Communication-aware Multi-agent Systems Control Based on k-hop Distributed Observers

Tommaso Zaccherini, Siyuan Liu and Dimos V. Dimarogonas

***Abstract*— We propose a distributed control strategy to allow the control of a multi-agent system requiring k-hop interactions based on the design of distributed state and input observers. In particular, we design for each agent a finite time convergent state and input observer that exploits only the communication with the 1-hop neighbors to reconstruct the information regarding those agents at a 2-hop or more distance. We then demonstrate that if the k-hop based control strategy is set-Input to State Stable with respect to the set describing the goal, then the observer information can be adopted to achieve the team objective with stability guarantees.**

I. Introduction

A multi-agent system consists of multiple autonomous agents, each with individual goals, capabilities, and decision-making processes, that interact to achieve collective or individual objectives. Their advantages in redundancy and flexibility over single-agent systems have led to extensive research over the years, particularly due to their ability to perform multiple simultaneous actions [1]–[4]. The main drawback compared to single-agent systems is the increased complexity in coordination and communication. Without centralized memory, agents rely on local 1-hop neighbor information, even though goals depend on the global system state. For this reason, when communication is limited, allowing agents to estimate state of agents beyond their 1-hop neighborhood can improve the overall system performance.

Several works concerning distributed state estimation in network systems are available in the literature [5]–[7]. In [5], a decentralized observer for a system of agents with discrete-time dynamics is proposed, where knowledge of the model and local information are exploited to estimate the plant state by means of a consensus based filter. In [6] instead, an asymptotic observer in which each agent estimates the global state by exploiting the communication interaction with its 1-hop neighbors is proposed. The main drawback of these approaches is the need to estimate the entire system state, irrespective of the specific information required by each agent. As a result, in large-scale network applications where agents may only require partial information about the global state, these approaches become difficult to implement due to their poor scalability with respect to the number of agents. Alternative approaches rely on the decomposition of the network into subsystems with local controllers where the information exchange between subsystems may or not be allowed as in [8] and [9], respectively. Other results regarding decentralized observers include [10], where distributed Kalman filters are adopted, [11] where both linear and nonlinear interconnected systems are considered and [12], which introduces a distributed, finite-time convergent k-hop observer where each agent in the network only needs to communicate with its 1-hop neighbors to estimate its own state and input, as well as those of the agents which resides up to k-hops away.

This paper is inspired by the k-hop observer-based distributed control strategy proposed in [12]. By the observation that each agent in the network knows its own state and input and may be able to receive those of its 1-hop neighbors through communication, we develop a distributed observer by restricting the estimation only to the state and input of those agents which are 2-hop distant or more. Compared to the work in [12], our results increase the speed of convergence of each agent's observer estimation by exploiting the value, provided by the common 1-hop neighbors, of the state of those elements that are 2-hop distant. In particular, we propose a finite time k-hop distributed observer for nonlinear systems where each agent estimates the state and the input of the agents 2-hop or more distant, by interacting only with the agents belonging to its 1-hop neighborhood. Moreover, we show that under a bounded input, the state observer convergence results to be independent of the input observer dynamics. We also show that by adopting a set-ISS feedback control law it is possible to exploit the state estimation information to drive the system towards an equilibrium representing the team objective. Furthermore, given the reduced number of estimates that each observer needs to update compared to those in [12], the proposed solution results more scalable while dealing with large-scale networks.

The paper is organized as follow: Section II presents the preliminaries and the problem setting. Section III and Section IV respectively propose the results concerning the k-hop distributed state and input observers. In Section V we introduce the feedback control structure and provide the conditions that guarantee the convergence of a k-hop estimation-based feedback controller towards the team objective. In Section VI we provide a simulation result to demonstrate the convergence towards the goal when k-hop estimation is used in the feedback controller, and in Section VII we provide final remarks and future work.

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II. Preliminaries and Problem Setting

Notation: We denote by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ the set of real and non-negative real numbers, respectively. Let $|S|$ be the cardinality of a set S, $\mathbb{R}^n$ be an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ be a space of real matrices with n rows and m columns. Denote by I_n the identity matrix of size n and by 1_n the vector of ones of size n. Given a matrix $B \in \mathbb{R}^{n\times n}$, we represent with $\lambda_i(B)$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ respectively the i-th, minimum and maximum eigenvalues belonging to the spectrum $\sigma(B)$ of matrix B. Given a positive definite matrix $B = B^\top \succ 0$ and a vector $x \in \mathbb{R}^n$, $\|x\|_B = \sqrt{x^\top B x}$, with the convention $\|x\| = \|x\|_I$. Additionally, $\|x\|_1 = \sum_{i=1}^n |x_i|$. Given a matrix B, we use $B \prec 0$ to denote that B is negative definite. Let $\text{diag}(a_1,\dots,a_n)$ be the diagonal matrix with diagonal elements $a_1, ..., a_n$ and let $\otimes$ be the Kronecker product. We represent with $\text{sign}(x)$ the non-smooth function defined as: $\text{sign}(x) = 1$ if $x \geq 0$ and $\text{sign}(x) = -1$ if $x < 0$. Given the presence of the $\text{sign}(\cdot)$ function in the observers' dynamics, non-smooth analysis is required to study their convergence. For this purpose, as in [13, (1.2a)], we denote by $K[f] : \mathbb{R}^m \to \mathbb{R}^m$ the set-valued map of a measurable, locally bounded function $f(y) : \mathbb{R}^m \to \mathbb{R}^m$, i.e. the function defined as $K[f](y) := \bigcap_{\delta>0}\bigcap_{\mu\{M\}=0} \overline{\text{co}}\{f(\mathcal{B}(y,\delta)/M)\}$, where $\mathcal{B}(y,\delta)$ denotes the ball of radius δ centered at y, $\cap_{\mu\{M\}=0}$ the intersection over all sets M of Lebesgue measure zero, $\mathcal{B}(y,\delta)/M$ the set difference between $\mathcal{B}(y,\delta)$ and M and $\overline{\text{co}}$ the convex closure. Moreover, we further define $\|K[f]\| = \sup_{z\in R(K[f])}\|z\|$, where $R(K[f]) = \bigcup_{y\in\mathbb{R}^m} K[f](y)$. We use notations $\mathcal{K}$ and $\mathcal{KL}$ to denote the different classes of comparison functions, as follows: $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \gamma$ is continuous, strictly increasing and $\gamma(0) = 0\}$; $\mathcal{KL} = \{\beta : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$: for each fixed s, the map $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty\}$.

A. Multi-agent systems

Consider a multi-agent system composed of a set of n interacting agents $\mathcal{V} = \{1,2,\dots,n\}$. Suppose each agent $i \in \mathcal{V}$ behaves according to the nonlinear dynamics:

$$\dot{x}_i(t) = f(x_i) + Ax_i + u_i, \tag{1}$$

where $x_i \in \mathbb{X} \subset \mathbb{R}^N$ with $\mathbb{X}$ denoting the state space, $A \in \mathbb{R}^{N\times N}$, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz nonlinear function with Lipschitz constant l_f and $u_i \in \mathbb{U} \subset \mathbb{R}^N$ is a measurable and essentially locally bounded function satisfying Assumption 1.

Assumption 1: One of the following conditions holds:

1) u_i is bounded with known upper bound $d_{u_i} \in \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{V}$.
2) The derivative $K[\dot{u}_i](\cdot)$ is bounded with known upper bound $d_{\dot{u}_i} \in \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{V}$.

