Edge-based funnel control for multi-agent systems using relative position measurements

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***Abstract*— In this work we consider the problem of control under Signal Temporal Logic specifications (STL) that depend on relative position information among neighboring agents. In particular, we consider STL tasks for given pairs of agents whose satisfaction is translated into a set of setpoint output tracking problems with transient and steady-state constraints. Contrary to existing work the proposed framework does not require initial satisfaction of the funnel constraints but can ensure their satisfaction within a pre-specified finite time. Given a tree topology in which agents sharing a STL task form an edge, we show that the resulting control laws ensure the satisfaction of the STL task as well as boundedness of all closed loop signals using only local information.**

I. Introduction

Multi-agent systems have been deployed in a plethora of highly complex environments such as underwater or underground environments, the space or in industrial settings. In such environments communication with central entities responsible for control and planning is often hard to establish or costly. To that end, great emphasis is given on decision making strategies that are based on local information obtained by onboard sensors such as range sensors and/or cameras.

Motivated by such applications, in this work we focus on the design of control strategies under complex, time-constrained tasks that depend on relative-position information among agents. These tasks are expressed in Signal Temporal Logic (STL) [1], a formal language that allow us to encode complex spatial tasks that need to performed within given time intervals. Contrary to other logics, STL is evaluated over continuous-time signals and is equipped with a robustness metric [2], [3] that allow us to quantify how well the STL task is satisfied. In the context of multi-agent control, distributed control strategies have been discussed in [4]–[7]. In [4] a hierarchical approach is proposed for control of local motion and safety tasks as well as global communication constraints. In [5] decentralized control laws are designed based on assume-guarantee contracts designed in a centralized manner. In [6] a distributed MPC scheme is proposed for single integrator systems under reach-avoid specifications with recursive feasibility guarantees while [7] proposes an iterative algorithm for control under coupled reach-avoid specifications using MILP programming. Closer to our approach is the work proposed in [8], [9], where prescribed performance control strategies have been applied to nonlinear [8] or interconnected systems [9]. In [8] a funnel-based switching control law is proposed for single-agent relative degree one systems while [9] employs a contract-based funnel control strategy to ensure decentralized control of interconnected systems subject to local STL tasks.

In all the aforementioned works the STL tasks are local and expressed in terms of the absolute position of the agents. Contrary to existing work, in this paper we consider STL tasks that depend on relative position information among neighboring agents. STL satisfaction is enforced by means of a set of output tracking objectives that need to be achieved with a prescribed transient and steady state behavior. Assuming a tree sensing topology, we design a switching control law that ensures the satisfaction of the STL task with a desired robustness as well as boundedness of all closed loop signals based only on local information. Contrary to the majority of works in prescribed performance control literature, here agents may initially violate the funnel constraints which are guaranteed to be satisfied after a pre-specified finite time instant thanks to appropriately designed shifting functions.

This work was partially supported by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation, the ERC CoG LEAFHOUND, EU SymAware and the Swedish Research Council.

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II. Notation and Preliminaries

True and false are denoted by $\top, \bot$ respectively. Scalars and vectors are denoted by non-bold and bold letters respectively. $A \otimes B$ denotes the Kronecker product of A, B. The cardinality of a set $\mathcal{V}$ is denoted by $|\mathcal{V}|$ and the identity matrix of dimension n by I_n. The block diagonal matrix of $A_1, \ldots, A_p$ is denoted by $\text{diag}(A_1, \ldots, A_p)$. The minimum eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $\lambda_{\min}(A)$ and $A \succ 0$ denotes that A is positive definite. The weighted Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is given by $\|\mathbf{x}\|_Q := \sqrt{\mathbf{x}^T Q \mathbf{x}}$, where $Q \succ 0$. Given subsets $\mathcal{I}_k \subset \mathbb{N}$ with $k \in \mathcal{M} \subseteq \mathbb{N}$ we denote the minimum and maximum of $\mathcal{I}_k$ by $\hat{i}^k_{\min} := \min \mathcal{I}_k$ and $\hat{i}^k_{\max} := \max \mathcal{I}_k$, respectively. In addition, for $i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\min}\}$, we define $\lfloor i \rfloor_k = \max\{i' \in \mathcal{I}_k : i' < i\}$ and for $i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\max}\}$, $\lceil i \rceil_k = \min\{i' \in \mathcal{I}_k : i < i'\}$.

A. Signal Temporal Logic (STL)

Signal Temporal Logic (STL) determines whether a predicate μ is true or false. The validity of each predicate μ is evaluated based on a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ as follows: $\mu = \top$, if $h(\mathbf{x}) \geq 0$, or $\mu = \bot$, otherwise. The basic STL formulas are given by the grammar: $\phi := \top \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathcal{G}_{[a,b]}\phi \mid \mathcal{F}_{[a,b]}\phi \mid \phi_1\ \mathcal{U}_{[a,b]}\ \phi_2$, where ϕ_1, ϕ_2 are STL formulas and $\mathcal{G}_{[a,b]}$, $\mathcal{F}_{[a,b]}$, $\mathcal{U}_{[a,b]}$ is the always, eventually and until operator defined over the interval $[a,b]$ with $0 \leq a \leq b < \infty$. Let $\mathbf{x} \models \phi$ denote the satisfaction of the formula ϕ by a signal $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. The formula ϕ is satisfiable if $\exists\, \mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\mathbf{x} \models \phi$. STL is equipped with robustness metrics determining how robustly an STL formula ϕ is satisfied at time t by a signal $\mathbf{x}$. The STL semantics and robust semantics are defined in [1] and [10], respectively. Note that $\mathbf{x} \models \phi$, if $\rho^{\phi}(\mathbf{x}, 0) > 0$.

B. Graph Theory

An *undirected* graph G is defined as a pair $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, R\} \subset \mathbb{N}$ is a finite set of nodes and $\mathcal{E} \subseteq \{(r, r') \in \mathcal{V} \times \mathcal{V} : r \neq r'\}$. A *path* is a sequence of edges connecting two distinct vertices. A graph is *connected*, if there exists a path between any pair of vertices. Given a numbering $k \in \mathcal{M} = \{1, \ldots, M\}$ of the edges $\mathfrak{e}_k \in \mathcal{E}$, after assigning an orientation to each edge in G we may define the incidence matrix $D = [d_{ij}]$ of G as follows: $d_{ij} = 1$, if the node i is the head of edge j, $d_{ij} = -1$, if i is the tail of edge j and $d_{ij} = 0$, otherwise. The edge Laplacian matrix of G is given by $L_e = D^T D$.

C. Prescribed Performance Control

Prescribed Performance Control (PPC) is a control method initially proposed in [11] ensuring that the tracking error $e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ remains at all times within a bounded region determined by a-priori known time-varying functions that impose a prescribed transient and steady state performance. Specifically, $-\gamma(t) < e(t) < \gamma(t)$ should hold for every $t \geq t_0 \geq 0$, where $\gamma(t)$ is a smooth, bounded, and monotonically decreasing function satisfying $\lim_{t \to +\infty} \gamma(t) = \gamma_\infty > 0$. An example of such function is $\gamma(t) = (\gamma_0 - \gamma_\infty)\exp(-lt) + \gamma_\infty$, where $\gamma_0, \gamma_\infty, l$ are positive parameters chosen such that $|e(t_0)| < \gamma(t_0)$ and $\gamma_\infty < \gamma_0$. The value of γ_∞ determines the maximum allowable size of the tracking error at steady state and can be chosen arbitrarily small while the parameter l determines a lower bound on the speed of convergence of the tracking error.

