Symbolic Abstractions for Periodic Event-triggered Linear Control Systems

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Abstract— **This paper studies the construction of symbolic abstractions for periodic event-triggered systems. To construct symbolic abstractions, the original event-triggered mechanism is over- and under-approximated, and thus the abstract event-triggered mechanisms are different from the original one, which leads to the asynchronous phenomenon between the original system and the constructed symbolic abstractions. To deal with this issue, an interface is proposed to guarantee the synchronization between the original and abstract event-triggering mechanisms and the equivalence relations between the original system and the constructed symbolic model. Furthermore, we study the controller refinement based on these two constructed symbolic models. Finally, the obtained results are illustrated via a numerical example.**

I. Introduction

The use of discrete abstractions [1], [2] has gradually become a standard approach for the design of hybrid systems. Because of discrete abstractions of continuous dynamics, controller synthesis problems can be studied efficiently via the techniques developed in the fields of supervisory control or algorithmic game theory. With an inclusion or equivalence relationship between the original system and the discrete abstraction, the synthesized controller is guaranteed to be correct by design, and the formal verification is either not needed or can be reduced [3].

In the field of symbolic abstractions, there are two directions for further study. Since not all the dynamical systems possess symbolic abstractions, the first direction is to identify more classes of dynamical systems admitting symbolic models. In this direction, different types of dynamical systems have been studied, such as switched control systems [4], [5], time-delay control systems [1], event-triggered linear control systems [6], and stochastic systems [7]. The commonly-used approach is based on (bi)simulation relation and its variants, which lead to equivalences of dynamic systems in an exact or approximate setting; see e.g., [5], [8], [9]. Since the abstraction construction involves a huge computational complexity, the second direction is to reduce the computational complexity in the construction of symbolic abstractions. Along this direction, many researchers have proposed different construction and refinement approaches, such as the feedback refinement relation [10] and abstraction refinements [11]–[13]. For instance, a coarse abstraction is proposed initially and then refined iteratively to ensure the satisfaction of the desired specifications [11], [13].

In this paper, we follow the first direction and consider the symbolic abstractions of periodic event-triggered control (PETC) systems, which are a special class of event-triggered systems [6], [14]–[16]. In our work, both the state and input sets are approximated first, and thus the event-triggered condition is approximated to the abstract version, which is different from the original version. Then, an interface is proposed to synchronize the original and abstract periodic event-triggered mechanisms (PETMs). Using different approximation techniques, two symbolic models are constructed, and both approximate input-output simulation relation and feedback refinement relation are established. Due to the differences between the original and abstract PETMs, we further study the controller synthesis problem based on the constructed symbolic models. Finally, a numerical example is given to illustrate the obtained results.

The main contributions of this paper are two-fold. First, different from [6], [17] where convex polyhedral cones and quotient systems are applied to construct symbolic models for linear event-triggered systems, a novel symbolic model based on uniform quantization and transition systems is proposed for PETC systems. In particular, the original PETM is approximated via both over- or under-approximation techniques, which leads to two abstract PETMs and thus two symbolic models. Since the original and abstract PETMs are different, a synchronous interface is proposed here for the first time to ensure the equivalence relations between the PETC system and its symbolic models. Second, due to the difference between the original and abstract PETMs, it is necessary to study the controller synthesis problem. We show that the controller for the original system can be obtained by refining the abstract controllers for the constructed symbolic models. Because the over- and under-approximations of the original PETM are considered, we can compare the performances achieved by different PETMs.

The remainder of this paper is as follows. In Section II, Notation and preliminary definitions are introduced. Symbolic abstractions are constructed in Section III. The controller synthesis problem is studied in Section IV. A numerical example is given in Section V. Finally, conclusion and future works are presented in Section VI.

II. Periodic Event-Triggered Control Systems

In this section, we start by introducing the notations, and then present the class of PETC systems to be studied, and finally recall the notion of approximate equivalence.

