Funnel-based Cooperative Control of Leader-follower Multi-agent Systems under Signal Temporal Logic Specifications

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***Abstract*— Control of multi-agent systems under temporal logical specifications has been popular due to its ability to tackle complex tasks that cannot be easily defined as classic control objectives. In this paper, a general class of leader-follower multi-agent systems subject to certain fragments of signal temporal logic (STL) specifications is considered. We first propose a funnel-based control strategy for the leader-follower multi-agent systems to enforce the satisfaction of the basic STL formulas by prescribing certain transient behavior on the funnels that constrain the closed-loop trajectories. A hybrid control strategy is then leveraged to satisfy the sequential STL formulas. Finally, a simulation example is given to illustrate the results.**

I. Introduction

Control of multi-agent systems has been significantly studied due to its wide applications in multi-robot coordination [18], manufacturing and transportation systems [6]. A recent trend is to apply the formal methods [2] based approaches within the multi-agent system context in order to specify more complex and high-level task specifications that cannot be easily defined as classic control objectives. Temporal logics, such as Linear Temporal Logic (LTL) [10] offer a formal way to specify such high-level tasks from a computer science perspective. Signal Temporal Logic (STL), which is based on continuous time signals, has the added feature of formulating both time and space constraints, and thus provides potentials to deal with quantitative transient constraints for multi-agent systems.

In this work, we focus on a leader-follower multi-agent framework which takes the different capabilities among the agents into account. For example, agents that show advanced actuation, computation and communication capabilities are selected as leaders in order to drive the whole agent team to fulfill the task specifications. These complex and high-level task specifications are represented by STL formulas in order to tackle both time and space constraints. Recent research within the leader-follower framework mainly focuses on controllability of leader-follower multi-agent systems [8] and leader selection for optimal performance [17], [7], however, complex tasks in the form of STL formulas are largely unexplored in the leader-follower setup.

Prescribed performance control (PPC), which utilises a funnel-based approach, was originally proposed in [1] to prescribe the evolution of system output or the tracking error within some predefined region. In this work, we use such funnel-based approach in order to enforce the satisfaction of the STL formulas by prescribing the transient behavior of the funnels that constrain the system trajectories. In our previous work [3], we have proposed distributed control laws for leader-follower multi-agent systems using PPC to achieve the classical consensus or formation tasks within predefined transient constraints. However, more complex and high-level specifications with quantitative time and space constraints have not been treated. In [15], the authors propose a control strategy based on a time-varying fixed-time convergent control barrier functions for a class of coupled multi-agent systems under STL tasks. Prescribed performance control of multi-agent systems under STL formulas is addressed in [11]. These papers use transient approaches such as PPC and control barrier functions in order to prescribe certain transient behavior that enforces the satisfaction of the STL specifications. However, they do not consider the different capabilities among the agents, and thus the control strategy is applied for all agents. In contrast, our contribution in this work is considering funnel-based control in a leader-follower framework such that the closed-loop system also satisfies complex high-level tasks represented by the STL specifications. It is also the first time that the funnel-based approach for the satisfaction of STL specifications in a leader-follower setup is considered.

The rest of the paper is organized as follows. In Section II, preliminaries are introduced and the problem is formulated. Section III proposes a funnel-based approach for leader-follower multi-agent systems under basic STL formulas. Further results for sequential STL formulas are discussed in Section IV. The results are verified by a simulation in Section V. Section VI includes conclusions and future work.

II. Preliminaries and problem statement

A. Leader-follower Multi-agent Systems

We consider a multi-agent system under an undirected communication graph [14] $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the vertices set $\mathcal{V} = \{1, 2, \dots, N\}$ and the edges set $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i\}$ where $\mathcal{N}_i$ denotes the neighbor of agent i such that agent $j \in \mathcal{N}_i$ can communicate with i. Suppose that we have N_F follower agents and N_L leader agents with the respective vertices set as $\mathcal{V}_F = \{1, 2, \dots, N_F\}$ and $\mathcal{V}_L = \{N_F+1, N_F+2, \dots, N_F+N_L\}$ where it holds that $N = N_F + N_L$.

Let $x_i \in \mathbb{R}^n$ be the state of agent i, and the state evolution of each follower $i \in \mathcal{V}_F$ be governed by the first order

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agreement protocol:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i), \tag{1}$$

while the state evolution of each leader $i \in \mathcal{V}_L$ is governed by the first order agreement protocol with an assigned external input $u_i \in \mathbb{R}^n$:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i) + u_i. \tag{2}$$

Let $x_F = [x_1, \dots, x_{N_F}]^T \in \mathbb{R}^{nN_F}$ and $x_L = [x_{N_F+1}, \dots, x_N]^T \in \mathbb{R}^{nN_L}$ be the respective stacked state vector of all follower and leader agents, denote $x = [x_F^T, x_L^T]^T \in \mathbb{R}^{nN}$. Let $u = [u_{N_F+1}, \dots, u_N]^T \in \mathbb{R}^{nN_L}$ be the control input vector. Stacking (1) and (2), we derive the following dynamics of the leader-follower multi-agent system:

$$\Sigma : \dot{x} = -(L \otimes I_n)x + (B \otimes I_n)u, \tag{3}$$

where L is the graph Laplacian [14], I_n is the $n \times n$ identity matrix and $B = \begin{bmatrix} 0_{N_F\times N_L} \\ I_{N_L} \end{bmatrix}$.