D. Problem Formulation

In this work we consider a multi-agent team of R agents whose dynamics are given by:

$$\dot{\mathbf{x}}_r = f_r(\mathbf{x}_r) + g_r(\mathbf{x}_r)\mathbf{u}_r + \mathbf{w}_r, \tag{1}$$

where $f_r : \mathbb{R}^n \to \mathbb{R}^n$, $g_r : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz functions, $\mathbf{x}_r \in \mathbb{R}^n, \mathbf{u}_r \in \mathbb{R}^m$ is the state and input of the r-th agent, respectively, $\mathbf{w}_r \in \mathbb{R}^n$ is a piecewise continuous and bounded disturbance acting on the r-th agent and $r \in \mathcal{V} := \{1, \ldots, R\}$.

Assumption 1. *The matrix $g_r : \mathbb{R}^n \to \mathbb{R}^{n \times m}, r \in \mathcal{V}$ is full row-rank for every $\mathbf{x}_r \in \mathbb{R}^n$.*

A direct implication of Assumption 1 is that $n \leq m$, i.e., the number of inputs is at least as equal as the number of states. Examples of $g_r(\cdot)$ satisfying Assumption 1 are constant, full row-rank matrices or $n \times n$ and invertible matrices for every $\mathbf{x}_r$. Let $\mathbf{x} := \begin{bmatrix}\mathbf{x}_1^T & \ldots & \mathbf{x}_R^T\end{bmatrix}^T \in \mathbb{R}^{Rn}$, $\mathbf{u} := \begin{bmatrix}\mathbf{u}_1^T & \ldots & \mathbf{u}_R^T\end{bmatrix}^T \in \mathbb{R}^{Rm}$, $\mathbf{w} := \begin{bmatrix}\mathbf{w}_1^T & \ldots & \mathbf{w}_R^T\end{bmatrix}^T \in \mathbb{R}^{Rn}$, denote the stacked vector of the states, inputs and disturbances of the multi-agent system, respectively. Then, the dynamics of the multi-agent team are given as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} + \mathbf{w}, \tag{2}$$

where $f(\mathbf{x}) := \begin{bmatrix} f_1^T(\mathbf{x}_1), \ldots, f_R^T(\mathbf{x}_R)\end{bmatrix}^T$ and $g(\mathbf{x}) := \text{diag}(g_1(\mathbf{x}_1), \ldots, g_R(\mathbf{x}_R))$. Each agent r is assumed to have relative position information with respect to a limited number of its peers. Let $G = (\mathcal{V}, \mathcal{E})$ be the sensing graph where $\mathcal{V}$ is the set of agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. Here, an edge $\mathfrak{e}_k = (r_k, r'_k) \in \mathcal{E}$ exists iff r_k, r'_k have access to the relative position vector $\mathbf{x}_{r_k} - \mathbf{x}_{r'_k}$. In the following we make the following assumption on the sensing graph G :

Assumption 2. *The graph $G = (\mathcal{V}, \mathcal{E})$ is a static, undirected tree.*

Assumption 2 requires agents to form a tree sensing graph which ensures that the edge Laplacian L_e is positive definite [12], a property which will be later used to ensure boundedness of the error signals.

Here, the multi-agent team is subject to a global STL task described by the following fragment:

$$\psi := \top \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2, \tag{3a}$$
$$\varphi := \mathcal{G}_{[a,b]}\psi \mid \mathcal{F}_{[a,b]}\psi \mid \psi_1 \,\mathcal{U}_{[a,b]}\, \psi_2, \tag{3b}$$
$$\phi := \varphi_1 \wedge \ldots \wedge \varphi_{q'}, \tag{3c}$$

where ψ_1, ψ_2 are STL formulas of the form (3a), $\varphi_i, i = 1, \ldots, q'$ are STL formulas of the form (3b), $a \leq b < \infty$. In the following we will assume that ϕ is defined as a conjunction of always and eventually STL tasks of the form $\mathcal{G}_{[a,b]}(h(\mathbf{x}) \geq 0)$ and $\mathcal{F}_{[a,b]}(h(\mathbf{x}) \geq 0)$, respectively, as the satisfaction of more complex formulas of (3a)-(3c) is ensured by conjunctions of such tasks. Given the set $\mathcal{I} := \{1, \ldots, q\}$, where $q > 1$, in this work we consider the global STL task:

$$\phi = \bigwedge_{i \in \mathcal{I}} \varphi_i. \tag{4}$$

Let $\beta_k(\mathbf{x}) := \mathbf{x}_{r_k} - \mathbf{x}_{r'_k}$ be the relative position among agents r_k, r'_k forming the k-th edge of G, where $k \in \mathcal{M} := \{1, \ldots, M\}$, $M := |\mathcal{E}|$. An example of such task is $\mathcal{G}_{[0,5]}(\|\mathbf{x}_1 - \mathbf{x}_2\| \leq 1)$, which requires agents 1,2 to be no more than 1m apart for every $t \in [0, 5]$. Here, we focus on STL tasks $\varphi_i, i \in \mathcal{I}$ that depend on $\beta_k(\mathbf{x})$ for some $k \in \mathcal{M}$. More specifically, we consider STL tasks that satisfy the following assumption:

Assumption 3. *The predicate functions $h_i : \mathbb{R}^n \to \mathbb{R}, i \in \mathcal{I}$ have the following properties: (i) $h_i(\cdot)$ is a function of the relative position among agents r_{k_i}, r'_{k_i} for some $k_i \in \mathcal{M}$, i.e., $h_i = h_i(\beta_{k_i})$, (ii) $h_i(\beta_{k_i})$ is continuously differentiable and $h_i(\beta_{k_i}) \to \infty$ as $\|\beta_{k_i}\| \to +\infty$, and (iii) $\frac{\partial h_i}{\partial \beta_{k_i}}$ is bounded with $\frac{\partial h_i}{\partial \beta_{k_i}} \neq \mathbf{0}$ in some known set $\mathcal{B}_i \subseteq \mathbb{R}^n$ with $\overline{\nu}_i := \sup_{\beta_{k_i} \in \mathcal{B}_i} h_i(\beta_{k_i}) > 0$. In addition, given the time intervals $[a_i, b_i]$ corresponding to the formulas $\varphi_i, i \in \mathcal{I}$ the following hold: (i) $a_{\hat{i}^k_{\min}} > 0$, for every $k \in \mathcal{M}$, (ii) $b_i < a_{\hat{i}'} - \epsilon$, for* *every* $i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\max}\}, k \in \mathcal{M}$ *where* $\mathcal{I}_k := \{i' \in \mathcal{I} : h_{i'} = h_{i'}(\beta_k)\}$*, is the index set of the STL formulas involving the agents forming the k-th edge of* G, $\hat{i}' = \lceil i \rceil_k$ *and* $\epsilon > 0$ *is a positive parameter.*