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A. Notations

Denote $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}_0^+ := [0, +\infty)$; $\mathbb{R}^+ := (0, +\infty)$; $\mathbb{N} := \{0, 1, \dots\}$; $\mathbb{N}^+ := \{1, 2, \dots\}$. Given $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a \leq b$, we denote by $[a, b]$ a closed interval. Given a vector $x \in \mathbb{R}^n$, denote by $\|x\|$ the infinite norm of x. The closed ball centered at $x \in \mathbb{R}^n$ with the radius $\varepsilon \in \mathbb{R}^+$ is defined by $\mathbf{B}(x, \varepsilon) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\}$. Id denotes the identity function. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum (sup norm) of f is denoted by $\|f\|_\infty$; $\|f\|_\infty := \operatorname{ess\,sup}\{\|f(t)\| : t \in \mathbb{R}_0^+\}$; $f(t^+) = \lim_{s \searrow t} f(s)$; $f^+ = f(t^+)$ when the time argument is omitted. A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if $\beta(s, t)$ is of class $\mathcal{K}$ for each fixed $t \in \mathbb{R}_0^+$ and decreases to zero as $t \to \infty$ for each fixed $s \in \mathbb{R}_0^+$. Given two sets $A, B \subset \mathbb{R}^n$, a relation $\mathcal{R} \subset A \times B$ is the map $\mathcal{R} : A \to 2^B$ defined by $b \in \mathcal{R}(a)$ if and only if $(a, b) \in \mathcal{R}$. $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e., $\mathcal{R}^{-1} := \{(b, a) \in B \times A : (a, b) \in \mathcal{R}\}$.

B. Periodic Event-Triggered Control Systems

Consider the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \forall t \in \mathbb{R}^+, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are nonzero and constant matrices, $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^m$ is the control input. For the system (1), the controller is given by

$$u(t) = K\bar{x}(t_k), \quad t \in (t_k, t_{k+1}], \tag{2}$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain, $\bar{x} \in \mathbb{R}^n$ is the most recently transmitted state measurement to the controller at the transmission time $t_k > 0$. The initial value of $\bar{x}$ can be given *a priori*. A periodic sampling sequence is given by

$$\mathcal{T}_s := \{t_k : t_k := k\tau, k \in \mathbb{N}\}, \tag{3}$$

where $\tau > 0$ is the sampling interval. A zero-order hold mechanism is applied in the sampling intervals. At each sampling time $t_k \in \mathcal{T}_s$, the state measurement $\bar{x} \in \mathbb{R}^n$ applied to (2) is updated as follows: for all $t \in (t_k, t_{k+1}]$,

$$\bar{x}(t) = \begin{cases} x(t_k), & \text{if } \|x(t_k) - \bar{x}(t_k)\| \geq \sigma\|x(t_k)\|, \\ \bar{x}(t_k), & \text{if } \|x(t_k) - \bar{x}(t_k)\| < \sigma\|x(t_k)\|, \end{cases} \tag{4}$$

where $\sigma > 0$ is a given constant parameter. Therefore, $t_{k+1} - t_k \geq \tau > 0$, which implies that Zeno phenomena are excluded. Following (3) and (4), we obtain the periodic event-triggered mechanism (PETM):

$$t_{k+1} := \min\{t \in \mathcal{T}_s : t > t_k, \|x(t) - \bar{x}(t)\| \geq \sigma\|x(t)\|\}, \tag{5}$$

which implies that the event-triggered condition $\mathbb{C}(x, \bar{x}) := \|x(t) - \bar{x}(t)\| - \sigma\|x(t)\| \geq 0$ is verified periodically. The PETM (5) has been applied in [14], [17].

Similarly to the general control system in [8], a curve $\xi : (a, b) \to X \subseteq \mathbb{R}^n$ is said to be a *trajectory* of (1)-(2), if there exists $\mathbf{u} \in \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{R}^m$ is a subset of all piecewise continuous functions of time from $(a, b) \subset \mathbb{R}$ to U with $a < 0 < b$ and which depends on the PETM (5), such that $\dot{\xi}(t) = A\xi(t) + B\mathbf{u}(t_k)$ holds for all $t \in (a, b) \cap [t_k, t_{k+1})$. We further define the trajectory $\mathbf{x} : [0, \tau] \to X$ on a closed interval $[0, \tau]$ with $\tau \in \mathbb{R}^+$ such that $\mathbf{x} = \xi|_{[0,\tau]}$. Denote by $\mathbf{x}(t, x, \mathbf{u})$ the point reached at $t \in (a, b)$ under $\mathbf{u} \in \mathcal{U}$ and the PETM (5) from $x \in X$. Such a point is uniquely determined for linear systems; see [18]. A PETC system is said to be *forward complete*, if every trajectory is defined on an interval of the form $(a, +\infty)$.

