On the discrepancy between the task and communication graphs in multi-agent control systems

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Abstract— **Motivated by applications in which agents need to perform collaborative tasks under limited communication, in this work we consider the asymptotic satisfaction of relative-position based spatial tasks by a leader-follower network. As opposed to existing approaches, the spatial tasks may involve non-communicating agents and/or a subset of the multi-agent team. In order to ensure the satisfaction of the constraints using local information, as a first contribution, we express the incidence matrix of the task graph in terms of the corresponding matrix of the communication graph. In addition, for undirected spanning tree communication graphs, we show that the relation of the incidence matrices of these graphs is unique. Building upon this relation, as a second contribution we propose a two-hop communication based distributed feedback control law that ensures asymptotic satisfaction of the constraints with a predetermined robustness. The proposed control law employs the line graph of the communication graph and does not require knowledge of a complete row of the constraint matrix.**

I. Introduction

Over the last decades multi-agent systems have been considered in a variety of applications in ground, space and underwater environments. Compared to the single agent case, the presence of multiple agents allows for higher task performance as well as for robustness in presence of failures. Existing work on networked control systems has focused on the formation and consensus control problem under various communication topologies and system dynamics [1]–[3]. A standing assumption in the state of the art is that agents assigned to coupled tasks are also capable of exchanging information with one another. This imposes excessive requirements on the communication capabilities of the agents rendering the proposed approaches either extremely costly or non-applicable in practice.

Motivated by applications in which agents have limited communication capabilities, in this work we distinguish between the task and communication graph considering tasks involving the relative position of non-neighboring agents. As a first step, we express the edges of the task graph in terms of the edges of the communication graph by identifying paths along the latter. To that end, as opposed to existing graph matching approaches [4]–[7] that aim at determining a mapping among edges, here we consider the notion of *path-homeomorphism* and derive algebraic conditions for two undirected graphs to be path-homeomorphic in terms of their incidence matrices. In addition, we show that paths along tree communication graphs are uniquely defined for each edge of the task graph. Later we extend these results for general communication graphs assuming that a spanning tree of theirs is known.

In the context of control, the interplay between the task and communication graph was recently considered in [8] for time-constrained tasks by means of a novel decomposition scheme that allows the design of local tasks in terms of the edges of the communication graph such that the satisfaction of the initial task is ensured. Building upon this decomposition scheme, the authors have also proposed a distributed controller in [9] using control barrier functions.

In this work, we consider the asymptotic satisfaction of the task constraints expressed in terms of the communication graph without requiring tailored decomposition algorithms. In particular, we propose a distributed control law to asymptotically solve a set of equality constraints in the edge-space that guarantees a desired level of task satisfaction. Distributed algorithms for solving linear systems of equations have been widely considered in the literature [10]. Nevertheless, the majority of these algorithms requires agents to have knowledge of a complete row of the matrices which might become restrictive for large-scale problems. To that end, motivated by [11]–[13], we design a distributed control law for leader-follower networks that requires agents to have knowledge of only the columns of the constraint matrix corresponding to the edges they are involved in. The proposed control law ensures exponential convergence to a solution of the desired equation under a two-hop communication scheme induced via the line graph of the communication graph.

The remainder of this paper is organized as follows: Section II includes notation and preliminaries, Section III introduces the problem formulation, Section IV presents the proposed control approach and Section V a numerical example. Finally, Section VI provides a summary of the work and future directions.

II. Notation and Preliminaries

$A \otimes B$ denotes the Kronecker product of A, B as defined in [14, Ch. 4]. The column vector of all ones is denoted by $\mathbf{1}_d \in \mathbb{R}^d$ and the identity matrix of dimension n by I_n. $|\mathcal{S}|$ represents the cardinality of the set $\mathcal{S}$. The block diagonal matrix of $A_1, \dots, A_n$ is denoted by $\text{diag}(A_1, \dots, A_n)$. A vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ is equivalently written as $\mathbf{x} = \text{col}(\{x_j\}_{j \in \{1,\dots,n\}})$. The composition operator $\circ$ of two functions $f : \mathcal{V} \to \mathcal{V}$, $g : \mathcal{V} \to \mathcal{V}$ returns the function $h : \mathcal{V} \to \mathcal{V}$ satisfying $h(v) = (f \circ g)(v) = f(g(v)), v \in \mathcal{V}$.

This work was partially supported by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation, the ERC CoG LEAFHOUND, EU SymAware and the Swedish Research Council.

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A. Graph Theory

An *undirected* graph G is defined as a pair $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, R\} \subset \mathbb{N}$ is a finite set of nodes and $\mathcal{E} \subseteq \{(r, r') \in \mathcal{V} \times \mathcal{V} : r \neq r'\}$. A *path* is a sequence of edges connecting two distinct vertices. A graph is *connected*, if there exists a path between any pair of vertices. Given a numbering $k \in \mathcal{M} = \{1, \ldots, M\}$ of the edges $e_k \in \mathcal{E}$, after assigning an orientation to each edge in G we may define the *incidence matrix* $D = [\hat{d}_k^r] \in \mathbb{R}^{R \times M}$ of G as follows: $\hat{d}_k^r = 1$, if r is the head of edge k, $\hat{d}_k^r = -1$, if r is the tail of edge k and $\hat{d}_k^r = 0$, otherwise. The *edge Laplacian* matrix of G is given by $L_e = D^\top D$. In addition, the Laplacian matrix of G, denoted by $L \in \mathbb{R}^{R \times R}$, is defined in terms of D as $L = DD^\top$. The line graph of an undirected graph $G = (\mathcal{V}, \mathcal{E})$ is defined as $\mathcal{L}(G) := (\mathcal{E}, \mathcal{E}_{\mathcal{L}})$, where $(e_1, e_2) \in \mathcal{E}_{\mathcal{L}}$ if and only if the edges $e_1, e_2 \in \mathcal{E}$ of G have a node of G in common.