Assumption 3 ensures that the STL tasks φ_i involving the agents forming the same edge in G are successive tasks, i.e., the time intervals within which they need to be satisfied are not overlapping. In this work each predicate function $h_i(\cdot), i \in \mathcal{I}$ is subject to prescribed performance constraints given as follows:

$$-\underline{\eta}_i \gamma_i(t) < h_i(\beta_{k_i}(t)) - \overline{\rho}_i < \overline{\eta}_i \gamma_i(t), \ t \in [c_i, b_i + \epsilon), \quad (5)$$

where $\overline{\rho}_i \geq \overline{\rho}$, $\overline{\eta}_i, \underline{\eta}_i > 0$ are positive tuning parameters, $\overline{\rho} \in (0, \min_{i \in \mathcal{I}} \overline{\nu}_i)$ is fixed and given, $\epsilon > 0$ is the same as in Assumption 3, and c_i is a time instant to be chosen such that for $i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\min}\}, k \in \mathcal{M}$, $c_i \in (b_{\hat{i}} + \epsilon, a_i)$, if $\varphi_i = \mathcal{G}_{[a_i, b_i]} \psi_i$ or $c_i \in (b_{\hat{i}} + \epsilon, b_i)$, otherwise, where $\hat{i} = \lfloor i \rfloor_k$. If $i = \hat{i}^k_{\min}$, for some $k \in \mathcal{M}$, then $c_i \in (0, a_i)$, if $\varphi_i = \mathcal{G}_{[a_i, b_i]} \psi_i$ or $c_i \in (0, b_i)$, otherwise. In addition, $\gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ are smooth, positive and strictly decreasing functions defined as:

$$\gamma_i(t) := (1 - \gamma_{i,\infty}) \exp(-l_i t) + \gamma_{i,\infty}, \quad (6)$$

where $\gamma_{i,\infty} := \frac{\bar{\gamma}_{i,\infty}}{\max(\underline{\eta}_i, \overline{\eta}_i)} > 0$ and $\bar{\gamma}_{i,\infty}, l_i \in \mathbb{R}_{>0}$ are tuning parameters determining the desired transient and steady state behavior of $h_i(\beta_{k_i}) - \overline{\rho}_i$. Intuitively, (5) enforces a desired behavior for the agents forming the k_i-th edge towards satisfying φ_i with a desired robustness $\overline{\rho}_i$. In particular, choosing $c_i > 0$ as mentioned above and after appropriately tuning the parameters of $\gamma_i(t)$ we can ensure that the predicate function $h_i(\beta_{k_i})$ can reach close to $\overline{\rho}_i$ within $[c_i, b_i + \epsilon)$. Based on the above, the problem considered in this work is expressed as follows:

Problem 1. *Consider a team of* R *agents that is subject to a global STL task defined by* (4)*. The states of each agent evolve over time according to* (1)*. Let Assumptions 1-3 hold. Then, design* $\mathbf{u}_r, r \in \mathcal{V}$ *(if possible) using only local information such that the satisfaction of* (5) *for each* $i \in \mathcal{I}$ *ensures* $\rho^{\phi}(\mathbf{x}, 0) \geq \overline{\rho}$*, where* $\overline{\rho} > 0$ *is a designer's choice.*

III. Main Results

In this section we will design a switching control law that ensures $\rho^{\phi}(\mathbf{x}, 0) \geq \overline{\rho}$. In Section III-A, we will design control laws that ensure the satisfaction of (5) using only local information and then in Section III-B we propose how to choose the design parameters in order to ensure the desired robustness of satisfaction of ϕ.

A. Control Design

We begin with the control design assuming that the parameters determining the funnel constraints as well as $\overline{\rho}_i, i \in \mathcal{I}$ are given. By Assumption 3 the tasks involving the agents of the k-th edge of G are sequential. Since $a_i \leq b_i$ for every $i \in \mathcal{I}$, then if $i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\max}\}$, for some $k \in \mathcal{M}$, it follows that $b_i + \epsilon < b_{\hat{i}'}$, where $\hat{i}' = \lceil i \rceil_k$. Therefore, for every $k \in \mathcal{M}$ we can define the ordered set $\Sigma_k := \{b_i + \epsilon : i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\max}\}\} \cup \{0, +\infty\}$. Based on Σ_k we can assign to each $i \in \mathcal{I}_k, k \in \mathcal{M}$ an interval $\mathcal{T}_i$ representing the time interval at which agents r_{k_i}, r'_{k_i} will move towards satisfying φ_i as follows:

$$\mathcal{T}_i := \begin{cases} [0, b_i + \epsilon), & \text{if } i = \hat{i}^k_{\min} \\ [b_{\hat{i}} + \epsilon, b_i + \epsilon), & \text{if } i \in \mathcal{I}_k \backslash \{\hat{i}^k_{\min}, \hat{i}^k_{\max}\} \\ [b_{\hat{i}} + \epsilon, +\infty), & \text{if } i = \hat{i}^k_{\max} \end{cases}, \quad (7)$$

where $\hat{i} = \lfloor i \rfloor_k$. Note that $[c_i, b_i + \epsilon) \subset \mathcal{T}_i$ hold for each $i \in \mathcal{I}$, where $[c_i, b_i + \epsilon)$ is the time interval considered in (5). This property wil be shortly considered towards introducing a modified funnel constraint (given in (8)) for each $i \in \mathcal{I}$ that needs to be satisfied for every $t \in \mathcal{T}_i$. Traditional PPC strategies require the state of the system at t_0 to be within the performance bounds, i.e., $-\underline{\eta}_i \gamma_i(t_0) < e_i(t_0) < \overline{\eta_i} \gamma_i(t_0)$ (see Section II-C). Nevertheless, in case of multiple switches among different objectives this condition may not be met. To that end, we will enforce a modified funnel-based constraint for each $i \in \mathcal{I}$ given as follows:

$$-\underline{\eta}_i \gamma_i(t) < e_i(t) < \overline{\eta}_i \gamma_i(t), \quad t \in \mathcal{T}_i \quad (8)$$

where $\mathcal{T}_i \subset \mathbb{R}_{\geq 0}$ is defined in (7), $e_i(t) := \omega_i(t)(h_i(\beta_{k_i}(t)) - \overline{\rho}_i)$ for $t \geq 0$, and $\omega_i : \mathbb{R}_{\geq 0} \to (0, 1]$, is a shifting function that satisfies the following assumption:

Assumption 4. *The functions* $\omega_i : \mathbb{R}_{\geq 0} \to (0, 1], i \in \mathcal{I}$ *are continuously differentiable in* $\mathcal{T}_i$ *and strictly increasing for* $t \in [\min \mathcal{T}_i, c_i)$ *with* $\omega_i(\min \mathcal{T}_i) = \delta_i > 0$ *and* $\omega_i(t) = 1$*, for every* $t \geq c_i$*, where* δ_i *is a positive tuning parameter,* $\mathcal{T}_i \subset \mathbb{R}_{\geq 0}$ *is defined in* (7)*, and* $\min \mathcal{T}_i = 0$*, if* $i = \hat{i}^k_{\min}$*, or* $\min \mathcal{T}_i = b_{\hat{i}} + \epsilon$*, otherwise. In addition,* $\dot{\omega}_i(t), i \in \mathcal{I}$ *are bounded for* $t \in \mathcal{T}_i$*.*

The designed shifting function $\omega_i(t)$ translates a bounded value of $h_i(\beta_{k_i}(\min \mathcal{T}_i)) - \overline{\rho}_i$ to $e_i(\min \mathcal{T}_i)$ such that $-\underline{\eta}_i \gamma_i(\min \mathcal{T}_i) < e_i(\min \mathcal{T}_i) < \overline{\eta}_i \gamma_i(\min \mathcal{T}_i)$ is satisfied when δ_i is chosen sufficiently small within $(0, 1)$. In addition, since $\omega_i(t) = 1$, for every $t \geq c_i$, it follows that $e_i(t) = h_i(\beta_{k_i}(t)) - \overline{\rho}_i$, for every $t \geq c_i$. Due to the latter, and since $[c_i, b_i + \epsilon) \subset \mathcal{T}_i$ holds for every $i \in \mathcal{I}$, it follows that the satisfaction of (8) ensures the satisfaction of (5). Similar assumptions to Assumption 4 have been made in [13], where shifting functions have been considered for output reference tracking of higher relative degree systems.

