ s has been selected. To guarantee $|\tilde{x}_{i,l}^{N_j^i}(t)| < \delta(t)$ and $|\tilde{u}_{i,l}^{N_j^i}(t)| < \theta(t)$ to hold, with $\delta(t) = 13.96e^{-5t} + 0.117$ and $\theta(t) = 230e^{-5t} + 1.39$, $\rho_{N_j^i,l}^i(t)$ and $\omega_{N_j^i,l}^i(t)$ are designed as in Section III-A for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $l \in \{1, 2\}$. To satisfy the initialization condition, each component $\omega_{N_j^i,l}^i(0)$ and $\rho_{N_j^i,l}^i(0)$, respectively of $\omega_{N_j^i,l}^i(0)$ and $\rho_{N_j^i,l}^i(0)$, has been tuned such that $|\xi_{N_j^i,l}^i(0)| < \rho_{N_j^i,l}^i(0)$ and $|\mu_{N_j^i,l}^i(0)| < \omega_{N_j^i,l}^i(0)$ hold for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $l \in \{1, 2\}$.

Fig. 2(a) illustrates the closed-loop system behavior when the state estimates provided by the proposed observer are used in the controller. As expected from Theorem 4, the MAS achieves consensus. Although the goal is achieved, the agents are not stabilized around a stationary mean. As introduced earlier, this behavior is not caused by the use of estimated states in u_i , but by the nonlinearity of f . Given the large number of estimates involved in the network, and due to space limitation, Fig. 2(b) and Fig. 2(c) present only the estimation results regarding Agent 4. For this agent, $\rho_{N_j^4,l}^4(t) = \rho(t) = 2.8e^{-5t} + 0.02$ and $\omega_{N_j^4,l}^4(t) = \omega(t) = 39.27e^{-5t} + 0.033$ for all $N_j^4 \in \mathcal{N}_4^{k\text{-hop}}$ and $l \in \{1, 2\}$. While Fig. 2(b) shows the evolution of the maximum absolute disagreement and estimation error for the estimate of Agent 4's state (obtained by agents $N_j^4 \in \mathcal{N}_4^{k\text{-hop}}$), Fig. 2(c) shows those regarding the input estimates. In accordance with Theorems 1 and 3, Fig. 2 shows that under proper initialization, if $|\xi_{N_j^i,l}^i(t)| < \rho_{N_j^i,l}^i(t)$ and $|\mu_{N_j^i,l}^i(t)| < \omega_{N_j^i,l}^i(t)$ hold for all t , then $|\tilde{x}_{i,l}^{N_j^i}(t)| < \delta(t)$ and $|\tilde{u}_{i,l}^{N_j^i}(t)| < \theta(t)$ are satisfied for all $i \in \mathcal{V}$, $N_j^i \in \mathcal{N}_i^{k\text{-hop}}$ and $l \in \{1, 2\}$.

VIII. CONCLUSION AND FUTURE WORK

We proposed k -hop Prescribed Performance Observers that enable each agent to estimate the state and input of agents up to k hops away while ensuring predefined transient and steady-state performance. The proposed solution is robust to bounded external disturbances and does not require global knowledge of the network. Furthermore, we proved that under the set-ISS condition of the feedback control law, the state estimates can be used in the controller to drive the agents toward the team objective.

Future work will address observer design for time-varying and directed graphs.

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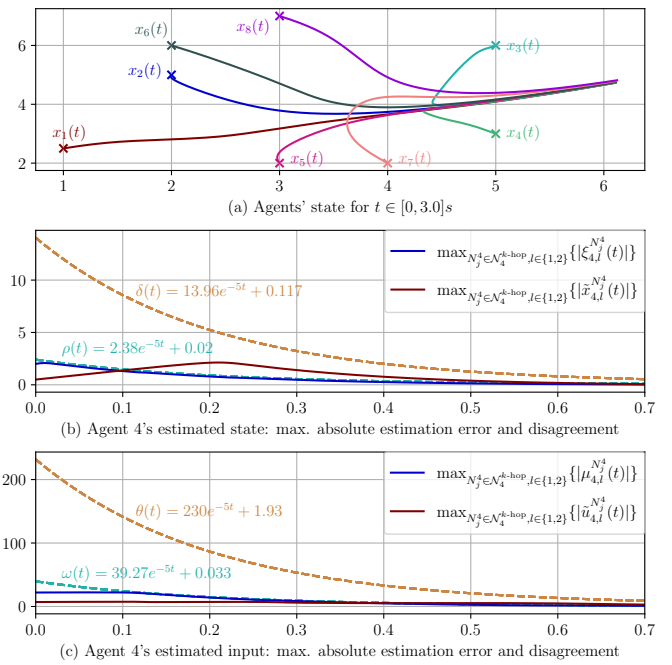


Fig. 2: (a) Agents' state evolution; initial states are represented by crosses. (b) and (c) Maximum absolute estimation error and estimation disagreement on Agent 4's state and input. The performance bounds are represented by dashed lines.

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