Lemma 1. *[15] The line graph of a connected, undirected graph is connected.*

III. Problem Formulation

Consider a team of R agents with limited communication capabilities. Let $G = (\mathcal{V}, \mathcal{E})$ denote the communication graph where $\mathcal{V} := \{1, \ldots, R\}$ is the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, where $(r, r') \in \mathcal{E}$, if and only if agents r and r' are capable of exchanging information. Here, we make the following assumption on G :

Assumption 1. *The graph $G = (\mathcal{V}, \mathcal{E})$ is static, undirected and connected.*

Here, the agents are assumed to evolve according to the first-order leader-follower formation protocol given by:

$$\dot{\mathbf{p}}_r := -\sum_{r' \in \mathcal{N}_r} (\mathbf{p}_r - \mathbf{p}_{r'} - \mathbf{p}_{rr'}^{\text{des}}) + b_r \mathbf{u}_r,$$

where $\mathbf{p}_r, \mathbf{u}_r \in \mathbb{R}^n$ is the state and input of the r-th agent, respectively, $\mathcal{N}_r := \{r' : (r, r') \in \mathcal{E}\}$ is the set of neighbors of agent r, $\mathbf{p}_{rr'}^{\text{des}} := \mathbf{p}_r^{\text{des}} - \mathbf{p}_{r'}^{\text{des}} \in \mathbb{R}^n$ is a desired relative position vector among agents r, r' and $b_r = 1$, if r is a leader agent or $b_r = 0$, if r is a follower agent. Let $\mathcal{V}_L \subset \mathcal{V}$, and $\mathcal{V}_F \subset \mathcal{V}$, denote the set of leaders and followers in the network, respectively. Let $\mathbf{p} := \text{col}(\{\mathbf{p}_r\}_{r \in \mathcal{V}}) := \text{col}(\mathbf{p}^F, \mathbf{p}^L)$, $\mathbf{p}^L := \text{col}(\{\mathbf{p}_r\}_{r \in \mathcal{V}_L})$, $\mathbf{p}^F := \text{col}(\{\mathbf{p}_r\}_{r \in \mathcal{V}_F})$, $\mathbf{p}^{\text{des}} := \text{col}(\{\mathbf{p}_r^{\text{des}}\}_{r \in \mathcal{V}})$, $\mathbf{u} := \text{col}(\{\mathbf{u}_r\}_{r \in \mathcal{V}_L})$ denote the stack vector of the positions of the multi-agent system, the leaders' and followers' positions, the desired positions and the control inputs of the multi-agent system, respectively. Then, the agents' dynamics can be written in stack form as follows:

$$\dot{\mathbf{p}} = -(L \otimes I_n)(\mathbf{p} - \mathbf{p}^{\text{des}}) + (B \otimes I_n)\mathbf{u}, \tag{1}$$

where $L \in \mathbb{R}^{R \times R}$ is the Laplacian matrix of G and $B := \begin{bmatrix} \mathbf{0}_{R_F \times R_L} \\ I_{R_L} \end{bmatrix}$, where $R_L > 0$ and $R_F > 0$ are the number of leaders and followers, respectively. Let $\mathbf{x}_k := \overline{\mathbf{x}}_k - \overline{\mathbf{p}}_k^{\text{des}}$, where $k \in \mathcal{M} := \{1, \ldots, M\}$, $M = |\mathcal{E}|$, $\overline{\mathbf{x}}_k := \mathbf{p}_{r_k} - \mathbf{p}_{r'_k}$, $\overline{\mathbf{p}}_k^{\text{des}} := \mathbf{p}_{r_k r'_k}^{\text{des}}$ and $(r_k, r'_k) \in \mathcal{E}$. Let further $\mathbf{x} := \text{col}(\{\mathbf{x}_k\}_{k \in \mathcal{M}})$. It can be easily shown that $(L \otimes I_n)\mathbf{p} = (D \otimes I_n)\overline{\mathbf{x}}$, where $D \in \mathbb{R}^{R \times M}$ is the incidence matrix of G and $\overline{\mathbf{x}} := \text{col}(\{\overline{\mathbf{x}}_k\}_{k \in \mathcal{M}})$. In addition, $\overline{\mathbf{x}} = (D^\top \otimes I_n)\mathbf{p}$. Given the structure of $\mathbf{p}$ the incidence matrix D can be written as $D = \begin{bmatrix} D_F^\top & D_L^\top \end{bmatrix}^\top$ where D_F, D_L denote the part of the incidence matrix corresponding to the follower and leader agents, respectively. Then, based on (1) we have:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{x}}_F \\ \dot{\mathbf{x}}_L \end{bmatrix} = -(L_e \otimes I_n)\mathbf{x} + (D_L^\top \otimes I_n)\mathbf{u}, \tag{2}$$

where $L_e \in \mathbb{R}^{M \times M}$ is the edge Laplacian of G.

In this work, the agents need to asymptotically satisfy a set of relative position based constraints:

$$h_i(\tilde{\mathbf{x}}_i) := \mathbf{C}_i \tilde{\mathbf{x}}_i - \mathbf{c}_i \geq \mathbf{0}, \quad i \in \mathcal{I}, \tag{3}$$

where $\mathcal{I} := \{1, \ldots, M_\tau\}$, $\mathbf{C}_i \in \mathbb{R}^{q_i \times n} \backslash \{\mathbf{0}\}$, $\mathbf{c}_i \in \mathbb{R}^{q_i}$, $\tilde{\mathbf{x}}_i := \mathbf{p}_{r_i} - \mathbf{p}_{r'_i}$ and (r_i, r'_i) is a pair of agents which may not satisfy $(r_i, r'_i) \in \mathcal{E}$. We are particularly motivated by applications in which agents need to stay close together due to potential communication constraints and collaborate towards asymptotically achieving a known relative-state based task as for example a desired formation expressed by another undirected graph and/or desired inter-agent distances.