Remark 1. *For* $i = \hat{i}^k_{\min}, k \in \mathcal{M}$*, the shifting function can be chosen as* $\omega_i(t) = 1, t \geq 0$ *when* $e_i(\mathbf{x}(0)) \in (-\underline{\eta}_i \gamma_i(0), \overline{\eta}_i \gamma_i(0))$ *holds.*

Differentiating $e_i = \omega_i(h_i(\beta_{k_i}) - \overline{\rho}_i)$ we obtain:

$$\dot{e}_i = \dot{\omega}_i(h_i(\beta_{k_i}) - \overline{\rho}_i) + \omega_i \frac{\partial h_i^T}{\partial \beta_{k_i}}(\dot{\mathbf{x}}_{r_k} - \dot{\mathbf{x}}_{r'_k}). \quad (9)$$

Let $\Sigma := \bigcup_{k \in \mathcal{M}} \Sigma_k$ and $\mathbf{e} := \begin{bmatrix} e_1 & \dots & e_q \end{bmatrix}^T$. For each time interval $[\sigma_p, \sigma_{p+1})$, where $\sigma_p, \sigma_{p+1} \in \Sigma, p \in \mathcal{P} := \{0, \dots, |\Sigma| - 1\}$ are consecutive time instants let $\mathbf{e}_p := \begin{bmatrix} e_{i_1} & \dots & e_{i_p} \end{bmatrix}^T \in \mathbb{R}^{z_p}, z_p \leq q$, denote the vector of the error signals $e_i, i \in \mathcal{I}$ for which $\mathcal{T}_i \cap [\sigma_p, \sigma_{p+1}) \neq \emptyset$, i.e., the error signals corresponding to STL tasks for which the funnel constraints (8) are active. Then, based on (9) the derivative of the stacked error vector at each $[\sigma_p, \sigma_{p+1}), p \in \mathcal{P}$ is given by:

$$\dot{\mathbf{e}}_p = \Omega_p(t)\mathbf{e}_p + F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)\dot{\mathbf{x}}, \tag{10}$$

where $\Omega_p(t) := \text{diag}\big(\frac{\dot{\omega}_{i_1}(t)}{\omega_{i_1}(t)}, \dots, \frac{\dot{\omega}_{i_p}(t)}{\omega_{i_p}(t)}\big)$, $\widetilde{D}_p \in \mathbb{R}^{R \times z_p}$ is a matrix whose i-th column is equal to the k-th column of the incidence matrix D of G and $F_p : \mathbb{R}^{Rn} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{z_p \times z_p n}$ is the matrix defined as:

$$F_p(\mathbf{x}, t) := \begin{bmatrix} \omega_{i_1}(t)\frac{\partial h_{i_1}^T}{\partial \beta_{k_1}} & \dots & \mathbf{0}_{1\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times n} & \dots & \omega_{i_p}(t)\frac{\partial h_{i_p}^T}{\partial \beta_{k_p}} \end{bmatrix}.$$

In particular, if $z_p = M$ and for every $i, i' \in \{i_1, \dots, i_p\}$ with $i \neq i'$ it holds $k_i \neq k_{i'}$, then $\widetilde{D}_p = DP$ for some permutation matrix $P \in \{0,1\}^{M \times M}$. Next, given $e_i, i \in \mathcal{I}$ define the normalized errors with respect to the prescribed performance functions as $\overline{e}_i(t) := \frac{e_i(t)}{\gamma_i(t)} \in (-\underline{\eta}_i, \overline{\eta}_i)$. Using the transformation functions $T_i : (-\underline{\eta}_i, \overline{\eta}_i) \to \mathbb{R}, i \in \mathcal{I}$ that are smooth, strictly increasing and satisfy $T_i(0) = 0$, we may define the transformed errors $\varepsilon_i := T_i(\overline{e}_i), i \in \mathcal{I}$, where

$$T_i(\star) := \ln\left(\frac{1 + \frac{\star}{\underline{\eta}_i}}{1 - \frac{\star}{\overline{\eta}_i}}\right). \tag{11}$$

Considering the transformed errors $\varepsilon_i, i \in \mathcal{I}$ it can been shown that if ε_i is bounded, then $\overline{e}_i \in (-\underline{\eta}_i, \overline{\eta}_i)$, which in turn ensures that (8) is satisfied. Differentiating the transformed errors with respect to time we have $\dot{\varepsilon}_i = \mathcal{J}_i(\overline{e}_i, t)\big(\dot{e}_i + \alpha_i(t)e_i\big)$, for every $i \in \mathcal{I}$, where $\mathcal{J}_i(\overline{e}_i, t) := \frac{\partial T_i(\overline{e}_i)}{\partial \overline{e}_i}\frac{1}{\gamma_i(t)} > 0$ and $\alpha_i(t) := -\frac{\dot{\gamma}_i(t)}{\gamma_i(t)} > 0$. Let $\boldsymbol{\varepsilon} := \begin{bmatrix}\varepsilon_1 & \dots & \varepsilon_q\end{bmatrix}^T$ and $\boldsymbol{\varepsilon}_p := \begin{bmatrix}\varepsilon_{i_1} & \dots & \varepsilon_{i_p}\end{bmatrix}^T \in \mathbb{R}^{z_p}, p \in \mathcal{P}$. Then, the derivative of $\boldsymbol{\varepsilon}_p$ can be written in vector form as:

$$\dot{\boldsymbol{\varepsilon}}_p = \mathcal{J}_p'(\overline{\mathbf{e}}_p, t)\big(\dot{\mathbf{e}}_p + \boldsymbol{\alpha}_p(t)\mathbf{e}_p\big) \tag{12}$$

where $\mathcal{J}_p'(\overline{\mathbf{e}}_p, t) := \text{diag}(\mathcal{J}_{i_1}(\overline{e}_{i_1}, t), \dots, \mathcal{J}_{i_p}(\overline{e}_{i_p}, t))$, $\boldsymbol{\alpha}_p(t) := \text{diag}(\alpha_{i_1}(t), \dots, \alpha_{i_p}(t))$, and $\overline{\mathbf{e}}_p := \begin{bmatrix}\overline{e}_{i_1} & \dots & \overline{e}_{i_p}\end{bmatrix}^T$. Based on the above, we define $\mathbf{u}_r$ for every $r \in \mathcal{V}$ as follows:

$$\mathbf{u}_r = -g_r^T(\mathbf{x}_r)\sum_{k \in \mathcal{M}}\sum_{i \in \mathcal{I}_k}\overline{o}_i(t)\overline{G}_i d_{rk}\mathcal{J}_i(\overline{e}_i, t)\varepsilon_i\omega_i(t)\frac{\partial h_i(\beta_k)}{\partial \beta_k}, \tag{13}$$

where $D = [d_{rk}]$, $\overline{G}_i > 0$ are gains to be tuned and $\overline{o}_i(t) := 1$, if $t \in \mathcal{T}_i$, or $\overline{o}_i(t) := 0$, otherwise. Then, $\mathbf{u}$ can be written in stack vector form as:

$$\mathbf{u} = -g^T(\mathbf{x})\sum_{p \in \mathcal{P}} o_p(t)(\widetilde{D}_p \otimes I_n)F_p^T(\mathbf{x}, t)\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p, \tag{14}$$

where $G_p := \text{diag}(\overline{G}_{i_1}, \dots, \overline{G}_{i_p})$, and $o_p(t) := 1$, if $t \in [\sigma_p, \sigma_{p+1})$, or $o_p(t) := 0$, otherwise and $p \in \mathcal{P}$. Under the proposed control law, we can show the satisfaction of (8) for each $i \in \mathcal{I}$ as depicted in the following theorem:

Theorem 1. *Consider the multi-agent system* (2) *that is subject to the STL task defined in* (4) *and let Assumptions 1-4 hold. Assume further that* $\|f_r(\mathbf{x}_r)\| < +\infty$, *for every* $\mathbf{x}_r \in \mathbb{R}^n, r \in \mathcal{V}$ *and* $\beta_{k_i}(\sigma_p)$ *satisfies* (8) *for every* $i \in \{i_1, \dots, i_p\}$, $\sigma_p \in \Sigma$ *and* $p \in \mathcal{P}$. *Given* $\omega_i : \mathbb{R}_{\geq 0} \to (0, 1]$ *and parameters* $\underline{\eta}_i, \overline{\rho}_i, \overline{\eta}_i, \bar{\gamma}_{i,\infty}, l_i$, *let* $\{\beta_{k_i} : e_i(\beta_{k_i}, t) \in (-\underline{\eta}_i\gamma_i(\min \mathcal{T}_i), \overline{\eta}_i\gamma_i(\min \mathcal{T}_i))\} \subset \mathcal{B}_i$, *for every* $t \in \mathcal{T}_i$ *and* $i \in \mathcal{I}$, *where* $\mathcal{B}_i$ *are the same sets as in Assumption 3. Then, the control law* (14) *ensures that* (8) *is satisfied for every* $i \in \mathcal{I}$ *and the closed-loop signals at each* $[\sigma_p, \sigma_{p+1}), p \in \mathcal{P}$ *are bounded.*

Proof. For each $p \in \mathcal{P}$ we consider the Lyapunov function $V_p : \mathcal{D}_p \to \mathbb{R}_{\geq 0}$, defined as $V_p(\overline{\mathbf{e}}_p) = \frac{1}{2}\boldsymbol{\varepsilon}_p^T(\overline{\mathbf{e}}_p)G_p\boldsymbol{\varepsilon}_p(\overline{\mathbf{e}}_p)$, where $\mathcal{D}_p := (-\underline{\eta}_{i_1}, \overline{\eta}_{i_1}) \times \dots \times (-\underline{\eta}_{i_p}, \overline{\eta}_{i_p})$. Differentiating V_p and after substitution of (12) we get $\dot{V}_p = \boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)\big(\dot{\mathbf{e}}_p + \boldsymbol{\alpha}_p(t)\mathbf{e}_p\big)$. Using (10), (2) and after substitution of the proposed control law (14) $\dot{V}_p$ becomes:

$$\begin{aligned}\dot{V}_p = \;&\boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)(f(\mathbf{x}) + \mathbf{w}) - \\ &- \boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)g(\mathbf{x})g^T(\mathbf{x}) \times \\ &\times (\widetilde{D}_p \otimes I_n)F_p^T(\mathbf{x}, t)\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p + \boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t) \times \\ &\times (\Omega_p(t) + \boldsymbol{\alpha}_p(t))\mathbf{e}_p.\end{aligned}$$

By Assumption 1, $g(\mathbf{x})$ is full row rank, thus $g(\mathbf{x})g^T(\mathbf{x})$ is positive definite. In addition, since G is a tree graph, $\text{rank}(D) = M$ [12, Lem.1]. Due to Assumption 3 and by definition of Σ, within each $[\sigma_p, \sigma_{p+1})$ there exists at most one $i \in \mathcal{I}$ involving the k-th edge of G, thus $z_p \leq M$ and the columns of $\widetilde{D}_p$ are linearly independent which in turn implies that $\text{rank}(\widetilde{D}_p) = z_p$. Therefore, by virtue of [14, Obs. 7.1.8] and the properties of the Kronecker product it follows that $(\widetilde{D}_p^T \otimes I_n)g(\mathbf{x})g^T(\mathbf{x})(\widetilde{D}_p \otimes I_n)$ is positive definite. Furthermore, by Assumption 3 and since $\omega_i(t) > 0$ as well as $\{\beta_{k_i} : e_i(\beta_{k_i}, t) \in (-\underline{\eta}_i\gamma_i(\min \mathcal{T}_i), \overline{\eta}_i\gamma_i(\min \mathcal{T}_i))\} \subset \mathcal{B}_i$, for every $t \in [\sigma_p, \sigma_{p+1})$ and $i \in \{i_1, \dots, i_p\}$ it follows that $\text{rank}(F_p(\mathbf{x}, t)) = z_p$, for every $\overline{\mathbf{e}}_p(t) \in \mathcal{D}_p, t \in [\sigma_p, \sigma_{p+1})$. Hence, invoking again [14, Obs. 7.1.8] we can conclude that $A(\mathbf{x}, t) := F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)g(\mathbf{x})g^T(\mathbf{x})(\widetilde{D}_p \otimes I_n)F_p^T(\mathbf{x}, t)$ is positive definite. Let $\lambda := \inf_{t \in [\sigma_p, \sigma_{p+1})}\lambda_{\min}(A(\mathbf{x}(t), t))$ and note that $\lambda > 0$. Then, $\dot{V}_p$ satisfies: $\dot{V}_p \leq \boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)\big(F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)(f(\mathbf{x}) + \mathbf{w}) + (\Omega_p(t) + \boldsymbol{\alpha}_p(t))\mathbf{e}_p\big) - \lambda\|\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p\|^2$. For a parameter $0 < \xi < \lambda$ and after adding and subtracting $\xi\|\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p\|^2$ to the right-hand side of the aforementioned inequality we obtain:

$$\begin{aligned}\dot{V}_p \leq \;&\boldsymbol{\varepsilon}_p^T G_p\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)\big(F_p(\mathbf{x}, t)(\widetilde{D}_p^T \otimes I_n)(f(\mathbf{x}) + \mathbf{w}) \\ &+ (\Omega_p(t) + \boldsymbol{\alpha}_p(t))\mathbf{e}_p\big) - 2\kappa(\lambda - \xi)V_p \\ &- \xi\|\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p\|^2,\end{aligned}$$