Furthermore, denote with $\boldsymbol{x}$ and $\boldsymbol{u}$ the stacked vector of agents' state and input:

$$\boldsymbol{x} = \left[x_1^\top, \dots, x_n^\top\right]^\top, \ \boldsymbol{u} = \left[u_1^\top, \dots, u_n^\top\right]^\top. \tag{2}$$

B. Communication graph

Let the communication capabilities among the agents be described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges representing the set of established communication among the agents.

Define a path between agent $i \in \mathcal{V}$ and $j \in \mathcal{V}$ as the set of non-repeating edges through which j can be reached by i. Under this definition, a k-hop path between agents $i, j \in \mathcal{V}$ is a path involving k edges from i to j.

Denote with $\mathcal{N}_i^{k\text{-hop}}$ the set of k-hop neighbors of agent $i \in \mathcal{V}$, i.e., the set of nodes $j \in \mathcal{V}$ for which there exists a p-hop path from j to i with $2 \leq p \leq k$. Moreover, indicate the elements of this set with $\mathcal{N}_i^{k\text{-hop}} = \{n_1^i, \dots, n_{\eta_i}^i\}$, where each n_j^i with $j \in \{1,\dots,\eta_i\}$ is the global index of the j-th k-hop neighbor of i and where $\eta_i = |\mathcal{N}_i^{k\text{-hop}}|$ represents its cardinality. For brevity, we indicate with $\mathcal{N}_i$ the set of 1-hop neighbors of agent $i \in \mathcal{V}$.

Denote with $\boldsymbol{x}^i$ and $\boldsymbol{u}^i$ the vectors containing respectively the state x_j and input u_j of the agents $j \in \mathcal{N}_i^{k\text{-hop}}$:

$$\boldsymbol{x}^i = \left[x_{n_1^i}^\top, \dots, x_{n_{\eta_i}^i}^\top\right]^\top, \ \boldsymbol{u}^i = \left[u_{n_1^i}^\top, \dots, u_{n_{\eta_i}^i}^\top\right]^\top \tag{3}$$

and let:

$$\hat{\boldsymbol{x}}^i = \left[\hat{x}_{n_1^i}^{i\,\top}, \dots, \hat{x}_{n_{\eta_i}^i}^{i\,\top}\right]^\top, \hat{\boldsymbol{u}}^i = \left[\hat{u}_{n_1^i}^{i\,\top}, \dots, \hat{u}_{n_{\eta_i}^i}^{i\,\top}\right]^\top \tag{4}$$

be the vectors containing their estimate carried out by the agent i, i.e. $\hat{x}_j^i$ and $\hat{u}_j^i$ are the estimates of the state x_j and input u_j of agent $j \in \mathcal{N}_i^{k\text{-hop}}$ done by i. Furthermore, denote with $\tilde{\boldsymbol{x}}^i$ and $\tilde{\boldsymbol{u}}^i$ the estimation errors:

$$\tilde{\boldsymbol{x}}^i = \boldsymbol{x}^i - \hat{\boldsymbol{x}}^i, \quad \tilde{\boldsymbol{u}}^i = \boldsymbol{u}^i - \hat{\boldsymbol{u}}^i. \tag{5}$$

Indicate with $\boldsymbol{x}_i$ and $\boldsymbol{u}_i$ the vectors defined as $\boldsymbol{x}_i = 1_{\eta_i} \otimes x_i$, $\boldsymbol{u}_i = 1_{\eta_i} \otimes u_i$ and denote with $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{u}}_i$ the stacked vector estimates of x_i and u_i computed by each of the agents $j \in \mathcal{N}_i^{k\text{-hop}}$:

$$\hat{\boldsymbol{x}}_i = \left[\hat{x}_i^{n_1^i\,\top}, \dots, \hat{x}_i^{n_{\eta_i}^i\,\top}\right]^\top, \hat{\boldsymbol{u}}_i = \left[\hat{u}_i^{n_1^i\,\top}, \dots, \hat{u}_i^{n_{\eta_i}^i\,\top}\right]^\top. \tag{6}$$

Similar to (5), we can define the estimation errors on $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{u}}_i$, computed by all $j \in \mathcal{N}_i^{k\text{-hop}}$ as:

$$\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - \hat{\boldsymbol{x}}_i, \quad \tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i - \hat{\boldsymbol{u}}_i. \tag{7}$$

For each $i \in \mathcal{V}$, denote with $\hat{\boldsymbol{P}}_i$ the matrix selecting the states estimated by agent i, i.e. $\boldsymbol{x}^i = \hat{\boldsymbol{P}}_i\boldsymbol{x}$, where $\hat{\boldsymbol{P}}_i = \hat{P}_i \otimes I_N$ and $\hat{P}_i = [e_{n_1^i}\ e_{n_2^i} \dots e_{n_{\eta_i}^i}]^\top$ is a $\eta_i \times n$ binary matrix where each e_j with $j \in \mathcal{N}_i^{k\text{-hop}}$ is a vector with all zeros except from the j-th element which is equal to 1. In a similar way, let $\boldsymbol{P}_i$ be the $|\mathcal{N}_i| \times n$ matrix selecting the components of the states of the 1-hop neighbors of agent i.

Example 1: Consider a network of 4 agents communicating according to the graph in Fig. 1. Suppose $N = 1$ and $k = 3$. Denote the global state with $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^\top$. Then, the set of 3-hop neighbors of agent 1 is $\mathcal{N}_1^{3\text{-hop}} = \{n_1^1, n_1^2\} = \{3, 4\}$, $\hat{\boldsymbol{x}}^1 = \begin{bmatrix}\hat{x}_3^1 & \hat{x}_4^1\end{bmatrix}^\top$, $\hat{\boldsymbol{x}}_1 = \begin{bmatrix}\hat{x}_1^3 & \hat{x}_1^4\end{bmatrix}^\top$, $\hat{\boldsymbol{P}}_1 = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ and $\boldsymbol{P}_1 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}$.

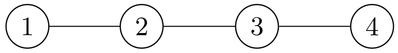


Fig. 1: Example for a path communication graph and k=3.

Suppose the following hold for the communication graph:

Assumption 2: Each $i \in \mathcal{V}$ knows which are the agents belonging to $\mathcal{N}_i$ and $\mathcal{N}_i^{k\text{-hop}}$.

Assumption 3: Each agent $i \in \mathcal{V}$ has access and can propagate at each time instant the state and input of the agents $j \in \mathcal{N}_i$ to its 1-hop neighbors $\mathcal{N}_i$.

Note that, while the last assumption on the state is reasonable if we consider sensing capabilities of the agents, the one on the input requires the communication to be fast enough with negligible time delays.

With the given notation, the first problem is formalized as:

***Problem** 1:* Design a finite-time convergent distributed observer such that there exists $T_{x,i} > 0$ for which $\|\tilde{\boldsymbol{x}}^i(t)\| = 0$ for all $t \geq T_{x,i}$ and all $i \in \mathcal{V}$.