Let $G_\tau = (\mathcal{V}, \mathcal{E}_\tau)$ be a static, undirected graph expressing the agents' couplings induced by the tasks in (3). In particular, $(r, r') \in \mathcal{E}_\tau$, if and only if there exists at least one $i \in \mathcal{I}$, such that $h_i := h_i(\mathbf{p}_r - \mathbf{p}_{r'})$, i.e., $h_i(\cdot)$ is an explicit function of $\mathbf{p}_r - \mathbf{p}_{r'}$. Let $D_\tau \in \mathbb{R}^{R \times M_\tau}$ denote the incidence matrix corresponding to G_τ, where $M_\tau := |\mathcal{E}_\tau|$. Then, (3) can be written as:

$$\mathbf{C}(D_\tau^\top \otimes I_n)\mathbf{p} - \mathbf{c}' \geq \mathbf{0}, \tag{4}$$

where $\mathbf{C} := \text{diag}(\mathbf{C}_1, \ldots, \mathbf{C}_{M_\tau})$, $\mathbf{c}' := \text{col}(\{\mathbf{c}_i\}_{i \in \mathcal{I}}) \in \mathbb{R}^q$ and $q := \sum_{i \in \mathcal{I}} q_i$. Let $\tilde{\mathbf{x}} := \text{col}(\{\tilde{\mathbf{x}}_i\}_{i \in \mathcal{I}})$. Here, we make the following assumption on the feasibility of the tasks:

Assumption 2. *The set $\mathcal{H} := \{\mathbf{x} \in \mathbb{R}^{Mn} : h_i(\tilde{\mathbf{x}}_i) \geq \mathbf{0}, i \in \mathcal{I}\}$ has a non-empty interior.*

Based on the above, the problem considered here can be expressed as follows:

Problem 1. *Consider the system dynamics* (1) *and let Assumptions 1-2 hold. Design $\mathbf{u}_r : \mathbb{R}_{\geq 0} \to \mathbb{R}^n, r \in \mathcal{V}_L$ such that $\mathbf{x}^* \in \mathcal{H}$, where $\mathbf{x}^* := \lim_{t \to +\infty} \mathbf{x}(t)$.*

IV. Main Results

In this section we will design the leaders' control laws that ensure the satisfaction of (3) as $t \to +\infty$ based on local information according to G. In Section IV-A we will first express the relative states involved in the spatial tasks with respect to the available relative information and in Section IV-B we will design the control laws for the leaders to ensure task satisfaction.

A. Graph Relations

If G_τ is a subgraph of G, then expressing the tasks in terms of the agents' states is straightforward. Neverthe-

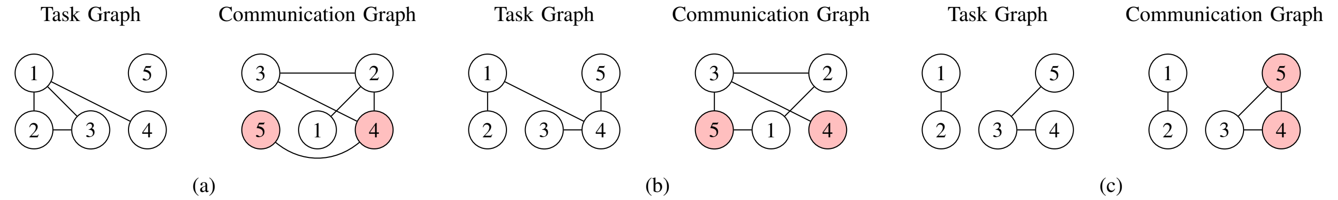

Fig. 1: (a)-(b) Examples of pairs of path homeomorphic graphs with $\sigma(3)=1, \sigma(5)=2, \sigma(1)=3, \sigma(4)=4, \sigma(2)=5$. (c) Example of path identical graphs. In all cases the pink nodes represent the network's leaders.

less, if there exists an $i \in \mathcal{I}$ such that $h_i(\cdot)$ depends on $(r, r') \notin \mathcal{E}$, then, an appropriate path along G needs to be identified connecting agents r, r'. Throughout this section we assume that G_τ, G are possibly disconnected unless otherwise stated. Motivated by [16], we define the notion of path-homeomorphism as follows:

Definition 1. *A graph $G_1 = (\mathcal{V}, \mathcal{E}_1)$ is said to be path-homeomorphic (p-hom) with respect to $G_2 = (\mathcal{V}, \mathcal{E}_2)$, denoted as $G_1 \preceq_p G_2$, if there exists a 1-1 mapping $\sigma : \mathcal{V} \to \mathcal{V}$ such that for each $(r, r') \in \mathcal{E}_1$, there exists a path $(\sigma(r_1), \sigma(r_2))(\sigma(r_2), \sigma(r_3)) \ldots (\sigma(r_{s-1}), \sigma(r_s))$ in G_2 with $\sigma(r_1) = r$ and $\sigma(r_s) = r'$, i.e., a path connecting r, r'. If $\sigma : \mathcal{V} \to \mathcal{V}$ is the identity mapping, then G_1 is said to be path-identical (p-id) with respect to G_2, denoted as $G_1 \preceq G_2$.*

Examples of path-homeomorphic and path-identical graphs are shown in Figure 1. The following proposition is an immediate result of Definition 1:

Proposition 1. *Consider the graphs $G = (\mathcal{V}, \mathcal{E})$ and $G_\tau = (\mathcal{V}, \mathcal{E}_\tau)$ and let Assumption 1 hold. Then, $G_\tau \preceq G$.*

According to Proposition 1 the couplings graph G_τ is path-identical to the communication graph G, even when it is disconnected, as long as G is connected. This conclusion allows us to encode a wide range of tasks that are possibly assigned to a subset of agents $r \in \mathcal{V}$.

Definition 2. *[17, Def. 2.18] An automorphism of the graph $G = (\mathcal{V}, \mathcal{E})$ is a permutation $\sigma : \mathcal{V} \to \mathcal{V}$ such that $(\sigma(r), \sigma(r')) \in \mathcal{E} \Leftrightarrow (r, r') \in \mathcal{E}$.*

The set of all automorphisms of G equipped with the composition operator constitutes the automorphism group of G, denoted by $\mathrm{Aut}(G)$.