where $\kappa = \frac{\overline{\kappa}}{\lambda_{\max}(G_p)}$ and $\overline{\kappa} := \min_{i \in \{i_1, \dots, i_p\}}\big(\frac{4\overline{G}_i}{\underline{\eta}_i + \overline{\eta}_i}\big)^2$, satisfying $\inf_{t \in [\sigma_p, \sigma_{p+1})}\lambda_{\min}(\mathcal{J}_p'^2(\overline{\mathbf{e}}_p, t)G_p^2) \geq \overline{\kappa}$ since $\gamma_i(t) \leq 1$ and $\frac{\partial T_i(\overline{e}_i)}{\partial \overline{e}_i} \geq \frac{4}{\underline{\eta}_i + \overline{\eta}_i}$, for every $i \in \{i_1, \dots, i_p\}$. Then, after completing squares we have $\dot{V}_p \leq -\hat{\lambda}V_p - \big(\frac{1}{2\sqrt{\xi}}\mathbf{y}(t) - \sqrt{\xi}\mathcal{J}_p'(\overline{\mathbf{e}}_p, t)G_p\boldsymbol{\varepsilon}_p\big)^T\big(\frac{1}{2\sqrt{\xi}}\mathbf{y}(t) -$

$\sqrt{\xi}\mathcal{J}_p'(\overline{\mathbf{e}}_p,t)G_p\boldsymbol{\varepsilon}_p\big) + \frac{1}{4\xi}\mathbf{y}^T(t)\mathbf{y}(t)$, where $\hat{\lambda} := 2\kappa(\lambda-\xi)$, and $\mathbf{y}(t) := F_p(\mathbf{x},t)(\widetilde{D}_p^T \otimes I_n)(f(\mathbf{x})+\mathbf{w}) + (\Omega_p(t)+\boldsymbol{\alpha}_p(t))\mathbf{e}_p$. The latter inequality then leads to $\dot{V}_p \leq -\hat{\lambda}V_p + \zeta(t)$, where we make use of the simplified notation $\zeta(t) := \frac{1}{4\xi}\mathbf{y}^T(t)\mathbf{y}(t)$. Let $W_p : \mathcal{D}_p \to \mathbb{R}_{\geq 0}$, be defined as $W_p(\overline{\mathbf{e}}_p) := 1 - \exp\left(-V_p(\overline{\mathbf{e}}_p)\right)$ which has the following properties: 1)$W_p(\mathbf{0}) = 0$, 2) $0 < W_p(\overline{\mathbf{e}}_p) < 1$, for every $\overline{\mathbf{e}}_p \in \mathcal{D}_p \backslash \{\mathbf{0}\}$ and 3) $W_p(\overline{\mathbf{e}}_p) \to 1$ as $\overline{\mathbf{e}}_p \to \partial\mathcal{D}_p$, where $\partial\mathcal{D}_p$ denotes the boundary of $\mathcal{D}_p$. Differentiating W_p we get $\dot{W}_p = \dot{V}_p(1-W_p)$ which leads to $\dot{W}_p \leq -\hat{\lambda}\ln\left(\frac{\exp\left(-\frac{\zeta(t)}{\hat{\lambda}}\right)}{1-W_p}\right)(1-W_p)$, where we have made use of the definition of $W_p(\overline{e}_p)$ and the fact that $\dot{V}_p \leq -\hat{\lambda}V_p + \zeta(t)$. Let $\mathcal{S} := \big\{\overline{\mathbf{e}}_p \in \mathcal{D}_p : W_p \leq 1 - \exp\left(-\frac{\overline{\zeta}}{\hat{\lambda}}\right)\big\} \subset \mathcal{D}_p$, where $\overline{\zeta} \geq \frac{1}{4\xi}\mathbf{y}^T(t)\mathbf{y}(t)$, for every $t \in [\sigma_p, \min(\tau_{\max}, \sigma_{p+1}))$, where $\tau_{\max} \in \mathbb{R}_{\geq 0}$ is the upper bound of the interval of the maximal solution of $\overline{\mathbf{e}}_p(t) \in \mathcal{D}_p$. Note that the time-invariant bound $\overline{\zeta}$ exists due to boundedness of $\omega_i(t), \dot{\omega}_i(t), \gamma_i(t), \dot{\gamma}_i(t), \mathbf{w}(t), f(\mathbf{x})$ for every $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^{Rn}$, respectively, as well as due to the boundedness of $\frac{\partial h_i}{\partial \beta_{k_i}}$ within $\mathcal{B}_i$ for every $i \in \{i_1,\ldots,i_p\}$ by Assumption 3. By assumption, $\overline{\mathbf{e}}_p(\sigma_p) \in \mathcal{D}_p$ which implies that $W(\overline{\mathbf{e}}_p(\sigma_p)) < 1$. Let $y := W(\overline{\mathbf{e}}_p(\sigma_p))$ and $\mathcal{S}_y := \{\overline{\mathbf{e}}_p \in \mathcal{D}_p : W(\overline{\mathbf{e}}_p) \leq y\}$. We consider two cases. First, let $y \geq 1 - \exp\left(-\frac{\overline{\zeta}}{\hat{\lambda}}\right)$. Then, $\mathcal{S} \subseteq \mathcal{S}_y$. Since $\dot{W}_p(\overline{\mathbf{e}}_p) \leq 0$ for every $\overline{\mathbf{e}}_p \in \mathcal{S}_y \backslash \mathcal{S}$ it follows that $\overline{\mathbf{e}}_p \to \mathcal{S}$. Next, let $y < 1 - \exp\left(-\frac{\overline{\zeta}}{\hat{\lambda}}\right)$ which implies $\mathcal{S}_y \subseteq \mathcal{S}$. Since $\dot{W}_p \leq 0$, for every $\overline{\mathbf{e}}_p \notin \mathcal{S}$, it follows that $\overline{\mathbf{e}}_p \in \mathcal{S}$. Hence, in both cases $W_p(\overline{\mathbf{e}}_p) < 1$. This implies that there exists a compact set $\mathcal{D}_p' \subset \mathcal{D}_p$ such that $\overline{\mathbf{e}}_p \in \mathcal{D}_p'$ which by means of the inverse transformation functions $T_i^{-1}(\cdot), i \in \{i_1,\ldots,i_p\}$ ensures the boundedness of $\boldsymbol{\varepsilon}_p$ and $\tau_{\max} \geq \sigma_{p+1}$. As a result, $e_i, i \in \{i_1,\ldots,i_p\}$ satisfies (8). ■

Theorem 1 ensures the satisfaction of (8) for every $t \in \mathcal{T}_i, i \in \mathcal{I}$, when $\beta_{k_i}(\sigma_p), i \in \{i_1,\ldots,i_p\}, p \in \mathcal{P}$ satisfy (8) and "within each funnel" the gradient of the corresponding predicate function remains bounded and non-zero.