B. Signal Temporal Logic (STL)

The boolean true and false values are represented by $\top$ and $\bot$, respectively, and $\mathbb{B} := \{\top, \bot\}$. Signal temporal logic (STL) [13] consists of predicates $\mu : \mathbb{R}^n \to \mathbb{B}$ which are obtained by evaluating a continuously differentiable predicate function $h : \mathbb{R}^n \to \mathbb{R}$ and assigning the respective true or false boolean value as: $\mu = \top$, if $h(x) \geq 0$; $\mu = \bot$, if $h(x) < 0$, where $x \in \mathbb{R}^n$. The STL syntax is defined as

$$\phi ::= \top \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathrm{F}_{[a,b]}\phi \mid \mathrm{G}_{[a,b]}\phi, \tag{4}$$

where ϕ_1, ϕ_2 are STL formulas and $\neg, \wedge, \mathrm{F}_{[a,b]}, \mathrm{G}_{[a,b]}$ are the respective negation, conjunction, eventually, always operators with $0 \leq a \leq b < \infty$. For a continuous-time signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $(x, t) \models \phi$ denotes the satisfaction relation, which holds if x satisfies ϕ at time t. Robust semantics have been introduced in [5] in order to quantify how robustly the signal x satisfies the STL formula ϕ at time t. Space robustness semantics [4] for STL are defined as: $\rho^{\mu}(x,t) := h(x(t))$; $\rho^{\neg\phi}(x,t) := -\rho^{\phi}(x,t)$; $\rho^{\phi_1\wedge\phi_2}(x,t) := \min(\rho^{\phi_1}(x,t), \rho^{\phi_2}(x,t))$; $\rho^{\mathrm{F}_{[a,b]}\phi}(x,t) := \max_{t_1\in[t+a,t+b]} \rho^{\phi}(x,t_1)$; $\rho^{\mathrm{G}_{[a,b]}\phi}(x,t) := \min_{t_1\in[t+a,t+b]} \rho^{\phi}(x,t_1)$. Note that it holds that $(x,t) \models \phi$ if $\rho^{\phi}(x,t) > 0$. In this work, we consider a fragment of the STL introduced above, which is defined as follows:

$$\psi := \top \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2 \tag{5a}$$

$$\phi := \mathrm{F}_{[a,b]}\psi \mid \mathrm{G}_{[a,b]}\psi \tag{5b}$$

$$\phi' := \bigwedge_{i=1}^{M} \phi_i \tag{5c}$$

where ψ in (5b) and ψ_1, ψ_2 in (5a) are non-temporal formulas of class ψ as in (5a), and where $\phi_i, i \in \{1, \cdots, M\}$ in (5c) are temporal formulas of class ϕ as in (5b) with time intervals $[a_i, b_i]$ satisfying $b_i \leq a_{i+1}, \forall i \in \{1, \cdots, M-1\}$. In this work, we focus on the fragment of STL in the form of (5b), (5c), which are expressive enough to tackle leader-follower multi-agent planning tasks, e.g. formation control, collision avoidance and connectivity maintenance. It is also possible to expand to full fragments that include more expressive tasks according to recent work, e.g., [12]. The non-smooth robust semantics $\rho^{\phi_1\wedge\phi_2}(x,t)$ can be replaced by a smooth approximation $\rho^{\phi_1\wedge\phi_2}(x,t) \approx -\frac{1}{\eta}\ln(\sum_{i=1}^{2}\exp(-\eta\rho^{\phi_i}(x,t)))$ with the parameter $\eta > 0$ determining the accuracy of the approximation. No matter the choice of η, it always holds that $-\frac{1}{\eta}\ln(\sum_{i=1}^{2}\exp(-\eta\rho^{\phi_i}(x,t))) \leq \min(\rho^{\phi_1}(x,t), \rho^{\phi_2}(x,t))$ with the equality holding as $\eta \to \infty$, which indicates that this smooth approximation is further an under approximation. Consequently, we can conclude that $(x,t) \models \phi_1 \wedge \phi_2$ as long as $-\frac{1}{\eta}\ln(\sum_{i=1}^{2}\exp(-\eta\rho^{\phi_i}(x,t))) > 0$.

C. Problem Statement

The aim of this work is to design the control strategy only for the leaders such that the leader-follower multi-agent system (3) can fulfill the target task which is represented by an STL formula. Formally, we define the problem as follows:

Problem 1. *Consider the leader-follower multi-agent system* (3)*, given an STL formula ϕ as in* (5b) *or ϕ' as in* (5c)*. Design a control strategy u for* (3) *such that the closed-loop trajectory $x : [0,\infty) \to \mathbb{R}^{nN}$ guarantees $\rho^{\phi}(x,0) > 0$ or $\rho^{\phi'}(x,0) > 0$.*

III. Satisfaction of STL using PPC

In this section, we show how to synthesise a control strategy u for the temporal formulas ϕ as in (5b) using prescribed performance control such that $\rho^{\phi}(x,0) > 0$ can be achieved where $x : [0,\infty) \to \mathbb{R}^{nN}$ is the closed-loop solution of (3). PPC is a funnel-based approach to prescribe the transient behavior within some predefined region, which can be described as follows according to our problem in hand:

$$-\mathrm{p}(t) + \rho^{\star} < \rho^{\psi}(x,0) < \rho^{\star}, \tag{6}$$

where ψ is the corresponding non-temporal formula inside the F, G operators as in (5b), which in combination with the temporal operators F, G forms the temporal formulas of class ϕ. Note that the robustness function $\rho^{\psi}(x,0)$ is time-dependent since it is evaluated based on the timed trajectory $x(t)$, but we sometimes replace $x(t)$ by x in the paper for simplicity. We aim to achieve $\rho^{\phi}(x,0) > 0$ by prescribing a temporal behavior to the corresponding $\rho^{\psi}(x,0)$ by appropriately designing the parameters $\mathrm{p}(t)$ and the positive scalar $\rho^{\star}$. Here $\mathrm{p}(t) : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is the so called performance function [1] which is positive, smooth and strictly decreasing, and introduces the funnel to prescribe the behavior of $\rho^{\psi}(x,0)$. We define $\mathrm{p}(t) := (\mathrm{p}_0 - \mathrm{p}_\infty)e^{-lt} + \mathrm{p}_\infty$ with $\mathrm{p}_0, \mathrm{p}_\infty, l$ as positive parameters and $\mathrm{p}_0 > \mathrm{p}_\infty$. We aim to design $\mathrm{p}(t), \rho^{\star}$ such that the satisfaction of (6) implies that $\rho^{\phi}(x,0) > 0$ holds. How to choose these parameters has been elucidated in [11]. We further define $e(x) = \rho^{\psi}(x,0) - \rho^{\star}$ and the modulated error $\bar{e}(x,t) = \frac{e(x)}{\mathrm{p}(t)}$ together with the corresponding prescribed performance region $\mathcal{D} := \{\bar{e} : \bar{e} \in (-1,0)\}$. Then the modulated error is transformed through a transformation function T that defines the smooth and strictly increasing mapping $T : \mathcal{D} \to \mathbb{R}$. The transformed error is then defined as

$$\varepsilon(x,t) = T(\bar{e}(x,t)) = \ln\left(-\frac{1+\bar{e}(x,t)}{\bar{e}(x,t)}\right). \tag{7}$$

It can be verified that if the transformed error $\varepsilon(x,t)$ is bounded, then the modulated error $\bar{e}(x,t)$ is constrained within the prescribed performance region $\mathcal{D}$ [1]. This also in turn implies the satisfaction of (6).

A. Control design for basic temporal formulas

In this subsection, we derive a control strategy for the leader-follower multi-agent system such that the prescribed behavior on $\rho^{\psi}(x,0)$, i.e., (6) can be achieved, which will be utilised later for the more complex sequential STL formulas. Before presenting the main results of this section, we first state the following assumption that is assumed in this paper.

Assumption 1. *The leaders have access to $\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$, which involves the knowledge of the robustness function $\rho^{\psi}(x,0)$ and the corresponding agent states it comprises. In addition $\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$ is a nonzero vector.*

Next, we propose a control strategy such that $\rho^{\psi}(x,0)$ is always within the funnel (6).

Theorem 1. *Consider the leader-follower multi-agent system* (3)*, given an STL formula ϕ as in* (5b)*, with the corresponding ψ satisfying Assumption 1. If the initial condition $\rho^{\psi}(x(0),0)$ is within the funnel* (6)*, then the control strategy*

$$u(x,t) = -\varepsilon(x,t)\frac{\partial \rho^{\psi}(x,0)}{\partial x_L} \tag{8}$$

guarantees the satisfaction of (6) *for all $t \geq 0$, where $\varepsilon(x,t)$ is the transformed error defined as in* (7).

Proof. The proof is based on the following three steps. We first show that there exists a maximal solution for $\bar{e}(x,t)$, which means that $\bar{e}(x,t)$ remains in $\mathcal{D}$ within the maximal time solution interval $[0,\tau_{\max})$. Next, we prove that the proposed control strategy (8) restricts $\bar{e}(x,t)$ in a compact subset of $\mathcal{D}$, which by contradiction results in $\tau_{\max} = \infty$ in the last step and the proof is completed. In the sequel, we show the proof in detail step by step.

Step 1. Since the initial condition $\rho^{\psi}(x(0),0)$ is within the funnel (6), this implies that the initial condition $\bar{e}(x(0),0)$ is within the prescribed performance region $\mathcal{D}$ according to equation (6) and the modulated error. Inserting (8) to (3), we obtain the closed-loop dynamics as $\dot{x} := f_1(x,\bar{e}) = -(L \otimes I_n)x - (B \otimes I_n)\ln\left(-\frac{1+\bar{e}}{\bar{e}}\right)\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$. By calculating the derivative of $\bar{e} = (\rho^{\psi}(x,0) - \rho^{\star})/\mathrm{p}(t)$, we have $\dot{\bar{e}} = \left(\frac{\partial \rho^{\psi}(x,0)}{\partial x}\dot{x}\mathrm{p}(t) - \dot{\mathrm{p}}(t)e(x)\right)/\mathrm{p}^2(t) = \left(\frac{\partial \rho^{\psi}(x,0)}{\partial x}\dot{x} - \dot{\mathrm{p}}(t)\bar{e}\right)/\mathrm{p}(t)$. Replacing $\dot{x}$, we further obtain $\dot{\bar{e}} := f_2(x,\bar{e},t) = \frac{1}{\mathrm{p}(t)}\left(\frac{\partial \rho^{\psi}(x,0)}{\partial x}f_1(x,\bar{e}) - \bar{e}\dot{\mathrm{p}}(t)\right)$. Let $z = [x^T, \bar{e}]^T$, then $\dot{z} = f(z,t) = [f_1^T(x,\bar{e}), f_2(x,\bar{e},t)]^T$. The initial condition $x(0)$ is such that $\bar{e}(x(0),0) \in \mathcal{D}$, which is an open set. We then define $\mathcal{D}_x := \{x \in \mathbb{R}^{nN} \mid \bar{e}(x(0),0) \in \mathcal{D}\}$, which is also an open, non-empty and bounded set. Therefore, $\mathcal{D}_z := \mathcal{D}_x \times \mathcal{D}$ is an open, non-empty and bounded set, and the initial condition satisfies $z(0) = [x^T(0), \bar{e}(x(0),0)]^T \in \mathcal{D}_z$. We now consider the initial value problem $\dot{z} = f(z,t)$ with $z(0) \in \mathcal{D}_z$. We can verify that $f(z,t)$ is continuous on t due to continuity of $\mathrm{p}(t)$ and $\dot{\mathrm{p}}(t)$. Moreover, since the transformed function $\ln\left(-\frac{1+\bar{e}}{\bar{e}}\right)$ is locally Lipschitz continuous and $\frac{\partial \rho^{\psi}(x,0)}{\partial x}$ is also locally Lipschitz continuous due to the smooth approximation discussed previously, we can conclude that $f(z,t)$ is locally Lipschitz on z. Hence, according to Theorem 54 of [16], there exists a maximal solution $z(t)$ of the initial value problem $\dot{z} = f(z,t)$ in a time interval $[0,\tau_{\max})$ such that $z(t) \in \mathcal{D}_z, \forall t \in [0,\tau_{\max})$.