III. Distributed k-hop State Observer

In this section a finite-time distributed observer to solve Problem 1 is introduced. For this purpose, assume that the state estimate $\hat{\boldsymbol{x}}^i$ is updated following (8):

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}^i &= \boldsymbol{f}(\hat{\boldsymbol{x}}^i) + \boldsymbol{A}^i\hat{\boldsymbol{x}}^i + \boldsymbol{\Omega}^i\boldsymbol{G}^i\boldsymbol{\xi}^i + \boldsymbol{\Theta}^i\text{sign}(\boldsymbol{G}^i\boldsymbol{\xi}^i) + \hat{\boldsymbol{u}}^i \\ \boldsymbol{\xi}^i &= \sum_{j\in\mathcal{N}_i}[\hat{\boldsymbol{P}}_i(-\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{P}}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{x}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{x}}^j) + \\ &\quad + \hat{\boldsymbol{P}}_i(-\boldsymbol{P}_j^\top\boldsymbol{P}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{x}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\boldsymbol{P}_j^\top\boldsymbol{P}_j\boldsymbol{x})],\end{aligned} \tag{8}$$

where $\boldsymbol{f}(\hat{\boldsymbol{x}}^i) = \left[f^\top(\hat{x}^i_{n_1^i})\dots, f^\top(\hat{x}^i_{n^i_{\eta_i}})\right]^\top$, $\boldsymbol{A}^i = I_{\eta_i}\otimes A$, $\boldsymbol{\Omega}^i = \Omega^i \otimes I_N$, $\boldsymbol{\Theta}^i = \Theta^i \otimes I_N$ and $\boldsymbol{G}^i = I_{\eta_i}\otimes G$ with:

$$\Omega^i = \text{diag}(\omega_{n_1^i},\dots,\omega_{n^i_{\eta_i}})\ ,\ \Theta^i = \text{diag}(\theta_{n_1^i}\dots,\theta_{n^i_{\eta_i}}), \tag{9}$$

$\omega_j \in \mathbb{R}_{\geq 0}$ and $\theta_j \in \mathbb{R}_{\geq 0}$ observer parameters to be tuned for all $j \in \mathcal{V}$ and where $G \in \mathbb{R}^{N\times N}$ is a positive symmetric matrix to be designed. In (8), each agent $i \in \mathcal{V}$ updates its state estimate $\hat{\boldsymbol{x}}^i$ of all $l \in \mathcal{N}_i^{k\text{-hop}}$ based on the real state information x_l transmitted by the agents $j \in \mathcal{N}_i \cap \mathcal{N}_l$: $\hat{\boldsymbol{P}}_i(-\boldsymbol{P}_j^\top\boldsymbol{P}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{x}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\boldsymbol{P}_j^\top\boldsymbol{P}_j\boldsymbol{x})$ and on the estimation $\hat{x}_l^j$ shared by those $j \in \mathcal{N}_i \cap \mathcal{N}_l^{k\text{-hop}}$: $\hat{\boldsymbol{P}}_i(-\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{P}}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{x}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{x}}^j)$.

Remark 1: Since for any graph $\mathcal{G}$ and any value of k it is always possible to find a full rank permutation matrix T of proper dimension such that $\left[(\tilde{\boldsymbol{x}}^1)^\top,\dots,(\tilde{\boldsymbol{x}}^n)^\top\right]^\top = T\left[\tilde{\boldsymbol{x}}_1^\top,\dots,\tilde{\boldsymbol{x}}_n^\top\right]^\top$, the convergence of $\tilde{\boldsymbol{x}}_i(t)$ as per (7) implies the one of $\tilde{\boldsymbol{x}}^i(t)$ as per (5).

As a result, Problem 1 can be reformulated as:

***Problem** 2:* Design a finite-time convergent distributed observer such that there exists $T_{x,i} > 0$ for which $\|\tilde{\boldsymbol{x}}_i(t)\| = 0$ for all $t \geq T_{x,i}$ and all $i \in \mathcal{V}$.

Denote with $\tilde{x}_i^l$ and $\tilde{u}_i^l$ the l-th component of vectors $\tilde{\boldsymbol{x}}_i$ and $\tilde{\boldsymbol{u}}_i$, i.e., the estimation errors on the i-th agent state and input when the estimate is performed by agent $l \in \mathcal{N}_i^{k\text{-hop}}$ and define with ξ_i^l the update, associated to $\boldsymbol{\xi}^l$ in (8), on the estimation of the i-th agent state done by agent l. Then, from $\dot{\hat{\boldsymbol{x}}}^l$, (1) and (7), the dynamics of $\tilde{x}_i^l$ results into:

$$\begin{aligned}\dot{\tilde{x}}_i^l &= \bar{f}(\hat{x}_i^l) + A\tilde{x}_i^l - \omega_i G\xi_i^l - \theta_i\text{sign}(G\xi_i^l) + \tilde{u}_i^l, \\ \xi_i^l &= \sum_{k\in(\mathcal{N}_l\cap\mathcal{N}_i^{k\text{-hop}})}(\tilde{x}_i^l - \tilde{x}_i^k) + \sum_{k\in(\mathcal{N}_l\cap\mathcal{N}_i)}\tilde{x}_i^l,\end{aligned} \tag{10}$$

where $\bar{f}(\hat{x}_i^l) = f(x_i) - f(\hat{x}_i^l)$.

By defining the vector $\boldsymbol{\xi}_i := \left[(\xi_i^{n_1^i})^\top,\dots,(\xi_i^{n^i_{\eta_i}})^\top\right]^\top$:

$$\boldsymbol{\xi}_i = \left((L_i^{\text{kc}} + H_i^{\text{kc}})\otimes I_N\right)\tilde{\boldsymbol{x}}_i = (M_i^{\text{kc}}\otimes I_N)\tilde{\boldsymbol{x}}_i, \tag{11}$$

where L_i^{kc} is the Laplacian matrix of the sub-graph $\mathcal{G}_i = (\mathcal{N}_i^{k\text{-hop}},\mathcal{E}_i)$ induced by the k-hop neighbors of agent i, with $\mathcal{E}_i = \{(p,q)\in\mathcal{E} : \{p,q\}\in\mathcal{N}_i^{k\text{-hop}}\}$ [1, pp. 24], $H_i^{\text{kc}} = \text{diag}\left(|\mathcal{N}_{n_1^i}\cap\mathcal{N}_i|,\dots,|\mathcal{N}_{n^i_{\eta_i}}\cap\mathcal{N}_i|\right)$ and M_i^{kc} is defined as:

$$M_i^{\text{kc}} = L_i^{\text{kc}} + H_i^{\text{kc}}. \tag{12}$$

By means of the mixed-product property of the Kronecker product in [14] and (11), the vector form of (10) results into:

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}_i &= (\boldsymbol{f}(\boldsymbol{x}_i) - \boldsymbol{f}(\hat{\boldsymbol{x}}_i)) + (\boldsymbol{A}^i - \omega_i(M_i^{\text{kc}}\otimes G))\tilde{\boldsymbol{x}}_i + \\ &\quad - \theta_i\text{sign}((M_i^{\text{kc}}\otimes G)\tilde{\boldsymbol{x}}_i) + \tilde{\boldsymbol{u}}_i,\end{aligned} \tag{13}$$

where $\boldsymbol{f}(\hat{\boldsymbol{x}}_i)$, $\boldsymbol{A}^i$, G, ω_i and θ_i come from the dynamics in (8), and $\boldsymbol{f}(\boldsymbol{x}_i) = 1_{\eta_i}\otimes f(x_i)$.

Lemma 1: Consider an undirected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$. If $\mathcal{G}$ is connected, $|\mathcal{N}_i\cap\mathcal{N}_j^{k\text{-hop}}| > 0$ or $|\mathcal{N}_i\cap\mathcal{N}_j| > 0$ for all $j\in\mathcal{V}$ and all $i\in\mathcal{N}_j^{k\text{-hop}}$. Furthermore, for all $j\in\mathcal{V}$ and for each connected component in the sub-graph $G_j = \{\mathcal{N}_j^{k\text{-hop}},\mathcal{E}_j\}$, there exists at least one agent $i\in\mathcal{N}_j^{k\text{-hop}}$ for which $|\mathcal{N}_i\cap\mathcal{N}_j^{k\text{-hop}}| > 0$ and $|\mathcal{N}_i\cap\mathcal{N}_j| > 0$.