B. Funnel Design

Having shown the satisfaction of the funnel constraints under the proposed control law (14), we can now proceed with the design of the funnel constraints so as to ensure STL satisfaction. Recall from Section II that $\gamma_i(t) := (1-\gamma_{i,\infty})\exp\left(-l_i t\right) + \gamma_{i,\infty}$ and thus given (5), (6) the parameters to be tuned are $\overline{\eta}_i, \underline{\eta}_i, \overline{\rho}_i, \bar{\gamma}_{i,\infty}, l_i$, for every $i \in \mathcal{I}$, where $\gamma_{i,\infty} := \frac{\bar{\gamma}_{i,\infty}}{\max(\overline{\eta}_i,\underline{\eta}_i)}$. First, based on the nature of the temporal operator corresponding to $\varphi_i, i \in \mathcal{I}$ we define the time instant $t_i^* \in [a_i, b_i]$ for every $i \in \mathcal{I}$ as follows:

$$t_i^* \in \begin{cases} \{a_i\}, & \text{if } \varphi_i = \mathcal{G}_{[a_i,b_i]}\psi_i \\ [\max(a_i,c_i), b_i], & \text{if } \varphi_i = \mathcal{F}_{[a_i,b_i]}\psi_i. \end{cases} \tag{15}$$

Intuitively, t_i^* is the time instant at which φ_i should be satisfied with a minimum robustness $\overline{\rho}$ which is given by the designer. Motivated by [8] we choose the funnel parameters as follows:

$$\bar{\gamma}_{i,\infty} \in (0, \max(\overline{\eta}_i, \underline{\eta}_i)), \tag{16a}$$

$$\overline{\rho}_i \in (\overline{\rho} + \underline{\eta}_i\gamma_{i,\infty}, \inf(\overline{\nu}_i, \underline{\eta}_i + \overline{\rho})) \tag{16b}$$

$$l_i > -\frac{\ln\left(\frac{\max(\overline{\eta}_i,\underline{\eta}_i)\widetilde{\gamma}_i - \bar{\gamma}_{i,\infty}}{\max(\overline{\eta}_i,\underline{\eta}_i) - \bar{\gamma}_{i,\infty}}\right)}{t_i^*}, \tag{16c}$$

where $\widetilde{\gamma}_i := \frac{\overline{\rho}_i - \overline{\rho}}{\underline{\eta}_i}$, for every $i \in \mathcal{I}$ and $\overline{\nu}_i = \sup_{\beta_{k_i}\in\mathcal{B}_i} h_i(\beta_{k_i})$ as defined in Assumption 3. Constraint (16a) implies that $\gamma_{i,\infty} := \frac{\bar{\gamma}_{i,\infty}}{\max(\overline{\eta}_i,\underline{\eta}_i)} < 1$ which in turn guarantees that $\gamma_{i,\infty} := \lim_{t\to+\infty}\gamma_i(t) < \gamma_i(0)$. Constraint (16c) ensures that $\gamma_i(t_i^*) < \widetilde{\gamma}_i$ which by definition $\widetilde{\gamma}_i$ implies that $\overline{\rho}_i - \underline{\eta}_i\gamma_i(t_i^*) > \overline{\rho}$. Thus, if the funnel constraints (8) are satisfied, then $h_i(\beta_{k_i}(\mathbf{x}(t_i^*))) > \overline{\rho}$ and since $\gamma_i(t)$ is strictly decreasing, $h_i(\beta_{k_i}(\mathbf{x}(t))) > \overline{\rho}$, will hold for every $t \in [t_i^*, b_i]$. Note that by design of $\gamma_i(t)$, l_i chosen according to (16c) should satisfy $l_i > 0$ or equivalently $\frac{\max(\overline{\eta}_i,\underline{\eta}_i)\widetilde{\gamma}_i - \bar{\gamma}_{i,\infty}}{\max(\overline{\eta}_i,\underline{\eta}_i) - \bar{\gamma}_{i,\infty}} < 1$, provided that the left hand side of the aforementioned inequality is strictly positive in order for the argument of the logarithm to be well-defined. Given the definition of $\widetilde{\gamma}_i$ and due to (16a) these properties are always satisfied when $\overline{\rho}_i$ is chosen according to (16b). Based on the aforementioned discussion, we can deduce the following:

Theorem 2. *Consider the multi-agent system* (2) *and let* $\varphi_i, i \in \mathcal{I}$ *be the STL formulas given in* (4). *Let the assumptions of Theorem 1 hold. Given* $\overline{\rho} \in (0, \min_{i\in\mathcal{I}}\overline{\nu}_i)$, *if for every* $i \in \mathcal{I}$ *the positive parameters* $\overline{\eta}_i, \underline{\eta}_i, \overline{\rho}_i, \bar{\gamma}_{i,\infty}, l_i$, *determining* (8) *are chosen according to* (16a)-(16c) *such that* $\{\beta_{k_i} : e_i(\beta_{k_i}, t) \in (-\underline{\eta}_i\gamma_i(\min\mathcal{T}_i), \overline{\eta}_i\gamma_i(\min\mathcal{T}_i))\} \subset \mathcal{B}_i$, *then under the proposed control law* (14), $\rho^{\varphi_i}(\mathbf{x}, 0) \geq \overline{\rho}$, *for every* $i \in \mathcal{I}$ *and thus* $\rho^{\phi}(\mathbf{x}, 0) \geq \overline{\rho}$.

Proof. In Theorem 1 it has been shown that the control law proposed in (14) ensures the satisfaction of (8). Choosing the parameters determining the funnel constraint according to (16a)-(16c) and since t_i^* is in the interior of $\mathcal{T}_i$ defined in (7), then $h_i(\beta_{k_i}(\mathbf{x}(t))) > \overline{\rho}$, for every $t \in [t_i^*, b_i]$. If $\varphi_i = \mathcal{F}_{[a_i,b_i]}\psi_i$, then for some $t_1 \in [a_i, b_i] \cap [t_i^*, b_i]$ it holds that $h_i(\beta_{k_i}(\mathbf{x}(t_1))) > \overline{\rho}$, which implies that $\max_{t\in[a_i,b_i]} h_i(\beta_{k_i}(\mathbf{x}(t))) \geq \overline{\rho}$ and the result follows. If $\varphi_i = \mathcal{G}_{[a_i,b_i]}\psi_i$, then choosing the funnel parameters as in (16a)-(16c) as well as by design of t_i^*, c_i, defined in (15) and (5), respectively, we have $h_i(\beta_{k_i}(\mathbf{x}(t))) > \overline{\rho}$, for every $t \in [a_i, b_i]$ which in turn implies that $\min_{t\in[a_i,b_i]} h_i(\beta_{k_i}(\mathbf{x}(t))) \geq \overline{\rho}$. Finally, $\rho^{\phi}(\mathbf{x}, 0) \geq \overline{\rho}$ follows by aforementioned analysis and the definition of $\rho^{\phi}(\mathbf{x}, 0)$. ■

IV. Numerical Example

The control strategy proposed in Section III will be applied to a time-varying formation control problem involving $R = 4$ agents with $f_r(\mathbf{x}_r) = \begin{bmatrix}\cos\left(x_r - r\right) - 2\sin\left(y_r\right) & \cos\left(0.2(x_r - y_r)\right)\end{bmatrix}^T, r \in \mathcal{V}\backslash\{2\}$, $g_r(\mathbf{x}_r) = \begin{bmatrix} 1 + 0.25x_r^2 & -1 \\ 0 & 2 + \sin\left(y_r\right)\end{bmatrix}$ for every

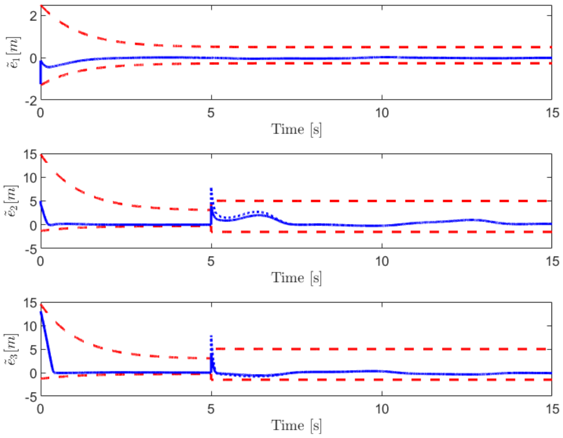

Fig. 1: Evolution of the errors corresponding to the k-th edge with and without the shifting functions defined as $\sum_{i\in\mathcal{I}_k}\overline{o}_i(t)e_i(t)$ (solid blue line) and $\sum_{i\in\mathcal{I}_k}\overline{o}_i(t)\frac{e_i(t)}{\omega_i(t)}$ (dotted blue line), respectively. The funnel constraints are shown in red.