Proposition 2. *Consider the graphs $G = (\mathcal{V}, \mathcal{E})$ and $G_\tau = (\mathcal{V}, \mathcal{E}_\tau)$ and assume that there exists $\sigma \in Aut(G)$ such that:*

$$\mathcal{E}_\tau \subseteq \{(r, r') \in \mathcal{V} \times \mathcal{V} : (\sigma(r), \sigma(r')) \in \mathcal{E}\}. \tag{5}$$

Then, $G_\tau \preceq_p G$.

Proof. By (5), for every $(r, r') \in \mathcal{E}_\tau$, under the automorphism function $\sigma : \mathcal{V} \to \mathcal{V}$, we have $(\sigma(r), \sigma(r')) \in \mathcal{E}$ which by Definition 2 implies that $(r, r') \in \mathcal{E}$. Then, $G_\tau \preceq_p G$ follows from Definition 1 when for each (r, r') the path along G is chosen to be $(\sigma(r), \sigma(r'))$ (or (r, r')). ∎

Notice that according to Proposition 2 the notion of path homeomorphism is less strong than graph automorphism [17, Def. 2.18] in the sense that graph automorphism is a subclass of path homeomorphism.

Here, we are interested in expressing $\tilde{\mathbf{x}}_i, i \in \mathcal{I}$ in terms of $\mathbf{x}_k, k \in \mathcal{M}$.

Proposition 3. *$G_\tau \preceq_p G$ if and only if there exist matrices $\Sigma \in \mathbb{R}^{R \times R}$ and $A \in \mathbb{R}^{M \times M_\tau}$, such that:*

$$D_\tau = \Sigma D A, \tag{6}$$

where $\Sigma = [\Sigma_{rr'}]_{r,r' \in \mathcal{V}}$ is the permutation matrix associated to $\sigma : \mathcal{V} \to \mathcal{V}$ of Definition 1, defined as:

$$\Sigma_{rr'} = \begin{cases} 1, & \text{if } \sigma(r') = r, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

If $\Sigma = I_R$, then $G_\tau \preceq G$ if and only if there exists a matrix $A \in \mathbb{R}^{M \times M_\tau}$, such that $D_\tau = DA$.

Proof (sketch). If $G_\tau \preceq_p G$, then if $e_i \in \mathcal{E}$, the i-th column of A is the i-th vector of the canonical basis of $\mathbb{R}^n$ up to a sign permutation determined by the orientation used in defining D. Otherwise, the i-th column of A can be defined according to the sign characteristic vector [18] of the cycle over the graph $G_i' := (\mathcal{V}, \mathcal{E} \cup \{e_i\})$ that is formed by the path in G and e_i. Next, assume that there exist matrices Σ, A such that $D_\tau = \Sigma D A$. First, it can be shown that the agents forming $e_i \in \mathcal{E}_\tau, i \in \mathcal{I}$ belong to the same connected component of G. Then, the existence of the path along G for each $e_i \in \mathcal{E}_\tau$ follows either trivially if $e_i \in \mathcal{E}$, or by [19, Cor. 14.2.3] after showing that for every $i \in \mathcal{I}$, $D_\tau = \Sigma D A$ implies the existence of a cycle in the graph determined by the corresponding connected component of G and e_i. ∎

For fixed Σ, using the properties of the Kronecker product and the matrix vectorization, (6) is equivalent to the solution of the linear equation:

$$(I_{M_\tau} \otimes \Sigma D)\mathrm{vec}(A) = \mathrm{vec}(D_\tau). \tag{8}$$

First, we consider the case where G is an undirected tree. Then, we can deduce the following:

Proposition 4. *Consider the communication and couplings graph $G = (\mathcal{V}, \mathcal{E})$ and $G_\tau = (\mathcal{V}, \mathcal{E}_\tau)$, respectively, and let Assumption 1 hold. Assume further that G is a spanning tree. If $G_\tau \preceq_p G$ with a known permutation matrix $\Sigma \in \{0, 1\}^{R \times R}$, then the elements of A in (6) are uniquely defined by:*

$$\mathrm{vec}(A) = W^\dagger \mathrm{vec}(D_\tau), \tag{9}$$

where $W^\dagger = I_{M_\tau} \otimes (\Sigma D)^\dagger = I_{M_\tau} \otimes (L_e^{-1} D^\top \Sigma^\top)$ is the Moore-Penroose matrix of $I_{M_\tau} \otimes \Sigma D$ and $L_e = D^\top D$ is the edge Laplacian of G.

Proof. By assumption G is a tree. Thus, $\mathrm{rank}(D) = R-1$. Since Σ is non-singular by the properties of the rank [14, Sec. 0.4.6], $\mathrm{rank}(\Sigma D) = \mathrm{rank}(D) = R-1$ which in turn implies that $\mathrm{rank}(I_{M_\tau} \otimes \Sigma D) = MM_\tau = (R-1)M_\tau$, i.e., the matrix $I_{M_\tau} \otimes \Sigma D$ is full-column rank. Then, the least squares solution, i.e., the solution that minimizes $\|\mathrm{vec}(D_\tau) - (I_{M_\tau} \otimes \Sigma D)\mathbf{y}\|_2^2$ is given by $\mathbf{y}^* = W^\dagger \mathrm{vec}(D_\tau)$. Since W is full-column rank, this solution is unique. In addition, by the properties of the Kronecker product and the Moore-Penroose matrix it holds that $W^\dagger = (I_{M_\tau} \otimes \Sigma D)^\dagger = I_{M_\tau}^\dagger \otimes (\Sigma D)^\dagger = I_{M_\tau} \otimes (\Sigma D)^\dagger$. Since Σ has orthonormal columns and is non-singular (being a permutation matrix), it follows that $(\Sigma D)^\dagger = D^\dagger \Sigma^\dagger$ which in turn leads to $(\Sigma D)^\dagger = D^\dagger \Sigma^{-1} = D^\dagger \Sigma^\top$. Given, the aforementioned analysis and since D is full-column rank, we have $(\Sigma D)^\dagger = (D^\top D)^{-1} D^\top \Sigma^\top = L_e^{-1} D^\top \Sigma^\top$ and thus $W^\dagger = I_{M_\tau} \otimes (L_e^{-1} D^\top \Sigma^\top)$. Given that $G_\tau \preceq_p G$ with a known permutation matrix Σ, by Proposition 3 it follows that $D_\tau = \Sigma D A$, or equivalently, $\mathrm{vec}(D_\tau) = (I_{M_\tau} \otimes \Sigma D)\mathrm{vec}(A)$, where D_τ, D are the incidence matrices of G_τ, G, respectively. Assume for a moment that $\mathrm{vec}(A) \neq \mathbf{y}^*$. Then, there exists another vector, namely $\mathrm{vec}(A)$, such that $0 = \|\mathrm{vec}(D_\tau) - (I_{M_\tau} \otimes \Sigma D)\mathrm{vec}(A)\|_2^2 < \|\mathrm{vec}(D_\tau) - (I_{M_\tau} \otimes \Sigma D)\mathbf{y}^*\|_2^2$, which leads to contradiction since $\mathbf{y}^*$ is the unique minimizer of $\|\mathrm{vec}(D_\tau) - (I_{M_\tau} \otimes \Sigma D)\mathbf{y}\|_2^2$. Therefore, $\mathrm{vec}(A) = \mathbf{y}^* = W^\dagger \mathrm{vec}(D_\tau)$ follows. This concludes the proof. ■