Step 2. Based on Step 1, we know that $\rho^{\psi}(x(t),0)$ satisfies (6) for all $t \in [0,\tau_{\max})$. This is due to the fact that $z(t) \in \mathcal{D}_z, \forall t \in [0,\tau_{\max})$, thus $\bar{e}(x(t),t) \in \mathcal{D}, \forall t \in [0,\tau_{\max})$, which in turn implies the satisfaction of (6) for all $t \in [0,\tau_{\max})$. We now consider the Lyapunov function candidate $V(\varepsilon) = \frac{1}{2}\varepsilon^2$. Taking the derivative on (7), we have $\dot{\varepsilon} = \frac{\bar{e}}{1+\bar{e}}\frac{\dot{\bar{e}}\bar{e} - \dot{\bar{e}}(1+\bar{e})}{\bar{e}^2} = -\frac{\dot{\bar{e}}}{\bar{e}(1+\bar{e})}$. By replacing $\dot{\bar{e}}$, which is derived in Step 1, $\dot{V}$ can be obtained as

$$\dot{V} = \varepsilon\dot{\varepsilon} = \varepsilon\left(-\frac{1}{\mathrm{p}\bar{e}(1+\bar{e})}\left(\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}\dot{x} - \bar{e}\dot{\mathrm{p}}\right)\right). \tag{9}$$

Replacing $\dot{x}$ by (3), we further obtain

$$\dot{V} = -\frac{\varepsilon}{\mathrm{p}\bar{e}(1+\bar{e})}\Bigg(\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}\big(-(L \otimes I_n)x + (B \otimes I_n)u\big) - \bar{e}\dot{\mathrm{p}}\Bigg). \tag{10}$$

Then, since the performance function $\mathrm{p}(t) := (\mathrm{p}_0 - \mathrm{p}_\infty)e^{-lt} + \mathrm{p}_\infty$ is strictly decreasing, we have $\mathrm{p}_\infty \leq \mathrm{p}(t) \leq \mathrm{p}_0$. Moreover, since $\bar{e} \in (-1,0)$, we can verify that $\frac{4}{\mathrm{p}_0} \leq -\frac{1}{\mathrm{p}_0\bar{e}(1+\bar{e})} \leq -\frac{1}{\mathrm{p}\bar{e}(1+\bar{e})} \leq -\frac{1}{\mathrm{p}_\infty\bar{e}(1+\bar{e})} < \infty$, $\forall \bar{e} \in \mathcal{D}$. Denote now $k_1 = -\frac{1}{\mathrm{p}\bar{e}(1+\bar{e})}$ which is always positive and bounded since $0 < \frac{4}{\mathrm{p}_0} \leq k_1 < \infty$, and we then can further upper bound $\dot{V}$ by

$$\begin{aligned}\dot{V} \leq{}& |\varepsilon|k_1\left(\|\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}\|\|-(L \otimes I_n)x\| + |\bar{e}\dot{\mathrm{p}}|\right)\\ &+ \varepsilon k_1\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}(B \otimes I_n)u.\end{aligned} \tag{11}$$

Since $\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}$ and $(L \otimes I_n)x$ are both bounded for all $t \in [0,\tau_{\max})$, we can additionally upper bound the term $\|\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}\|\|-(L \otimes I_n)x\| + |\bar{e}\dot{\mathrm{p}}|$ by a positive constant k_2, which results in

$$\dot{V} \leq |\varepsilon|k_1k_2 + \varepsilon k_1\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}(B \otimes I_n)u. \tag{12}$$

Next, we replace the control law (8) for the leaders in (12) and derive that

$$\dot{V} \leq |\varepsilon| k_1 k_2 - \varepsilon^2 k_1 \frac{\partial \rho^{\psi}(x,0)^T}{\partial x}(B \otimes I_n)\frac{\partial \rho^{\psi}(x,0)}{\partial x_L} \quad (13)$$

Due to the structure of the B matrix, we have that $\frac{\partial \rho^{\psi}(x,0)^T}{\partial x}(B \otimes I_n) = \frac{\partial \rho^{\psi}(x,0)^T}{\partial x_L}$. Based on Assumption 1 that $\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$ is a nonzero vector, we can obtain that $\|\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}\|^2 \geq k_3 > 0$. Therefore, $\dot{V}$ can be further upper bounded by

$$\dot{V} \leq |\varepsilon| k_1 k_2 - \varepsilon^2 k_1 k_3. \quad (14)$$

From (14), we know that $\dot{V} \leq 0$ as long as $|\varepsilon| \geq \frac{k_2}{k_3}$. Therefore, we can conclude that the transformed error is upper bounded by $|\varepsilon| \leq \varepsilon^{\star} = \max\left\{|\varepsilon(0)|, \frac{k_2}{k_3}\right\}$, $\forall t \in [0, \tau_{\max})$ [9]. Due to the boundedness of $|\varepsilon|$ in $t \in [0, \tau_{\max})$, we can restrict $\bar{e}$ in a compact subset of $\mathcal{D}$ as

$$\bar{e}(x,t) \in [\underline{\delta}, \bar{\delta}] \triangleq [T^{-1}(-\varepsilon^{\star}), T^{-1}(\varepsilon^{\star})] \subset \mathcal{D}, \quad (15)$$

where T^{-1} is the inverse function of T.