Proof: The first part of Lemma 1 can be proved by contradiction. Assume there exists $j\in\mathcal{V}$ and $i\in\mathcal{N}_j^{k\text{-hop}}$ such that $|\mathcal{N}_i\cap\mathcal{N}_j^{k\text{-hop}}| = 0$ and $|\mathcal{N}_i\cap\mathcal{N}_j| = 0$. If $|\mathcal{N}_i\cap\mathcal{N}_j| = 0$ and $i\in\mathcal{N}_j^{k\text{-hop}}$, agent i is connected to agent j by means of a path of length $l > 2$. This implies that $|\mathcal{N}_i\cap\mathcal{N}_j^{k\text{-hop}}| \neq 0$, which however results to be in contradiction with the initial assumption. In a similar way, $|\mathcal{N}_i\cap\mathcal{N}_j^{k\text{-hop}}| = 0$ and $i\in\mathcal{N}_j^{k\text{-hop}}$ implies that agent i is connected to j by means of a path of length 2. Therefore, there must exist an agent $k\in\mathcal{N}_i\cap\mathcal{N}_j$, which contradicts $|\mathcal{N}_i\cap\mathcal{N}_j| = 0$.

To prove the second statement, suppose by contradiction that for an agent $j\in\mathcal{V}$, all i in a connected component of $\mathcal{G}_j = (\mathcal{N}_j^{k\text{-hop}},\mathcal{E}_j)$ are such that $|\mathcal{N}_i\cap\mathcal{N}_j| = 0$. This implies that there does not exist a l-hop path, with $l\leq k$, between i and j, which contradicts the assumption $i\in\mathcal{N}_j^{k\text{-hop}}$. ■

Lemma 2: Consider an undirected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$. If $\mathcal{G}$ is connected, $M_i^{\text{kc}}\succ 0$ for all $i\in\mathcal{V}$ with $\mathcal{N}_i^{k\text{-hop}}\neq\emptyset$.

Proof: Since M_i^{kc} is the sum of two real positive semi-definite symmetric matrices (12), it results to be Hermitian [15, Def. 4.1.1]. Therefore, according to [15, Corollary 4.3.12], its eigenvalues satisfy $\lambda_j(L_i^{\text{kc}})\leq\lambda_j(M_i^{\text{kc}})$ for all $j\in\{1,\dots,\eta_i\}$, with equality for some j if and only if H_i^{kc} is singular and there exists a nonzero vector x such that $L_i^{\text{kc}}x = \lambda_j(L_i^{\text{kc}})x$, $H_i^{\text{kc}}x = 0$ and $M_i^{\text{kc}}x = \lambda_j(M_i^{\text{kc}})x$. Since L_i^{kc} is the Laplacian matrix of the sub-graph $\mathcal{G}_i = (\mathcal{N}_i^{k\text{-hop}}, \mathcal{E}_i)$, its smallest eigenvalue is equal to zero and the algebraic multiplicity of the null eigenvalue is related to the number of connected components in the graph $\mathcal{G}_i$. For this reason, since the eigenvectors associated to the zero eigenvalue of L_i^{kc} represent a base for the null space of L_i^{kc}, the positive definiteness of M_i^{kc} is directly guaranteed if none of the vectors in the null space of L_i^{kc} is orthogonal to H_i^{kc}. In this case indeed, from [15, Corollary 4.3.12], $0 < \lambda_j(M_i^{\mathrm{kc}})$ for all $j \in \{1, \dots, \eta_i\}$. Since L_i^{kc} represents the Laplacian matrix of a graph characterized by several connected components, each eigenvector associated with the zero eigenvalue belongs to the span of the vectors of ones representing the consensus among the agents of the connected sub-graphs. Therefore, since for Lemma 1 each connected component has at least one associated non-zero element in the diagonal matrix H_i^{kc}, none of the eigenvectors of L_i^{kc} is orthogonal to H_i^{kc}, leading to the strict inequality $0 < \lambda_j(M_i^{\mathrm{kc}})$ for all $j \in \{1, \dots, \eta_i\}$, and therefore to the positive definiteness of the matrix M_i^{kc} defined as in (12). ■

Lemma 3: Consider matrices $\boldsymbol{A}^i$ and M_i^{kc} as per (8) and (12), respectively. Assume (14), (15) holds, i.e.:

$$\omega_i \geq \frac{1}{\lambda_{\min}(M_i^{\mathrm{kc}})} \left(1 + \frac{l_f \|(M_i^{\mathrm{kc}} \otimes G)\|}{\lambda_{\min}(M_i^{\mathrm{kc}}) \lambda_{\min}(G^\top G)}\right), \quad (14)$$

$$G^\top A + A^\top G - 2G^\top G \prec 0. \quad (15)$$

Then:

$$(M_i^{\mathrm{kc}} \otimes G)(\boldsymbol{A}^i - \omega_i (M_i^{\mathrm{kc}} \otimes G)) + l_f \|(M_i^{\mathrm{kc}} \otimes G)\| I_{N\eta_i} \prec 0. \quad (16)$$

Proof: Given the positive definiteness of the symmetric matrix M_i^{kc}, it is always possible to find a matrix $T_i \in \mathbb{R}^{\eta_i \times \eta_i}$ with respect to which M_i^{kc} can be written as $T_i \Lambda_i^{\mathrm{kc}} T_i^\top = M_i^{\mathrm{kc}}$, where $\Lambda_i^{\mathrm{kc}} = \mathrm{diag}(\lambda_1, \dots, \lambda_{\eta_i})$ and $\lambda_j = \lambda_j(M_i^{\mathrm{kc}})$ for all $j \in \{1, \dots, \eta_i\}$ [15]. By replacing $M_i^{\mathrm{kc}} = T_i \Lambda_i^{\mathrm{kc}} T_i^\top$ in the term $(M_i^{\mathrm{kc}} \otimes G)(\boldsymbol{A}^i - \omega_i (M_i^{\mathrm{kc}} \otimes G))$ in (16), it results $(T_i \Lambda_i^{\mathrm{kc}} T_i^\top \otimes G)\boldsymbol{A}^i - \omega_i (T_i \Lambda_i^{\mathrm{kc}} T_i^\top \otimes G)(T_i \Lambda_i^{\mathrm{kc}} T_i^\top \otimes G)$.

Since for matrices A, B, C and D of proper dimensions $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ holds, $\boldsymbol{A}^i$, $I_{N\eta_i}$ and $(T_i \Lambda_i^{\mathrm{kc}} T_i^\top \otimes G)$ can be rewritten as $\boldsymbol{A}^i = (T_i \otimes I_N)(I_{\eta_i} \otimes A)(T_i^\top \otimes I_N)$, $I_{N\eta_i} = (T_i \otimes I_N)(T_i^\top \otimes I_N)$ and $(T_i \Lambda_i^{\mathrm{kc}} T_i^\top \otimes G) = (T_i \otimes I_N)(\Lambda_i^{\mathrm{kc}} \otimes G)(T_i^\top \otimes I_N)$. By introducing these formulations in (16), after some manipulation it results:

$$\begin{aligned}(T_i \otimes I_N)[(\Lambda_i \otimes GA) - \omega_i(\Lambda_i^2 \otimes G^\top G) + \\ + l_f \|(M_i^{\mathrm{kc}} \otimes G)\| I_{N\eta_i}](T_i^\top \otimes I_N) \prec 0,\end{aligned} \quad (17)$$

whose validity can be studied by neglecting the outer terms $T_i \otimes I_N$ and $T_i^\top \otimes I_N$. Then, since $[(\Lambda_i \otimes GA) - \omega_i(\Lambda_i^2 \otimes G^\top G) + l_f \|(M_i^{\mathrm{kc}} \otimes G)\| I_{N\eta_i}]$ is a block diagonal matrix with elements $\lambda_j \left(G^\top A - \omega_i \lambda_j G^\top G + \frac{l_f \|(M_i^{\mathrm{kc}} \otimes G)\|}{\lambda_j} I_N\right)$, to prove (17) it suffices to prove:

$$\lambda_j \left(G^\top A - \omega_i \lambda_j G^\top G + \frac{l_f \|(M_i^{\mathrm{kc}} \otimes G)\|}{\lambda_j} I_N\right) \prec 0, \quad (18)$$

for all eigenvalues $\lambda_j \in \sigma(M_i^{\mathrm{kc}})$. In this regard, if ω_i is tuned as in (14), the positive term $\frac{l_f \|(M_i^{\mathrm{kc}} \otimes G)\|}{\lambda_j} I_N$ in (18) is dominated and since, from Lemma 2, $\lambda_j > 0$ for all $j \in \{1, \dots, \eta_i\}$, (18) is satisfied for all $\lambda_j \in \sigma(M_i^{\mathrm{kc}})$ if $G^\top A - G^\top G \prec 0$. For this reason, since the negative definiteness of $G^\top A - G^\top G$ follows from its symmetric part $\frac{1}{2}(G^\top A + A^\top G - 2G^\top G)$ [15, pp. 231], to guarantee (18), and therefore (16), it suffices to design ω_i and G as in (14) and (15). ■

Remark 2: The existence of a matrix G that satisfies (15) is guaranteed by assuming (A, I_N) stabilizable and observable [16, Th. 2].

By Lemma 3, the convergence of the state estimation can be stated in Theorem 1.

Theorem 1: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed state observer (8). For $i \in \mathcal{V}$, consider the error dynamics in (13) and assume that $\|K[\tilde{\boldsymbol{u}}_i]\|$ is bounded by $d_{\tilde{\boldsymbol{u}}_i}$. Then, $\tilde{\boldsymbol{x}}_i(t)$ as per (7) reaches the origin in finite time $T_{x,i} > 0$ given that the gain θ_i as per (9) is tuned such that:

$$\theta_i > \frac{\lambda_{\max}(M_i^{\mathrm{kc}}) \lambda_{\max}(G)}{\lambda_{\min}(M_i^{\mathrm{kc}}) \lambda_{\min}(G)} d_{\tilde{\boldsymbol{u}}_i}, \quad (19)$$

and that ω_i and G are designed so that conditions (14) and (15) in Lemma 3 hold. Furthermore:

$$T_{x,i} \leq \frac{\lambda_{\max}(M_i^{\mathrm{kc}}) \lambda_{\max}(G)}{\phi_i} \|\tilde{\boldsymbol{x}}_i(0)\| \quad (20)$$

with:

$$\phi_i = \theta_i \lambda_{\min}(M_i^{\mathrm{kc}}) \lambda_{\min}(G) - \|(M_i^{\mathrm{kc}} \otimes G)\| \|K[\tilde{\boldsymbol{u}}_i]\|. \quad (21)$$

Proof: Since the proposed observer and the error dynamics are discontinuous, non-smooth analysis must be used to prove the finite-time convergence of (13) [13], [17].

Consider the following candidate Lyapunov function:

$$V_i(\tilde{\boldsymbol{x}}_i) = \frac{1}{2} \tilde{\boldsymbol{x}}_i^\top (M_i^{\mathrm{kc}} \otimes G) \tilde{\boldsymbol{x}}_i. \quad (22)$$

Given the continuous differentiability of (22), its time derivative satisfies $\dot{V}_i(\tilde{\boldsymbol{x}}_i) \overset{a.e.}{\in} \mathring{V}_i(\tilde{\boldsymbol{x}}_i)$, where the generalized derivative $\mathring{V}_i(\tilde{\boldsymbol{x}}_i)$ assumes the expression $\mathring{V}_i(\tilde{\boldsymbol{x}}_i) = \nabla V_i(\tilde{\boldsymbol{x}}_i)^\top K[\dot{\tilde{\boldsymbol{x}}}_i](\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{u}}_i)$, where $\nabla V_i(\tilde{\boldsymbol{x}}_i)$ denotes the gradient of $V_i(\tilde{\boldsymbol{x}}_i)$. By introducing (13) and the gradient expression, after some manipulations resulting from properties of the Kronecker product and of the set-valued map $K[\;](\cdot)$ [13, Th. 1], $\mathring{V}_i(\tilde{\boldsymbol{x}}_i)$ can be rewritten as $\mathring{V}_i(\tilde{\boldsymbol{x}}_i) \subset \tilde{\boldsymbol{x}}_i^\top (M_i^{\mathrm{kc}} \otimes G)(\boldsymbol{f}(\boldsymbol{x}_i) - \boldsymbol{f}(\hat{\boldsymbol{x}}_i)) + \tilde{\boldsymbol{x}}_i^\top (M_i^{\mathrm{kc}} \otimes G)(\boldsymbol{A}^i - \omega_i (M_i^{\mathrm{kc}} \otimes G))\tilde{\boldsymbol{x}}_i - \theta_i \|(M_i^{\mathrm{kc}} \otimes G)\tilde{\boldsymbol{x}}_i\|_1 + \tilde{\boldsymbol{x}}_i^\top (M_i^{\mathrm{kc}} \otimes G) K[\tilde{\boldsymbol{u}}_i]$. Then, by noticing $\|(M_i^{\mathrm{kc}} \otimes G)\tilde{\boldsymbol{x}}_i\|_1 \geq \|(M_i^{\mathrm{kc}} \otimes G)\tilde{\boldsymbol{x}}_i\| \geq \lambda_{\min}(M_i^{\mathrm{kc}}) \lambda_{\min}(G) \|\tilde{\boldsymbol{x}}_i\|$ and that Lemma 3 holds due to the validity of (15) and (16), i.e. $\tilde{\boldsymbol{x}}_i^\top [(M_i^{\mathrm{kc}} \otimes G)(\boldsymbol{A}^i - \omega_i (M_i^{\mathrm{kc}} \otimes G)) + l_f \|(M_i^{\mathrm{kc}} \otimes G)\| I_{N\eta_i}]\tilde{\boldsymbol{x}}_i \leq 0$ for all $\tilde{\boldsymbol{x}}_i \in \mathbb{R}^{N\eta_i}$, the generalized derivative of the Lyapunov function can be upper bounded by:

$$\mathring{V}_i(\tilde{\boldsymbol{x}}_i) \leq -\phi_i \|\tilde{\boldsymbol{x}}_i\|, \quad (23)$$

where ϕ_i is defined as per (21). Then, if θ_i is designed according to (19), ϕ_i results to be strictly positive, thus proving the convergence of (13).

By recalling the definition of the Lyapunov function in (22), $V_i(\tilde{\boldsymbol{x}}_i)^{\frac{1}{2}} \leq \sqrt{\frac{1}{2} \lambda_{\max}(M_i^{\mathrm{kc}}) \lambda_{\max}(G)} \|\tilde{\boldsymbol{x}}_i\|$, from which

(23) results into $\mathring{V}_i(\tilde{\boldsymbol{x}}_i) \le -V_i(\tilde{\boldsymbol{x}}_i)^{\frac{1}{2}} \frac{\phi_i\sqrt{2}}{\sqrt{\lambda_{\max}(M_i^{\text{kc}})\lambda_{\max}(G)}}$. By solving this inequality with respect to time, we can compute the upper bound on the convergence time, i.e. $T_{x,i} \le \frac{\lambda_{\max}(M_i^{\text{kc}})\lambda_{\max}(G)}{\phi_i}\|\tilde{\boldsymbol{x}}_i(0)\|$, which guarantees the finite time convergence of (13) [18]. ∎

Given the equivalence between Problem 1 and Problem 2, the following can be stated from Theorem 1:

Corollary 1: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed state observer (8). Under Assumption 1, $\|\tilde{\boldsymbol{x}}^i(t)\| \le \|\tilde{\boldsymbol{x}}^i(0)\|$ for all $t \ge 0$ and $i \in \mathcal{V}$. Furthermore, there exists a $T_x > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| = 0$ for all $t > T_x$ with $T_x = \max_{i\in\mathcal{V}}\{T_{x,i}\}$.