When the cycle space of G is non-empty, then $I_{M_\tau} \otimes \Sigma D$ has in general more columns than rows and thus the solution of (8) is not unique. Nevertheless, if a spanning tree of G is known we may deduce the following:

Proposition 5. *Consider the communication and couplings graph $G = (\mathcal{V}, \mathcal{E})$ and $G_\tau = (\mathcal{V}, \mathcal{E}_\tau)$, respectively, and let Assumption 1 hold. Let $\overline{G}_\mathcal{T}$ be a known spanning tree of G. Then, $G_\tau \preceq_p \overline{G}_\mathcal{T} \Rightarrow G_\tau \preceq_p G$.*

Proof. By Proposition 3, $G_\tau \preceq_p \overline{G}_\mathcal{T}$ implies that there exist $\Sigma, A_\mathcal{T}$ such that $D_\tau = \Sigma \overline{D}_\mathcal{T} A_\mathcal{T}$, where $\overline{D}_\mathcal{T} \in \mathbb{R}^{R \times R-1}$ is the incidence matrix of $\overline{G}_\mathcal{T}$. By [20, Eq. (11)] it is known that there exists a permutation matrix $P \in \mathbb{R}^{M \times M}$ such that the incidence matrix D can be written as $DP = \begin{bmatrix} \overline{D}_\mathcal{T} & \overline{D}_{G \setminus \mathcal{T}} \end{bmatrix}$, where $\overline{D}_{G \setminus \mathcal{T}}$ is the matrix containing the columns of D corresponding to the edges of the subgraph of G obtained after removing the edges of $\overline{G}_\mathcal{T}$. As a result, $\overline{D}_\mathcal{T} = DP \begin{bmatrix} I_{R-1} \\ \mathbf{0}_{(M-R+1)\times(R-1)} \end{bmatrix}$. Then, choosing $A = P \begin{bmatrix} I_{R-1} \\ \mathbf{0}_{(M-R+1)\times R-1} \end{bmatrix} A_\mathcal{T}$, we have $D_\tau = \Sigma D A$ and the result follows. ■

Proposition 5 suggests that a sufficient condition for $G_\tau \preceq_p G$ is G_τ to be p-homeomorphic to one of its spanning trees. If Σ is a-priori known, then, the matrix A can be uniquely determined using Proposition 4.

B. Control Design

In this section we will proceed with the control design. By Assumption 1, G is connected and thus $G_\tau \preceq G$ follows from Proposition 1. Then, based on Proposition 3 and the properties of the Kronecker product, (4) can be written as:

$$\mathbf{C}(A^\top \otimes I_n)\mathbf{x} - \mathbf{c} \geq \mathbf{0}, \tag{10}$$

where $\mathbf{c} := \mathbf{c}' - \mathbf{C}(D_\tau^\top \otimes I_n)\mathbf{p}^{\mathrm{des}}$. Notice that due to Assumption 2, there exists at least one $\mathbf{x} \in \mathbb{R}^{Mn}$ satisfying (10). Motivated by applications in which agents need to ensure the satisfaction of the task with a given robustness, i.e., such that $\mathbf{x} \in \mathrm{int}(\mathcal{H})$, here we will enforce the satisfaction of $\mathbf{C}(A^\top \otimes I_n)\mathbf{x} = \tilde{\mathbf{c}}$, where $\tilde{\mathbf{c}} > \mathbf{c}$ is an offline chosen parameter vector. In particular, we pose the following assumption on $\tilde{\mathbf{c}}$:

Assumption 3. *For a given vector $\tilde{\mathbf{c}} \in \mathbb{R}^q$ with $\tilde{\mathbf{c}} > \mathbf{c}$, there exists a non-negative constant α_1 such that $\mathcal{C} := \mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, where:*

$$\mathcal{C}_1 := \{\mathbf{x} \in \mathbb{R}^{Mn} : \mathbf{C}(A^\top \otimes I_n)\mathbf{x} = \tilde{\mathbf{c}}\},$$
$$\mathcal{C}_2 := \{\mathbf{x} \in \mathbb{R}^{Mn} : ((L_e + \alpha_1 D_L^\top D_L) \otimes I_n)\mathbf{x} = \mathbf{0}\},$$

and where $D_L \in \mathbb{R}^{R_L \times M}$ is the part of the incidence matrix of G corresponding to the leader agents $r \in \mathcal{V}_L$.

Assumption 3 requires a solution of $\mathbf{C}(A^\top \otimes I_n)\mathbf{x} = \tilde{\mathbf{c}}$ to be an equilibrium point of $\dot{\mathbf{x}} = -((L_e + \alpha_1 D_L^\top D_L) \otimes I_n)\mathbf{x}$. This assumption will be later used in establishing an equilibrium point of the closed loop system under the proposed control law given by (15), (17).