Step 3. Finally, we need to prove that $\tau_{\max}$ can be extended to ∞. According to (15), we know that $\bar{e}(x,t) \in \mathcal{D}', \forall t \in [0, \tau_{\max})$, where $\mathcal{D}' = [\underline{\delta}, \bar{\delta}]$. Hence, $\mathcal{D}' \subset \mathcal{D}$ is a nonempty and compact subset of $\mathcal{D}$ and it can be concluded that $\bar{e}(x,t) \in \mathcal{D}', \forall t \in [0, \tau_{\max})$. Let us now assume that $\tau_{\max} < \infty$. According to Proposition C.3.6 of [16], there then exists a $t' \in [0, \tau_{\max})$ such that $\bar{e}(x,t) \notin \mathcal{D}'$, which leads to a contradiction. Hence, we conclude that $\tau_{\max}$ is extended to ∞, that is $\bar{e}(x,t) \in \mathcal{D}' \subset \mathcal{D}, \forall t \geq 0$. Therefore ε is bounded for all $t \geq 0$ and the boundedness of the transformed error ε implies that $\rho^{\psi}(x,0)$ satisfies (6) for all $t \geq 0$. Therefore, we can conclude that the satisfaction of (6) is guaranteed when applying the control strategy (8). ☐

Remark 1. *In Assumption 1, we assume that $\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$ is a nonzero vector, which is used to avoid the local optima that may cause infeasibility issues. This assumption is not conservative: for example, in the popular connectivity maintenance [14] task (STL task $\phi := \mathrm{F}_{[0,4]}(||x_N - x_1|| < 1)$), $\frac{\partial \rho^{\psi}(x,0)}{\partial x_L}$ is a zero vector if and only if $x_N = x_1$ holds, which trivially satisfies the task specification. In addition, we design the funnel such that local optima can not be reached, which guarantees that Assumption 1 holds. Using the same example, we can choose $\rho^{\star} < 1$, and then the local optima is avoid. This assumption also requires that there exists at least one leader involved in the robustness function.*

We then derive conditions on the funnels such that the satisfaction of (6) will enforce that $0 < \rho^{\phi}(x,0) < \rho^{\star}$ holds. This is generally done by prescribing the transient behavior of the funnel. We first define the so called crossing time as

$$t_{\star} = \begin{cases} a & \text{if } \phi = \mathrm{G}_{[a,b]}\psi; \\ a' & \text{if } \phi = \mathrm{F}_{[a,b]}\psi, \end{cases} \quad (16)$$

where $a' \in [a,b]$. The crossing time $t_{\star}$ characterises when the funnel which is described by the function $-\mathrm{p}(t) + \rho^{\star}$ will traverse across zero. If we consider the "always" operator $\phi = \mathrm{G}_{[a,b]}\psi$, we need that $\rho^{\phi}(x(t),0) > 0$ for all $t \in [a,b]$, that is why we set the crossing time $t_{\star} = a$. As long as $-\mathrm{p}(t_{\star}) + \rho^{\star} = 0$, we know that $-\mathrm{p}(t) + \rho^{\star} > 0, \forall t \in (a,b]$ since the function $-\mathrm{p}(t) + \rho^{\star}$ is strictly increasing. Similarly, if we consider the "eventually" operator $\phi = \mathrm{F}_{[a,b]}\psi$, we only require that there exists $t \in [a,b]$ such that $\rho^{\phi}(x,0) > 0$, that is why we set the crossing time $t_{\star} = a' \in [a,b]$ for $\phi = \mathrm{F}_{[a,b]}\psi$. Now, recall the performance function $\mathrm{p}(t) := (\mathrm{p}_0 - \mathrm{p}_{\infty})e^{-lt} + \mathrm{p}_{\infty}$. The following theorem shows how to choose the parameters $\mathrm{p}_0, \mathrm{p}_{\infty}$ and l such that the satisfaction of (6) ensures $(x,t) \models \phi$.

Theorem 2. *Consider the leader-follower multi-agent system* (3)*, given an STL formula ϕ as in* (5b) *with the corresponding ψ satisfying Assumption 1. If the initial condition $\rho^{\psi}(x(0),0)$ is within the funnel* (6)*, and it further holds that*

- *for $t_{\star} = 0$, $\mathrm{p}_0 \in (\rho^{\star} - \rho^{\psi}(x(0),0), \rho^{\star}]$; $\mathrm{p}_{\infty} \in (0, \min(\mathrm{p}_0, \rho^{\star}))$; $l > 0$; $\rho^{\star} > \rho^{\psi}(x(0),0)$.*
- *for $t_{\star} > 0$, $\mathrm{p}_0 \in (\rho^{\star} - \rho^{\psi}(x(0),0), \infty)$; $\mathrm{p}_{\infty} \in (0, \min(\mathrm{p}_0, \rho^{\star}))$; $l = -\frac{1}{t_{\star}}\ln(\frac{\rho^{\star} - \mathrm{p}_{\infty}}{\mathrm{p}_0 - \mathrm{p}_{\infty}})$; $\rho^{\star} > \rho^{\psi}(x(0),0)$.*