Proof: Given Theorem 1 and the equivalence between the convergence of $\tilde{\boldsymbol{x}}_i(t)$ and $\tilde{\boldsymbol{x}}^i(t)$ from Remark 1, there exists a time $T_{x,i}$, for all $i \in \mathcal{V}$, that satisfies $\|\tilde{\boldsymbol{x}}_i(t)\| = 0$ for all $t > T_{x,i}$. As a result, for every $t > T_x$ with $T_x = \max_{i\in\mathcal{V}}\{T_{x,i}\}$, $\|\tilde{\boldsymbol{x}}_i(t)\| = 0$ for all $i \in \mathcal{V}$. ∎

IV. Distributed k-hop Input Observer

In this section, we present a finite-time distributed input observer to allow each agent $i \in \mathcal{V}$ to estimate the input of all $j \in \mathcal{N}_i^{k\text{-hop}}$, i.e. $\hat{\boldsymbol{u}}^i(t)$ as per (4). For this purpose, consider the following dynamics for the input estimations:

$$\begin{aligned} \dot{\hat{\boldsymbol{u}}}^i &= \boldsymbol{\Pi}^i \text{sign}(\boldsymbol{\rho}^i) \\ \boldsymbol{\rho}^i &= \sum_{j\in\mathcal{N}_i}[\hat{\boldsymbol{P}}_i(-\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{P}}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{u}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\hat{\boldsymbol{P}}_j^\top\hat{\boldsymbol{u}}^j) + \\ &\quad + \hat{\boldsymbol{P}}_i(-\boldsymbol{P}_j^\top\boldsymbol{P}_j\hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{u}}^i + \hat{\boldsymbol{P}}_i^\top\hat{\boldsymbol{P}}_i\boldsymbol{P}_j^\top\boldsymbol{P}_j\boldsymbol{u})], \end{aligned} \tag{24}$$

where $\boldsymbol{\Pi}^i = \Pi^i \otimes I_N$, $\Pi^i \in \mathbb{R}^{\eta_i\times\eta_i}$ is a diagonal matrix of the form $\Pi^i = \text{diag}(\pi_{n_1^i}, \dots \pi_{n_{\eta_i}^i})$, and $\pi_{n_j^i} \in \mathbb{R}_{\ge 0}$ with $j \in \{1,\dots,\eta_i\}$ is a design parameter to be tuned. Similarly to the state estimation case, the convergence of the input estimation error $\tilde{\boldsymbol{u}}^i$ can be equivalently formulated in terms of $\tilde{\boldsymbol{u}}_i$. Starting from (24) and following similar manipulations to those performed for the state observer from (10) to (13), the estimation error dynamics on the i-th agent input results into:

$$\dot{\tilde{\boldsymbol{u}}}_i = \dot{\boldsymbol{u}}_i - \pi_i \text{sign}((M_i^{\text{kc}} \otimes I_N)\tilde{\boldsymbol{u}}_i), \tag{25}$$

with M_i^{kc} defined as in (12). Then, the convergence behavior of (25) can be formulated as in Theorem 2.

Theorem 2: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed input observer (24). For $i \in \mathcal{V}$, consider the error dynamics in (25) and assume that $\|K[\dot{u}_i]\|$ is bounded by $d_{\dot{u}_i}$ as per Assumption 1. Then, $\tilde{\boldsymbol{u}}_i$ reaches the origin in finite time $T_{u,i} > 0$ given that the gain π_i is tuned such that $\pi_i > \frac{\lambda_{\max}(M_i^{\text{kc}})}{\lambda_{\min}(M_i^{\text{kc}})}\sqrt{\eta_i}d_{\dot{u}_i}$. Furthermore:

$$T_{u,i} \le \frac{\lambda_{\max}(M_i^{\text{kc}})}{\psi_i}\|\tilde{\boldsymbol{u}}_i(0)\|, \tag{26}$$

with $\psi_i = \left[\pi_i\lambda_{\min}(M_i^{\text{kc}}) - \|(M_i^{\text{kc}} \otimes I_N)\|\sqrt{\eta_i}d_{\dot{u}_i}\right]$.

Proof: The proof follows similar reasoning as the one of Theorem 1 with $V_i(\tilde{\boldsymbol{u}}_i) = \frac{1}{2}\tilde{\boldsymbol{u}}_i^\top(M_i^{\text{kc}} \otimes I_N)\tilde{\boldsymbol{u}}_i$ and is not reported here due to space limitation. ∎

Given the relation between $\tilde{\boldsymbol{u}}^i$ and $\tilde{\boldsymbol{u}}_i$, resulting from similar reasoning as the one in Remark 1, $\tilde{\boldsymbol{u}}^i$ satisfies Corollary 2.

Corollary 2: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed input observer (24). Under Assumption 1, $\|\tilde{\boldsymbol{u}}^i(t)\| \le \|\tilde{\boldsymbol{u}}^i(0)\|$ for all $t \ge 0$ and all $i \in \mathcal{V}$. Furthermore, there exists a $T_u > 0$ such that $\|\tilde{\boldsymbol{u}}^i(t)\| = 0$ for all $t > T_u$, with $T_u = \max_{i\in\mathcal{V}}\{T_{u,i}\}$.

Remark 3: Note that similarly to the matrix M_i in [12], M_i^{kc} is positive definite. However, thanks to the smaller number of required estimations, the spectrum of M_i^{kc} results to be improved in term of estimation requirements. Given the smaller maximum and the higher minimum eigenvalues indeed, the convergence time and observer parameters are smaller compared to those in [12].

Consider for example the graph in Fig. 1 with $k = 3$. Since the eigenvalue of a scalar is unique and is the scalar itself, and given the matrices definition in [12, (45)] and (12), M_2, M_2^{kc} and their eigenvalues are:

$$M_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \; M_2^{\text{kc}} = 1, \tag{27}$$

$\lambda_{\min}(M_2) = 0.17$, $\lambda_{\max}(M_2) = 3.96$ and $\lambda_{\min}(M_2^{\text{kc}}) = \lambda_{\min}(M_2^{\text{kc}}) = 1$. As a result, thanks to the smaller maximum and to the bigger minimum eigenvalues, for fixed θ_2, π_2, G and $\|K[\tilde{\boldsymbol{u}}_2]\|$, (20) and (26) prove smaller time convergence upper bounds compared to those obtained in [12].

V. k-hop Estimation-Based Feedback Controller

The k-hop distributed observer presented in the previous sections allows the control of each agent $i \in \mathcal{V}$ by indirect exploitation of the state of those agents $j \in \mathcal{N}_i^{k\text{-hop}}$. Although Corollaries 1 and 2 prove the convergence of the two observers, additional analysis needs to be performed for the composite behavior. For this purpose, in Lemma 4 we present the behavior of the k-hop estimation-based closed-loop controller.