To simplify notation, let $\widetilde{\mathbf{C}} := \mathbf{C}(A^\top \otimes I_n)$. Then, $\mathbf{C}(A^\top \otimes I_n)\mathbf{x} = \tilde{\mathbf{c}}$ can be written as:

$$\widetilde{\mathbf{C}}\mathbf{x} = \tilde{\mathbf{c}}. \tag{11}$$

In this work, we will assume that no agent has knowledge of the whole matrix $\widetilde{\mathbf{C}}$. In particular, we will assume that $\widetilde{\mathbf{C}}$ is decomposed column-wise to matrices $\widetilde{\mathbf{C}}_k \in \mathbb{R}^{q \times n}, k \in \mathcal{M}$, such that:

$$\widetilde{\mathbf{C}} := \begin{bmatrix} \widetilde{\mathbf{C}}_1 & \dots & \widetilde{\mathbf{C}}_M \end{bmatrix}. \tag{12}$$

In addition, we consider vectors $\mathbf{g}_k \in \mathbb{R}^q, k \in \mathcal{M}$, such that:

$$\sum_{k \in \mathcal{M}} \mathbf{g}_k = \tilde{\mathbf{c}}. \tag{13}$$

The vectors $\mathbf{g}_k, k \in \mathcal{M}$ can be chosen freely as long as (13) holds. Then, the control objective of this work is to design $\mathbf{u}_r, r \in \mathcal{V}_L$ such that $\sum_{k \in \mathcal{M}} (\widetilde{\mathbf{C}}_k \mathbf{x}_k^* - \mathbf{g}_k) = \mathbf{0}$, where $\mathbf{x}^* = \lim_{t \to +\infty} \mathbf{x}(t)$ and $\mathbf{x}^* := \mathrm{col}(\{\mathbf{x}_k^*\}_{k \in \mathcal{M}})$.

Motivated by [12], we introduce an auxiliary variable $\boldsymbol{\zeta}_k \in \mathbb{R}^q$ for each $k \in \mathcal{M}$ that provides an estimate of the error in the satisfaction of the objective. Let $\boldsymbol{\zeta}_k \in \mathbb{R}^q$ evolve over time according to:

$$\dot{\boldsymbol{\zeta}}_k = \widetilde{\mathbf{C}}_k \mathbf{x}_k - \mathbf{g}_k - \alpha_2 \sum_{k' \in \mathcal{N}_k^l} (\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k'}), \tag{14}$$

where $\mathcal{N}_k^l \subseteq \mathcal{M}$ is the set of neighbors of the k-th node of $\mathcal{L}(G)$ (the line graph of G), and α_2 is a positive gain to be tuned. Define $\boldsymbol{\zeta} := \mathrm{col}(\{\boldsymbol{\zeta}_k\}_{k\in\mathcal{M}})$. Then, $\boldsymbol{\zeta} \in \mathbb{R}^{Mq}$ evolves over time according to the following equation:

$$\dot{\boldsymbol{\zeta}} = F\mathbf{x} - \mathbf{g} - \alpha_2(L^l \otimes I_q)\boldsymbol{\zeta}, \tag{15}$$

where $F := \mathrm{diag}(\widetilde{\mathbf{C}}_1, \ldots, \widetilde{\mathbf{C}}_M) \in \mathbb{R}^{Mq\times Mn}$, $\mathbf{g} := \mathrm{col}(\{\mathbf{g}_k\}_{k\in\mathcal{M}}) \in \mathbb{R}^{Mq}$ and $L^l \in \mathbb{R}^{M\times M}$ is the Laplacian matrix of $\mathcal{L}(G)$.

Next, in order to ensure the desired objectives we propose the following control law for each $r \in \mathcal{V}_L$:

$$\mathbf{u}_r = -\sum_{k\in\mathcal{M}} \hat{d}_k^r\big[\alpha_1\mathbf{x}_k + \widetilde{\mathbf{C}}_k^\top\big(\widetilde{\mathbf{C}}_k\mathbf{x}_k - \mathbf{g}_k - \alpha_2\sum_{k'\in\mathcal{N}_k^l}(\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k'})\big)\big] \tag{16}$$

where $\hat{d}_k^r$ is the (r,k)-th element of the incidence matrix of G, α_1 is the same as in Assumption 3 and α_2 the same as in (14). Then, the proposed control law of (16) can be written in stack form as follows:

$$\mathbf{u} = -(D_L \otimes I_n)\bigg[\alpha_1\mathbf{x} + F^\top\bigg(F\mathbf{x} - \mathbf{g} - \alpha_2(L^l \otimes I_q)\boldsymbol{\zeta}\bigg)\bigg]. \tag{17}$$

Under the proposed control law (15), (17) we can show asymptotic convergence to the solution of (11) as follows:

Theorem 1. *Consider the system* (2) *that is subject to* (3). *Let Assumptions 1-3 hold. Assume further that the matrix S defined as:*

$$S := \begin{bmatrix} -(L_e \otimes I_n) - \alpha_1 E - EF^\top F & \alpha_2 EF^\top(L^l \otimes I_q) \\ F & -\alpha_2(L^l \otimes I_q) \end{bmatrix}, \tag{18}$$

where $E := D_L^\top D_L \otimes I_n$, has non-positive real eigenvalues and any zero eigenvalue is non-defective[1]. *Then, the control law defined by* (15), (17) *ensures that $\mathbf{x}(t) \to \mathbf{x}^*$ exponentially fast as $t \to +\infty$, where $\mathbf{x}^* \in \mathbb{R}^{Mn}$ satisfies* (11).

Proof. The proof will be divided in two parts. In the first part we will show the existence of an equilibrium point for the closed loop system and as well as the exponential convergence of the system towards an equilibrium set to be determined. Finally, in the second part we will show that every point in the equilibrium set is a solution to $\sum_{k\in\mathcal{M}}(\widetilde{\mathbf{C}}_k\mathbf{x}_k - \mathbf{g}_k) = \mathbf{0}$.