Then, the control strategy (8) *guarantees that $0 < \rho^{\phi}(x,0) < \rho^{\star}$ holds.*

Proof. According to Theorem 1, since the initial condition $\rho^{\psi}(x(0),0)$ is within the funnel (6), the control strategy (8) guarantees the satisfaction of (6) for all $t \geq 0$. What remains to prove is that by appropriately choosing the parameters $\mathrm{p}_0, \mathrm{p}_{\infty}$ and l as above, the satisfaction of (6) further ensures that $0 < \rho^{\phi}(x,0) < \rho^{\star}$ holds. In general, the choices of the parameters should guarantee the initial condition, i.e., $\rho^{\psi}(x(0),0)$ is within the funnel (6). Moreover, $-\mathrm{p}(t_{\star}) + \rho^{\star} \geq 0$ should hold in order to enforce the satisfaction of the STL formula (5b) by prescribing the transient behavior of the funnel. For $t_{\star} = 0$, $\mathrm{p}_0 \in (\rho^{\star} - \rho^{\psi}(x(0),0), \rho^{\star}]$ will ensure that $\mathrm{p}_0 > \rho^{\star} - \rho^{\psi}(x(0),0)$, which is equivalent to $-\mathrm{p}_0 + \rho^{\star} < \rho^{\psi}(x(0),0)$, and by further choosing $\rho^{\star}$ such that $\rho^{\psi}(x(0),0) < \rho^{\star}$, the initial condition is satisfied. Moreover, $\mathrm{p}_0 \leq \rho^{\star}$ means that $-\mathrm{p}(t_{\star}) + \rho^{\star} = -\mathrm{p}_0 + \rho^{\star} \geq 0$. Next, we use the fact that the function $-\mathrm{p}(t_{\star}) + \rho^{\star}$ is strictly increasing in order to conclude on the satisfaction of $\phi = \mathrm{G}_{[a,b]}\psi$ or $\phi = \mathrm{F}_{[a,b]}\psi$. For $t_{\star} > 0$, we can check similarly that the initial condition holds, i.e., the initial condition $\rho^{\psi}(x(0),0)$ is within the funnel (6). In addition $l = -\frac{1}{t_{\star}}\ln(\frac{\rho^{\star} - \mathrm{p}_{\infty}}{\mathrm{p}_0 - \mathrm{p}_{\infty}})$ results in $-\mathrm{p}(t_{\star}) + \rho^{\star} = -(\mathrm{p}_0 - \mathrm{p}_{\infty})e^{-lt_{\star}} - \mathrm{p}_{\infty} + \rho^{\star} = 0$, and thus we can also obtain the satisfaction of $\phi = \mathrm{G}_{[a,b]}\psi$ or $\phi = \mathrm{F}_{[a,b]}\psi$ according to the fact that $-\mathrm{p}(t_{\star}) + \rho^{\star}$ is strictly increasing. Therefore, we can conclude that the control strategy (8) guarantees that $0 < \rho^{\phi}(x,0) < \rho^{\star}$ holds for ϕ as in (5b) by appropriately choosing the parameters $\mathrm{p}_0, \mathrm{p}_{\infty}$ and l as above. ☐

IV. CONTROL FOR SEQUENTIAL STL FORMULAS

In this section, we design a hybrid control strategy for the leader-follower multi-agent system (3) such that the sequential STL formula as in (5c) is satisfied. The sequential STL formula $\phi' := \bigwedge_{i=1}^{M} \phi_i$ is composed by M STL formulas of the form (5b), for which the control strategy has been discussed in the previous section. The time intervals $[a_i, b_i]$ of the basic STL formulas satisfy $b_i \le a_{i+1}, \forall i \in \{1, \cdots, M-1\}$. For each basic STL formula $\phi_i, i \in \{1, \cdots, M\}$, i.e., $\phi_i = \mathrm{G}_{[a_i,b_i]}\psi_i$ or $\phi_i = \mathrm{F}_{[a_i,b_i]}\psi_i$ we denote its robustness function as $\rho^{\phi_i}(x, 0)$, and the corresponding robustness function $\rho^{\psi_i}(x, 0)$ with respect to ψ_i. For each $\rho^{\psi_i}(x, 0)$, the corresponding funnel is designed as

$$-\mathrm{p}_i(t) + \rho_i^\star < \rho^{\psi_i}(x, 0) < \rho_i^\star, \tag{17}$$

with $\rho_i^\star$ being a positive scalar and $\mathrm{p}_i(t) := (\mathrm{p}_{0,i} - \mathrm{p}_{\infty,i})e^{-l_i(t-\tau_i)} + \mathrm{p}_{\infty,i}$, where τ_i are the switching moments that will be defined afterwards. Next, We define $e_i(x) = \rho^{\psi_i}(x, 0) - \rho_i^\star$, the modulated error $\bar{e}_i(x, t) = \frac{e_i(x)}{\mathrm{p}_i(t)}$, and the transformed error $\varepsilon_i(x, t) = T_i(\bar{e}_i(x, t)) = \ln\left(-\frac{1+\bar{e}_i(x,t)}{\bar{e}_i(x,t)}\right)$. All these parameters are defined in a similar manner as in previous section. For each ϕ_i, we also define the crossing time as

$$t_{\star,i} = \begin{cases} a_i & \text{if } \phi_i = \mathrm{G}_{[a_i,b_i]}\psi_i; \\ a_i' & \text{if } \phi_i = \mathrm{F}_{[a_i,b_i]}\psi_i, \end{cases} \tag{18}$$

with $a_i' \in [a_i, b_i]$. There is only one basic STL formula ϕ_i active for every moment and the switch from ϕ_i to $\phi_{i+1}, i \in \{1, \cdots, M-1\}$ occurs once ϕ_i is satisfied. We then define the switching moments as $\tau_1, \cdots, \tau_M$ with $\tau_1 = 0, \tau_i \le \tau_{i+1}$: τ_i represents the moment that the basic STL formula ϕ_i becomes active, and for $i \in \{2, \cdots, M\}$, we have that

$$\tau_i = \begin{cases} b_{i-1} & \text{if } \phi_{i-1} = \mathrm{G}_{[a_{i-1},b_{i-1}]}\psi_{i-1}; \\ t_{\star,i-1} & \text{if } \phi_{i-1} = \mathrm{F}_{[a_{i-1},b_{i-1}]}\psi_{i-1}. \end{cases} \tag{19}$$