Lemma 4: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed state and input observers (8) and (24). Under the assumption $\|K[\dot{u}_i]\| \le d_{\dot{u}_i}$ for all $i \in \mathcal{V}$, there exists a $T_u > 0$ and $\mathcal{X} > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| < \mathcal{X}$ for all $t > T_u$, with $T_u = \max_{i\in\mathcal{V}}\{T_{u,i}\}$ and $\mathcal{X} = \max_{i\in\mathcal{V}}\left\{\sup_{0\le\tau\le T_u}\|\tilde{\boldsymbol{x}}^i(t)\|\right\}$. Furthermore, there exists $T_{xu} > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| = 0$ for all $t > T_{xu}$ with $T_{xu} = T_u + T_x$.

Proof: From Corollary 2, there exists a time T_u such that $\|\tilde{\boldsymbol{u}}_i(t)\| = 0$ for all $t > T_u$ and all $i \in \mathcal{V}$. Therefore, starting from T_u, (19) is satisfied independently of θ_i and $\tilde{\boldsymbol{x}}^i$ decreases for all $i \in \mathcal{V}$ according to Theorem 1, i.e., $\tilde{\boldsymbol{x}}_i(t) < \tilde{\boldsymbol{x}}_i(T_u)$ for all $t > T_u$.

To prove Lemma 4 however, further analysis is required for the time interval $[0, T_u]$, where there is no guarantee on the validity of (19). To this end, consider the inequality in (23) and note that if (19) is not valid, ϕ_i defined as in (21) turns out to be negative. As a result, the Lyapunov function in (22) can increase, and so can the state estimation error $\tilde{\boldsymbol{x}}_i$. However, even in this case, since the generalized Lyapunov derivative is upper-bounded from above by a continuous positive function, i.e. $\mathring{V}_i(\tilde{\boldsymbol{x}}_i) \leq -\phi_i \|\tilde{\boldsymbol{x}}_i\|$, and the time interval is finite, the Lyapunov function, and therefore $\tilde{\boldsymbol{x}}_i$, will remain finite over $[0, T_u]$. From this reasoning, given the equivalence between $\tilde{\boldsymbol{x}}^i$ and $\tilde{\boldsymbol{x}}_i$, it follows that an upper bound on the norm of the state estimation error $\tilde{\boldsymbol{x}}_i$ of any agent i can be found as $\mathcal{X} = \max_{i \in \mathcal{V}} \left\{ \sup_{0 \leq \tau \leq T_u} \|\tilde{\boldsymbol{x}}_i\| \right\}$, which is the largest value in norm that an error may have achieved over $[0, T_u]$. To prove the last part, note that for any time $t > T_u$, as said at the beginning of the proof, the conditions of Corollary 1 holds. Therefore, for all $i \in \mathcal{V}$, $\|\tilde{\boldsymbol{x}}_i(t)\| < \|\tilde{\boldsymbol{x}}_i(T_u)\| \leq \mathcal{X}$ for all $t > T_u$ and $\|\tilde{\boldsymbol{x}}_i(t)\| = 0$ for all $t > T_u + T_x$. ■

Remark 4: According to Theorem 1, in case of bounded input, the convergence of the state observer is guaranteed by the existence of an upper bound on $\|K[\tilde{\boldsymbol{u}}_i]\|$ resulting from the one on u_i. In this case, for all $i \in \mathcal{V}$, it is possible to adjust θ_i in accordance with (19), ensuring that the dynamics of the state estimates remains independent of the convergence of the input estimation errors. Alternatively, if only an upper bound on $\|K[\dot{u}_i]\|$ is known, the state observer convergence depends on the input observer, which will drive the input estimation errors towards values satisfying (19) even if an upper bound on $\|K[\tilde{\boldsymbol{u}}_i]\|$ is unknown a priori.

Consider the vectorized version of the multi-agent dynamics in (1):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + (I_n \otimes A)\boldsymbol{x} + \boldsymbol{u}, \tag{28}$$

where $\boldsymbol{x}$ is defined as in (2) and the input vector $\boldsymbol{u}$ is a general nonlinear state-feedback function of the form $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}) = \left[q_1(\bar{\boldsymbol{x}}_1, \boldsymbol{x}^1)^\top, \dots, q_n(\bar{\boldsymbol{x}}_n, \boldsymbol{x}^n)^\top\right]^\top$, where $\boldsymbol{x}^i$ for each i is given in (3) and each $\bar{\boldsymbol{x}}_i$ is the vector containing the state information of agent i and of its 1-hop neighbors. To satisfy the condition on the upper bound of the input derivative required for the input observers, let's assume there exists a known upper bound on $K[\dot{q}_i](\cdot)$.

Due to the lack of local information regarding the k-hop neighbors' state $\boldsymbol{x}^i$ for all $i \in \mathcal{V}$, the previous controller is implemented by using their estimates $\hat{\boldsymbol{x}}^i$:

$$\boldsymbol{u} = \boldsymbol{q}(\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}) = \left[q_1(\bar{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}^1)^\top, \dots, q_n(\bar{\boldsymbol{x}}_n, \hat{\boldsymbol{x}}^n)^\top\right]^\top. \tag{29}$$

However, by noticing that $\bar{\boldsymbol{x}}_i$ is a selection of the components of $\boldsymbol{x}$ and that $\hat{\boldsymbol{x}}^i = \boldsymbol{x}^i - \tilde{\boldsymbol{x}}^i$, (29) can be rewritten as $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x} - \tilde{\boldsymbol{x}})$, where $\tilde{\boldsymbol{x}} = [\tilde{\boldsymbol{x}}^{1\top}, \dots, \tilde{\boldsymbol{x}}^{n\top}]^\top$.

By defining $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \boldsymbol{f}(\boldsymbol{x}) + (I_n \otimes A)\boldsymbol{x} + \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x} - \tilde{\boldsymbol{x}})$, (28) becomes $\dot{\boldsymbol{x}} = \Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, where $\tilde{\boldsymbol{x}}$ is interpreted as an input disturbance for the system with nominal unforced dynamics $\dot{\boldsymbol{x}} = \Phi(\boldsymbol{x}, 0_{Nn})$.

Definition 1 ([19]): A system $\dot{x} = f(x, u)$, with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is set-Input to State Stable (set-ISS) with respect to a set $\mathcal{A}$ if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for each initial condition and any locally essentially bounded input u satisfying $\sup_{t \geq 0} \|u(t)\| \leq \infty$, $\|x(t)\|_{\mathcal{A}} \leq \beta(\|x(0)\|_{\mathcal{A}}, t) + \gamma\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right)$ holds with $\|x(t)\|_{\mathcal{A}} = \text{dist}(x, \mathcal{A}) = \inf_{a \in \mathcal{A}} \{\|x - a\|\}$.

Assumption 4: Under perfect state knowledge, the nonlinear state-feedback $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x})$ ensures convergence of the multi-agent system to an equilibrium representing the system objective, e.g. consensus, formation and flocking.

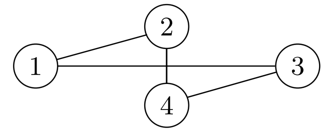


Fig. 2: Graph $\mathcal{G}_T$ used for consensus.

Theorem 3: Consider the multi-agent system (1) with connected graph $\mathcal{G}$ and distributed state and input observers (8) and (24). Assume that each agent runs the local control input (29) under the assumption of bounded $K[\dot{q}_i](\cdot)$ for all $i \in \mathcal{V}$. Then, if $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is set-ISS with respect to a set $\mathcal{A}$ representing the system objective and Assumption 4 holds, the multi-agent system reaches an equilibrium representing the team objective.