Part I: Substituting the control law given by (15)-(17) to (2) we obtain the closed loop system given by:

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\boldsymbol{\zeta}}\end{bmatrix} = S\begin{bmatrix}\mathbf{x} \\ \boldsymbol{\zeta}\end{bmatrix} + \boldsymbol{q}, \tag{19}$$

where $\boldsymbol{q} := \begin{bmatrix} EF^\top\mathbf{g} \\ -\mathbf{g}\end{bmatrix}$ and S is defined in (18). Let $\mathbf{z} := \mathrm{col}(\{\mathbf{z}_k\}_{k\in\mathcal{M}}) \in \mathbb{R}^{Mn}$ be a solution of (11) with $\mathbf{z} \in \mathcal{C}$, where $\mathcal{C} \subseteq \mathbb{R}^{Mn}$ is the set defined in Assumption 3. Considering the decomposition of $\widetilde{\mathbf{C}}, \tilde{\mathbf{c}}$ as determined in (12), (13), it follows that:

$$\widetilde{\mathbf{C}}\mathbf{z} = \tilde{\mathbf{c}} \Leftrightarrow \sum_{k\in\mathcal{M}}(\widetilde{\mathbf{C}}_k\mathbf{z}_k - \mathbf{g}_k) = \mathbf{0} \Leftrightarrow (\mathbf{1}_M^\top \otimes I_q)(F\mathbf{z} - \mathbf{g}) = \mathbf{0}. \tag{20}$$

By Assumption 1 and Lemma 1 it follows that the line graph of G is connected, thus $\mathbf{1}_M$ is an eigenvector of L^l corresponding to the zero eigenvalue, i.e., $L^l\mathbf{1}_M = \mathbf{0}$. This in turn ensures that $(\mathbf{1}_M^\top \otimes I_q)(L^l \otimes I_q) = \mathbf{0}$ implying that $\mathrm{Ker}(\mathbf{1}_M^\top \otimes I_q) = \mathrm{Im}(L^l \otimes I_q)$. Then, using (20) one can conclude that $F\mathbf{z} - \mathbf{g} \in \mathrm{Im}(L^l \otimes I_q)$. As a result there should exist a $\boldsymbol{\zeta}_z \in \mathbb{R}^{Mq}$ such that $F\mathbf{z} - \mathbf{g} - \alpha_2(L^l \otimes I_q)\boldsymbol{\zeta}_z = \mathbf{0}$. In addition, since $\mathbf{z} \in \mathcal{C}$ and thus $\mathbf{z} \in \mathcal{C}_2$ by Assumption 3, it follows that $(L_e \otimes I_n + \alpha_1 E)\mathbf{z} = \mathbf{0}$. Combining the aforementioned results, we can conclude that $\begin{bmatrix}\mathbf{z}^\top & \alpha_2\boldsymbol{\zeta}_z^\top\end{bmatrix}^\top$ is an equilibrium point of (19). Let the error vector $\boldsymbol{\eta} := \begin{bmatrix}\mathbf{x} \\ \boldsymbol{\zeta}\end{bmatrix} - \begin{bmatrix}\mathbf{z} \\ \alpha_2\boldsymbol{\zeta}_z\end{bmatrix}$, which evolves according to $\dot{\boldsymbol{\eta}} = S\boldsymbol{\eta}$. By Assumption, S has negative or zero eigenvalues. In addition, if S has a zero eigenvalue, then it is non-defective. As a result, $\boldsymbol{\eta}$ converges exponentially to either zero or a constant vector $\bar{\boldsymbol{\eta}} \in \mathbb{R}^{M(n+q)}$ satisfying $\bar{\boldsymbol{\eta}} \in \mathrm{Ker}(S)$. Therefore, $\mathrm{col}(\mathbf{x}(t), \boldsymbol{\zeta}(t))$ converges exponentially to a vector $\mathrm{col}(\mathbf{x}^*, \boldsymbol{\zeta}^*) \in \mathcal{X}$, where $\mathcal{X}$ is the equilibrium set of (19), defined as:

$$\mathcal{X} := \Big\{\begin{bmatrix}\mathbf{x}^* \\ \boldsymbol{\zeta}^*\end{bmatrix} : \begin{bmatrix}\mathbf{x}^* \\ \boldsymbol{\zeta}^*\end{bmatrix} = \begin{bmatrix}\mathbf{z} \\ \alpha_2\boldsymbol{\zeta}_z\end{bmatrix} + \bar{\boldsymbol{\eta}},\ \bar{\boldsymbol{\eta}} \in \mathrm{Ker}(S)\Big\}. \tag{21}$$

Part II: For any point $\mathrm{col}(\mathbf{x}^*, \boldsymbol{\zeta}^*) \in \mathcal{X}$, it holds $F\mathbf{x}^* - \mathbf{g} - \alpha_2(L^l \otimes I_q)\boldsymbol{\zeta}^* = \mathbf{0}$. Multiplying this equation with $\mathbf{1}_M^\top \otimes I_q$ from the left, we obtain $(\mathbf{1}_M^\top \otimes I_q)(F\mathbf{x}^* - \mathbf{g}) - \alpha_2(\mathbf{1}_M^\top \otimes I_q)(L^l \otimes I_q)\boldsymbol{\zeta}^* = \mathbf{0}$. Using the properties of the Kronecker product and the fact that $L^l\mathbf{1}_M = \mathbf{0}$, we have $(\mathbf{1}_M^\top \otimes I_q)(L^l \otimes I_q)\boldsymbol{\zeta}^* = \mathbf{0}$. Thus, we get $(\mathbf{1}_M^\top \otimes I_q)(F\mathbf{x}^* - \mathbf{g}) = \mathbf{0}$. Using the definitions of $F, \mathbf{g}$ in (15), we can deduce that $(\mathbf{1}_M^\top \otimes I_q)(F\mathbf{x}^* - \mathbf{g}) = \mathbf{0} \Leftrightarrow \sum_{k\in\mathcal{M}}(\widetilde{\mathbf{C}}_k\mathbf{x}_k^* - \mathbf{g}_k) = \mathbf{0}$, which in turn leads to $\widetilde{\mathbf{C}}\mathbf{x}^* = \tilde{\mathbf{c}}$, where $\mathbf{x}^* = \mathrm{col}(\{\mathbf{x}_k^*\}_{k\in\mathcal{M}})$. Therefore, every point of the equilibrium set $\mathcal{X}$ is a solution to the system of equations (11). This concludes the proof. ■

In the following we provide sufficient conditions for S to satisfy the properties of Theorem 1 in terms of the communication topology, the number of leaders and the constraint matrix under consideration. The proof of the result can be found in [21].