Now, based on the Theorems 1 and 2, we derive the following result that proposes a hybrid control strategy for the satisfaction of the sequential STL formulas as in (5c) and also completes the solution for Problem 1.

Theorem 3. *Consider the leader-follower multi-agent system* (3)*, given a sequential STL formula ϕ' as in* (5c) *such that Assumption 1 holds for each ψ_i. Assume that the following conditions hold for all $i \in \{1, \cdots, M\}$:*

- *for $t_{\star,i} - \tau_i = 0$, $\mathrm{p}_{0,i} \in (\rho_i^\star - \rho^{\psi_i}(x(\tau_i), 0), \rho_i^\star]$; $\mathrm{p}_{\infty,i} \in (0, \min(\mathrm{p}_{0,i}, \rho_i^\star))$; $l_i > 0$; $\rho_i^\star > \rho^{\psi_i}(x(\tau_i), 0)$.*
- *for $t_{\star,i} - \tau_i > 0$, $\mathrm{p}_{0,i} \in (\rho_i^\star - \rho^{\psi_i}(x(\tau_i), 0), \infty)$; $\mathrm{p}_{\infty,i} \in (0, \min(\mathrm{p}_{0,i}, \rho_i^\star))$; $l_i = -\frac{1}{t_{\star,i}-\tau_i}\ln(\frac{\rho_i^\star - \mathrm{p}_{\infty,i}}{\mathrm{p}_{0,i}-\mathrm{p}_{\infty,i}})$; $\rho_i^\star > \rho^{\psi_i}(x(\tau_i), 0)$.*

Let $i \in \{1, \cdots, M-1\}$, then the hybrid control strategy

$$u(x, t) = \begin{cases} -\varepsilon_i(x, t)\frac{\partial \rho^{\psi_i}(x,0)}{\partial x_L}, & t \in [\tau_i, \tau_{i+1}); \\ -\varepsilon_M(x, t)\frac{\partial \rho^{\psi_M}(x,0)}{\partial x_L}, & t \in [\tau_M, b_M]. \end{cases} \tag{20}$$

guarantees that $0 < \rho^{\phi_i}(x, 0) < \rho_i^\star$ holds for all $t \in [\tau_i, \tau_{i+1}], i \in \{1, \cdots, M-1\}$ and for all $t \in [\tau_M, b_M]$, thus $(x, t) \models \phi'$.

Proof. The proof is constructed iteratively based on the proofs of Theorems 1 and 2. Starting from $t = 0$ when the basic STL formula ϕ_1 is active and $\tau_1 = 0$. Based on the above conditions, we know that the initial condition $\rho^{\psi_1}(x(0), 0)$ is within the funnel $-\mathrm{p}_1(t) + \rho_1^\star < \rho^{\psi_1}(x, 0) < \rho_1^\star$. Then, according to Theorems 1 and 2, we can conclude that $0 < \rho^{\phi_1}(x(t), 0) < \rho_1^\star$ is satisfied for all $t \in [\tau_1, \tau_2]$. When $t = \tau_2$, the basic STL formula ϕ_2 is active and the the control strategy is also switched according to (20). By choosing appropriately the parameters $\mathrm{p}_{0,2}, \mathrm{p}_{\infty,2}, \rho_2^\star$ and l_2 as Theorem 3, the initial condition $\rho^{\psi_2}(x(\tau_2), 0)$ for this iteration is also within the funnel $-\mathrm{p}_2(t) + \rho_2^\star < \rho^{\psi_2}(x, 0) < \rho_2^\star$. Similarly, we can also conclude that $0 < \rho^{\phi_2}(x(t), 0) < \rho_2^\star$ is satisfied for all $t \in [\tau_2, \tau_3]$. Next, for each active basic STL formula $\phi_i, i \in \{3, \cdots, M\}$, we can use the similar arguments that the initial condition $\rho^{\psi_i}(x(\tau_i), 0)$ of each iteration is within the funnel $-\mathrm{p}_i(t) + \rho_i^\star < \rho^{\psi_i}(x, 0) < \rho_i^\star$ by appropriately designing the parameters $\mathrm{p}_{0,i}, \mathrm{p}_{\infty,i}, \rho_i^\star$ and l_i as Theorem 3. Then, according to Theorems 1 and 2, we can conclude that $0 < \rho^{\phi_i}(x(t), 0) < \rho_i^\star$ holds for all $t \in [\tau_i, \tau_{i+1}], i \in \{1, \cdots, M-1\}$ and for all $t \in [\tau_M, b_M]$, which means that the STL formula ϕ' is satisfied by applying the hybrid control strategy (20), i.e., $(x, t) \models \phi'$. ☐

V. Simulations

In this section, we show a simulation of the derived results. We consider 3 followers and 2 leaders in the plane as shown in Fig. 1, where the leaders and followers are represented by grey and white nodes, respectively. The edges are represented by the black links that indicate the neighboring relations. We denote the position of agent i as $x_i = [x_{i1}, x_{i2}]^T, i = \{1, 2, 3, 4, 5\}$, which is initialised as $x_1 = [0, 0]^T, x_2 = [6, 0]^T, x_3 = [12, 0]^T, x_4 = [3, 3]^T, x_5 = [9, 3]^T$.