Proof: From Lemma 4, there exists an upper bound $\mathcal{X} > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| \leq \mathcal{X}$ for all $i \in \mathcal{V}$ and $t \in \mathbb{R}_{\geq 0}$. As a result, $\|\tilde{\boldsymbol{x}}(t)\| \leq \sqrt{n}\mathcal{X}$ holds for all $t \in \mathbb{R}_{\geq 0}$. Furthermore, Lemma 4 guarantees the existence of $T_{xu} > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| = 0$ for all $t > T_{xu}$. Then, by exploiting the set-ISS assumption on $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ and that from $t = T_{xu}$ the multi-agent system evolves from $\boldsymbol{x}(T_{xu})$ under the dynamics $\Phi(\boldsymbol{x}, 0_{Nn})$, we can conclude that $\|\boldsymbol{x}(t)\|_{\mathcal{A}} \leq \beta(\|\boldsymbol{x}(T_{xu})\|_{\mathcal{A}}, t - T_{xu})$ for all $t > T_{xu}$. As a result, thanks to the convergence to the origin of $\beta(\|\boldsymbol{x}(T_{xu})\|_{\mathcal{A}}, t - T_{xu})$, resulting from the set-ISS definition, convergence towards the objective represented by the set $\mathcal{A}$ is guaranteed. ■

VI. Simulations

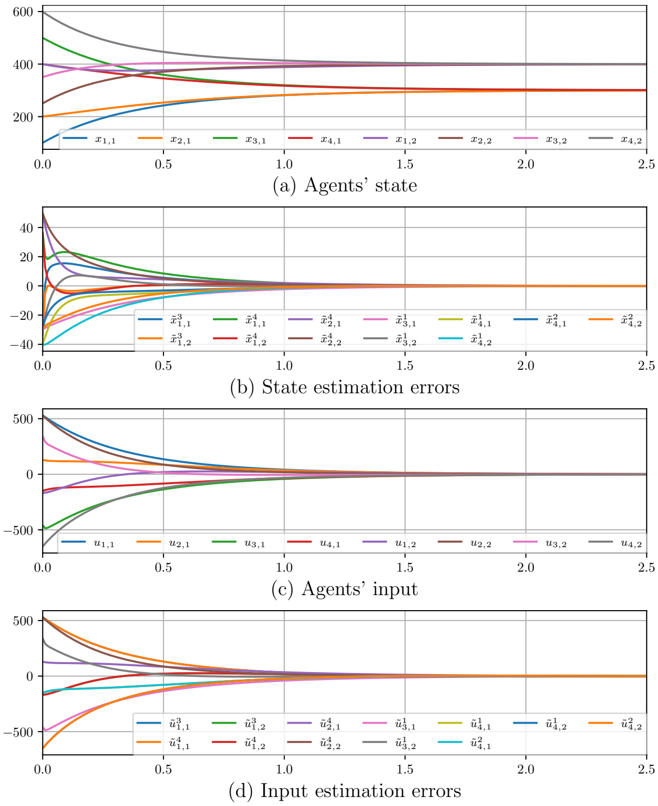
Consider a multi-agent system composed of $n = 4$ agents communicating according to the path graph $\mathcal{G}_C = (\mathcal{V}, \mathcal{E}_C)$ depicted in Fig. 1. Assume each agent behaves according to the single integrator dynamics $\dot{x}_i = u_i$, where $x_i \in [x_{\text{min}}, x_{\text{max}}] \subset \mathbb{R}^2$ and the input u_i is designed in order to drive the agents towards consensus by exploiting only the edges of the graph $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$ shown in Fig. 2, i.e.:

$$u_i(t) = \sum_{j \in (\mathcal{N}_i^C \cap \mathcal{N}_i^T)} (x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}_i^T / \mathcal{N}_i^C} (\hat{x}_j^i(t) - x_i(t)), \tag{30}$$

where $\mathcal{N}_i^C$ and $\mathcal{N}_i^T$ are the neighbors of agent $i \in \mathcal{V}$ respectively in graphs $\mathcal{G}_C$ and $\mathcal{G}_T$. It is worth noticing that, for the purpose of testing the observer-based controller, the proposed consensus approach differs from the traditional one, as edges not belonging to the communication graph are exploited to achieve the team objective.

Given the boundedness of the state and of the state estimations, it is possible to prove that $u_i(t) \leq |\mathcal{N}_i^T| d_{\text{max}}$, with $d_{\text{max}} = x_{\text{max}} - x_{\text{min}}$. As noted in Remark 4, this implies that θ_i can be tuned to satisfy (19) from initialization, ensuring that each agent's state observation converges independently of the input observer's dynamics.

To claim the applicability of Theorem 3 to this case study, Assumption 4 and the set-ISS property of u_i with respect to the set $\mathcal{A}$ representing the consensus along the state's components must be checked. For this purpose note that, given the connectivity of $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$, the input $u_i^{\text{ideal}}(t) = \sum_{j \in \mathcal{N}_i^T} (x_j(t) - x_i(t))$ guarantees the convergence of the multi-agent system towards consensus [1]. Furthermore,



(a) Agents' state

(b) State estimation errors

(c) Agents' input

(d) Input estimation errors

Fig. 3: Simulation results when the proposed k-hop observer-based control strategy is used to drive the system towards consensus. (a) and (c) contain respectively the agents' state and input, (b) and (d) the state and input estimation errors.

since u_i can be rewritten as $u_i(t) = \sum_{j\in\mathcal{N}_i^T}(x_j(t) - x_i(t)) - v_i$ with $v_i(t) = \sum_{j\in\mathcal{N}_i^T/\mathcal{N}_i^C} \tilde{x}_j^i$ bounded, it is possible to prove that the vectorized state dynamics $\dot{\boldsymbol{x}}(t) = -(L_T \otimes I_2)\boldsymbol{x}(t) - \boldsymbol{v}(t)$, where L_T is the Laplacian matrix of the graph $\mathcal{G}_T$ and $\boldsymbol{v}(t) = [v_1^\top(t), \ldots, v_n^\top(t)]^\top$, fulfill the following:

$$\|\boldsymbol{x}(t)\|_{\mathcal{A}} \le e^{-\lambda_2(L_T)t}\|\boldsymbol{x}(0)\|_{\mathcal{A}} + \frac{1}{\lambda_2(L_T)} \sup_{0\le\tau\le t} \|\boldsymbol{v}(\tau)\|, \quad (31)$$

where $\lambda_2(L_T)$ is the smallest positive eigenvalue of L_T. Then, given the convergence of the state observer, (31) is consistent with the set-ISS definition. As a result, Theorem 3 holds, and the state and input observers with $k = 3$ can be used to control the system towards consensus.

For simulation purposes, a sampling time $dt = 10^{-3}s$ and parameters satisfying Theorems 1 and 2 have been chosen. Fig. 3 shows the results obtained with $g = 20$, $\omega_1 = \omega_4 = 2.62$, $\omega_2 = \omega_3 = 1.0$, $\theta_1 = \theta_4 = 3.4$, $\theta_2 = \theta_3 = 0.5$, $\pi_1 = \pi_4 = 9.7$ and $\pi_2 = \pi_3 = 1.0$. While the agents' input vector is initialized using (30) and according to the states and state estimation information, the estimated input vector is initialized to zero for every agent. Fig. 3b shows the convergence of the state estimations, which is independent of the input observer behavior in Fig. 3d as per Remark 4.

VII. Conclusion and Future Work

We proposed a communication-based k-hop distributed observer in which each agent estimates only the state and input of agents within a k-hop distance. The distributed state and input observers result to be finite time convergent and provide state estimations that, under set-ISS condition of the feedback control law, can be used to drive the agents towards an equilibrium representing the team objective.

As presented in Section II, while Assumption 3 is reasonable for the state if agents are equipped with sensors, it seams more restricting for the input. For this reason, in addition to study possible disturbance effects, future works will be oriented towards analyzing how the delays on 2-hop input propagation may affect the observer convergence.

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