Lemma 2. *Consider the matrix S in* (18) *and let Assumption 1 hold. Assume further the following: (i)* $\mathrm{Ker}(EF^\top) \cap \mathrm{Ker}(L^l \otimes I_q) = \{\mathbf{0}\}$ *and (ii) the matrices S_5, S_6 commute, where:*

$$S_5 := \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & I_{Mq}\end{bmatrix},$$

$$S_6 := \begin{bmatrix} F^\top \\ -\alpha_2(L^l \otimes I_q)\end{bmatrix}\begin{bmatrix} F & -\alpha_2(L^l \otimes I_q)\end{bmatrix},$$

where E is defined as in Theorem 1. Then, S has negative or zero real eigenvalues and any zero eigenvalue is non-defective.

V. Numerical Example

Consider $R = 10$ agents evolving according to (2) in $\mathbb{R}^2$ with $R_L = 2$ leaders and $R_F = 8$ followers. Here, $\mathcal{E} := \{(1,2), (4,10), (5,6), (6,7), (7,8), (8,9), (2,3), (4,5), (3,4)\}$.

[1] An eigenvalue is non-defective if any only if its algebraic multiplicity is equal to its geometric multiplicity.

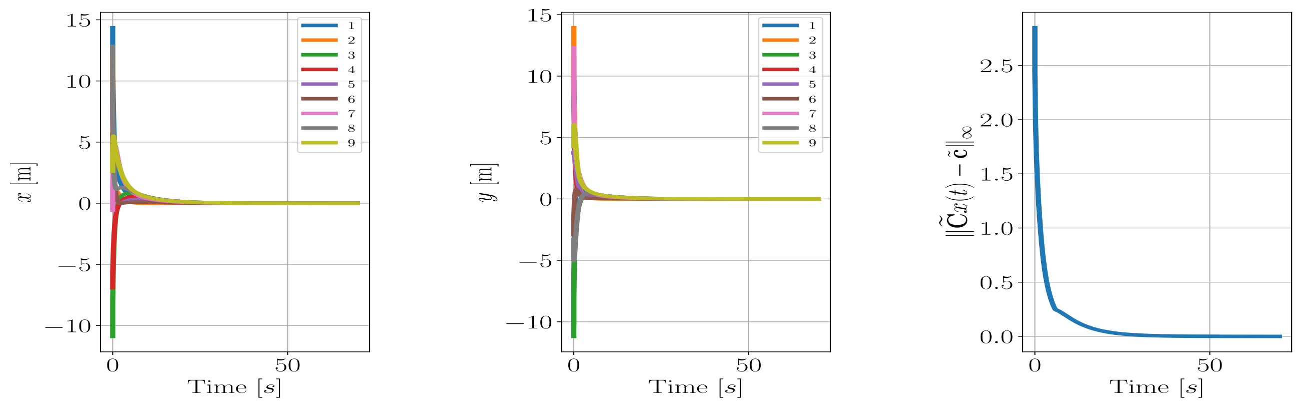

Fig. 2: Evolution of the edge states and the infinity norm of the constraints over t under the proposed control law.

The agents are subject to the constraints with $0.1\tilde{\mathbf{x}}_i \leq \mathbf{c}'_i,\ i \in \mathcal{I} := \{1,\dots,9\}$, where $\mathbf{c}'_i := \begin{bmatrix}-0.6 & 0.4\end{bmatrix}^\top,\ i \in \{1,\dots,5\}$, $\mathbf{c}'_i := \begin{bmatrix}0.4 & -0.6\end{bmatrix}^\top,\ i \in \{6,\dots,9\}$ and $\mathcal{E}_\tau := \{(7,9), (7,10), (1,10), (1,3), (3,5), (2,4), (4,10), (6,8), (8,10)\}$. We set $\alpha_1 = 1.2$ and $\alpha_2 = 2$. The vectors $\mathbf{g}_k, k \in \mathcal{M}$ are found as a solution to a QP problem that aims at keeping $\mathbf{g}_k$ as close as possible to $\tilde{\mathbf{c}} = \mathbf{0}$ while satisfying (13). The initial conditions are chosen as $\boldsymbol{\zeta}_k(0) = \mathbf{0}, k \in \mathcal{M}$ and $\mathbf{p}_r(0) = \begin{bmatrix}-3+r & \sin((r-1)\theta) - 3 + (2/3)(r-1)\end{bmatrix}^\top, r \in \mathcal{V}$, where $\theta = 2\pi/R$. The matrix A and the formation parameters can be found in [21]. In Figure 2 the evolution of the edge states and the infinity norm of $\widetilde{\mathbf{C}}\mathbf{x} - \tilde{\mathbf{c}}$ are shown. The satisfaction of the constraints is achieved exponentially fast as $t \to +\infty$ verifying the validity of Theorem 1.

VI. Conclusions

In this work, a distributed control law is proposed for leader-follower systems under coupled relative position based objectives involving non-neighboring agents. First, we express the incidence matrix of the task graph in terms of the one of the communication graph and obtain graph theoretic conditions under which the proposed expression is unique. Then, a distributed control law is designed for the leader-follower edge dynamics employing a two-hop communication scheme using the line graph of the communication graph. Future work will explore infeasible constraints and more general communication topologies.

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