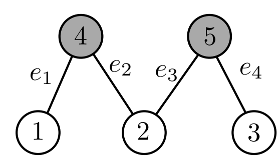

Fig. 1: Leader-follower graph topology.

The leader-follower multi-agent system is assigned a sequential STL task $\phi' = \phi_1 \wedge \phi_2$ with two basic STL sub-tasks given by $\phi_1 = \mathrm{F}_{[0,2]}(\psi_1 \wedge \psi_2)$ and $\phi_2 = \mathrm{G}_{[3,5]}(\psi_3 \wedge \psi_4)$, where $\psi_1 = (||x_4 - x_1||_2 < 2)$, $\psi_2 = (||x_4 - x_2||_2 < 2)$, $\psi_3 = (||x_5 - x_3||_2 < 2)$ and $\psi_4 = (||x_5 - x_2||_2 < 2)$. This sequential task means that 1) within 2 seconds, the leader indexed by 4 keeps a distance of 2 to the first follower, while in the mean-time it keeps a distance of 2 to the second follower. 2) Within 3 to 5 seconds, the leader indexed by 5 always keeps a distance of 2 to the second follower and simultaneously keeps a distance of 2 to the third follower.

The simulation results when applying the control strategy (20) are shown in Fig. 2 and 3. In the first task, the performance function regarding the first funnel is chosen as $\mathrm{p}_1(t) = 19e^{-l_1 t} + 1$ with $t_{\star,1} = 1, \rho_1^\star = 2, \mathrm{p}_{0,1} = 20, \mathrm{p}_{\infty,1} = 1$ and $l_1 = \ln(19)$. The evolution of the agents is shown as in Fig. 2, where the black lines show the initial formation, while the blue lines show the final positions of the agents. The dashed lines indicate the trajectories of the agents. Fig. 3 depicts the evolution of the robustness function (red curve) against the corresponding funnel (black curve). We can see that ϕ_1 is active until $t_{\star,1} = 1$ according to (19). The performance function $-\mathrm{p}_1(t) + \rho_1^\star$ enforces the satisfaction of ϕ_1 as it crosses zero (dashed line) at exactly $t_{\star,1} = 1$. We can conclude that the first task is fulfilled due to the fact that the robustness function evolves within the funnel. At $\tau_2 = t_{\star,1} = 1$, the second task ϕ_2 becomes active. The corresponding performance function is chosen as $\mathrm{p}_2(t) = 9e^{-l_2(t-1)} + 1$ with $t_{\star,2} = 3, \rho_2^\star = 2, \mathrm{p}_{0,2} = 10, \mathrm{p}_{\infty,2} = 1$ and $l_2 = 0.5\ln(9)$. Similarly, the evolution of the agents is shown as in Fig. 2, where the blue lines show the initial formation of the second task, which is the same as the final formation of the first task at $t = 1$, while the red lines show the final positions of the agents at $t = 5$. In Fig. 3, we can also check the evolution of the robustness function against the second funnel. The jumps at $t = 1$ for both the robustness function and the funnel indicate that ϕ_2 becomes active. The performance function $-\mathrm{p}_2(t) + \rho_2^\star$ crosses zero at $t_{\star,2} = 3$, which in turn enforces the satisfaction of ϕ_2. Since the evolution of the robustness function is always within the funnel for $t \in [1, 5]$, we can also conclude that ϕ_2 is fulfilled. Therefore, we can conclude that the sequential STL task ϕ' is satisfied by applying the control strategy (20).

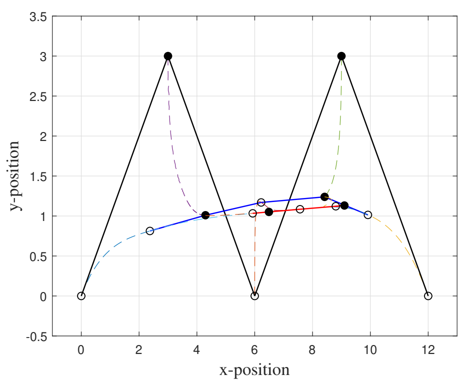

Fig. 2: Agents evolution for ϕ_1 and ϕ_2.

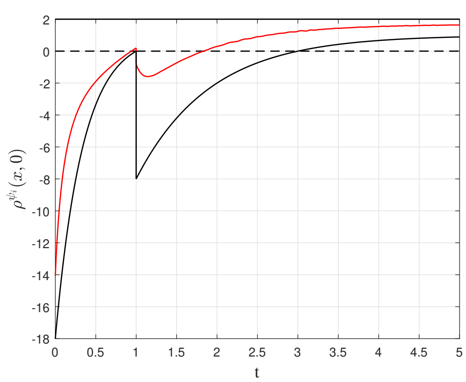

Fig. 3: Evolution of the robustness functions (red curves) and the funnels (black curves).

VI. Conclusions

In this paper, control of leader-follower multi-agent systems under certain fragments of signal temporal logic specifications has been investigated. In order to enforce the satisfaction the basic STL formulas, a funnel-based control strategy has been proposed through appropriately design the funnel parameters to prescribe certain transient behavior on the funnels that constrain the closed-loop trajectories. Then, a hybrid control strategy has been leveraged to satisfy the sequential STL formulas.

Future research directions include control of leader-follower multi-agent systems under a set of local STL formulas that may also present couplings and conflicts. Moreover, a more general class of STL formulas will be further studied.

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