$r \in \mathcal{V}$, $\mathbf{w}_r(t) = \cos(2t)\begin{bmatrix}-0.5 & 0.1\end{bmatrix}^T, r \in \mathcal{V}\backslash\{2,4\}$, $\mathbf{w}_4 = \cos(t)\begin{bmatrix}0.5 & -2\end{bmatrix}^T$, $f_2(\mathbf{x}_2) + \mathbf{w}_2 = \text{sat}_4(-2(\mathbf{x}_2 - \mathbf{x}_2^{\text{ref}}))$, $\mathbf{x}_2^{\text{ref}} = \begin{bmatrix}-1+\cos(t) & 1+\sin(t)\end{bmatrix}^T$, where $\mathbf{x}_r = \begin{bmatrix}x_r & y_r\end{bmatrix}^T$ is the position of the r-th agent. The team is subject to the STL task $\phi = \phi_1 \wedge \phi_2$, where $\phi_1 := \bigwedge_{i=1}^{2}(h_i(\beta_i(\mathbf{x})) \geq 0)\ \mathcal{U}_{[2,5]}(h_3(\beta_3(\mathbf{x})) \geq 0)$, and $\phi_2 := \mathcal{G}_{[12,15]}\big((h_4(\beta_2(\mathbf{x})) \geq 0) \wedge (h_5(\beta_3(\mathbf{x})) \geq 0)\big)$, $h_i(\beta_{k_i}(\mathbf{x})) := \vartheta_i - \|\beta_{k_i}(\mathbf{x})\|^2$, for every $i \in \mathcal{I}$, and $\vartheta_1 = 4$, $\vartheta_2 = \vartheta_3 = 16$ and $\vartheta_4 = \vartheta_5 = 25$. The edge set of G is given by $\mathcal{E} = \{\mathfrak{e}_1 = (1,2), \mathfrak{e}_2 = (1,3), \mathfrak{e}_3 = (3,4)\}$. Here, $\epsilon = 10^{-3}$ and $\overline{\rho} = \delta_i = 0.5$, for every $i \in \{4,5\}$. The shifting functions $\omega_i(t), i \in \{4,5\}$ are chosen as:

$$\omega_i(t) = \begin{cases} \mathfrak{q}_{i,1}t^2 + \mathfrak{q}_{i,2}t + \underline{\delta}, & 0 \leq t < \underline{c}_i \\ \mathfrak{q}_{i,3}(t-\underline{c}_i)^2 + \mathfrak{q}_{i,4}(t-\underline{c}_i) + \delta_i, & \underline{c}_i \leq t < c_i \\ 1, & t \geq c_i \end{cases}$$

where $\underline{c}_i := \min \mathcal{T}_i$, $\underline{\delta} = 0.01$ is a tuning parameter ensuring that $\omega_i(0) > 0$, $\mathfrak{q}_{i,1} := \frac{1}{\underline{c}_i}\left(\mathfrak{q}_{i,4} - \frac{\delta_i - \underline{\delta}}{\underline{c}_i}\right)$, $\mathfrak{q}_{i,2} := \mathfrak{q}_{i,4} - 2\mathfrak{q}_{i,1}\underline{c}_i$, $\mathfrak{q}_{i,3} := \frac{\delta_i - 1}{(c_i - \underline{c}_i)^2}$, $\mathfrak{q}_{i,4} := 2\frac{1-\delta_i}{c_i - \underline{c}_i}$ and $c_4 = c_5 = 10$. For $i \in \{1,2,3\}$ we set $\omega_i(t) = 1$, for every t. The initial conditions are chosen as $\mathbf{x}_1 = \begin{bmatrix}-2 & 1\end{bmatrix}^T$, $\mathbf{x}_2 = \begin{bmatrix}0 & 0.5\end{bmatrix}^T$, $\mathbf{x}_3 = \begin{bmatrix}-1 & 4\end{bmatrix}^T$, $\mathbf{x}_4 = \begin{bmatrix}0 & 3\end{bmatrix}^T$, and satisfy $e_i(\mathbf{x}(0)) \in (-\underline{\eta}_i\gamma_i(0), \overline{\eta}_i\gamma_i(0))$, for every $i \in \{1,2,3\}$. The funnel parameters are chosen as follows: $\underline{\eta}_i = 1.3,\ i \in \{1,2,3\}$, $\underline{\eta}_i = 3,\ i \in \{4,5\}$, $\overline{\eta}_1 = 2.5$, $\overline{\eta}_2 = 14.8$, $\overline{\eta}_3 = 14.5$, $\overline{\eta}_i = 10,\ i \in \{4,5\}$, $l_i = 0.9,\ i \in \{1,2,3\}$, $l_i = 0.92,\ i \in \{4,5\}$ $\bar{\gamma}_{1,\infty} = 0.5$, $\bar{\gamma}_{2,\infty} = 2.96$, $\bar{\gamma}_{3,\infty} = 2.9$, $\bar{\gamma}_{i,\infty} = 5,\ i \in \{4,5\}$, $\overline{\rho}_i = 1,\ i \in \{1,2,3\}$ and $\overline{\rho}_i = 2.1,\ i \in \{4,5\}$. In addition, we consider control gain matrices $G_p = 0.1I_3$, for $p \in \{1,2\}$ and set $t_3^* = 5$. In Figure 1 the evolution of the error signals corresponding to each edge $k \in \mathcal{M}$ is shown with and without the effect of the shifting functions. As shown in the figure the existence of the shifting functions ensures the satisfaction of the funnel constraints especially at $t = 5 + 10^{-3}$ sec when the desired formation shape changes. In addition, $\rho^{\phi}(\mathbf{x}, 0) = 0.9681$ holds which verifies the result of Theorem 2, i.e., that $\rho^{\phi}(\mathbf{x}, 0) \geq 0.5$.

V. Conclusions

In this work a distributed switching control strategy is designed to ensure the satisfaction of a conjunction of STL tasks that are based on relative position information among neighboring agents. The satisfaction of individual STL tasks is enforced by prescribed performance functions designed to ensure a desired level of robustness. Assuming a tree graph topology, we show the satisfaction of the funnel constraints after pre-specified time instants and the boundedness of the closed-loop signals. Future efforts will be directed towards considering more complex graph topologies and STL tasks as well higher-relative degree systems.

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