Definition 1 ([19]): The system (1) is *incrementally input-to-state stable (δ-ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\|\mathbf{x}(t, x_1, \mathbf{u}_1) - \mathbf{x}(t, x_2, \mathbf{u}_2)\| \leq \beta(\|x_1 - x_2\|, t) + \gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty)$ holds for all $t \in \mathbb{R}_0^+$, $x_1, x_2 \in \mathbb{R}^n$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$.

Assumption 1: There exists a feedback law $u = Kx$ such that the system $\dot{x} = (A+BK)x + BKe$ is δ-ISS with respect to $e = x - \bar{x}$.

Similar assumption can be found in [20, Section IV-C]. In particular, for linear systems, δ-ISS equals to the classic ISS, and thus Assumption 1 holds if the system (1) is stabilizable.

C. Approximate Equivalence Notions

Definition 2 ([9]): A *transition system* is a sextuple $T = (X, X^0, U, \Delta, Y, H)$ consisting of: (i) a set of states X; (ii) a set of initial states $X^0 \subseteq X$; (iii) a set of inputs U; (iv) a transition relation $\Delta \subseteq X \times U \times X$; (v) a set of outputs Y; (vi) an output function $H : X \to Y$. T is said to be *metric* if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$, and *symbolic* if the sets X and U are finite or countable.

The transition $(x, u, x') \in \Delta$ is denoted by $x' \in \Delta(x, u)$, which means that the system can evolve from a state x to a state x' under the input u. An input $u \in U$ belongs to *the set of enabled inputs* at the state x, denoted by $\operatorname{enab}(x)$, if $\Delta(x, u) \neq \varnothing$. If $\operatorname{enab}(x) = \varnothing$, then x is said to be *blocking*, otherwise, it is said to be *non-blocking*. The transition system T is said to be *deterministic*, if for all $x \in X$ and all $u \in \operatorname{enab}(x)$, $\Delta(x, u)$ has exactly one element. In this case, we write $x' = \Delta(x, u)$ with a slight abuse of notation.

Definition 3 ([21]): Let $T_i = (X_i, X_i^0, U_i, \Delta_i, Y, H_i)$, $i = 1, 2$, be two transition systems with the same output set Y and the metric $\mathbf{d}$. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_0^+$, a relation $\mathcal{R} := \mathcal{R}_X \times \mathcal{R}_U \subseteq X_1 \times X_2 \times U_1 \times U_2$ is said to be a $(\varepsilon_1, \varepsilon_2)$*-approximate input-output simulation relation ($(\varepsilon_1, \varepsilon_2)$-aIOSR)* from T_1 to T_2, if for all $(x_1, x_2, u_1, u_2) \in \mathcal{R}$,

(i) $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon_1$ and $\mathbf{d}(u_1, u_2) \leq \varepsilon_2$;
(ii) for all $(x_1, x_2) \in \mathcal{R}_X$ and $u_1 \in U_1(x_1)$, there exists $u_2 \in U_2(x_2)$ such that: $(x_1, x_2, u_1, u_2) \in \mathcal{R}$; and for each $x_1' \in \Delta_1(x_1, u_1)$, there exists $x_2' \in \Delta_2(x_2, u_2)$ such that $(x_1', x_2') \in \mathcal{R}_X$.

If $H_1 = \text{Id}$ and $H_2 = \text{Id}$, then $\mathcal{R} \subseteq X_1 \times X_2 \times U_1 \times U_2$ is said to be a $(\varepsilon_1, \varepsilon_2)$*-approximate input-state simulation relation ($(\varepsilon_1, \varepsilon_2)$-aISSR)* from T_1 to T_2.

Definition 4 ([22]): Let T_1 and T_2 be two transition systems with $T_i = (X_i, X_i^0, U_i, \Delta_i, Y_i, H_i)$ for $i \in \{1, 2\}$, and assume that $U_2 \subseteq U_1$. A relation $\mathcal{F} \subseteq X_1 \times X_2$ is a *feedback refinement relation* from T_1 to T_2, if for all $(x_1, x_2) \in \mathcal{F}$,

(i) $U_2(x_2) \subseteq U_1(x_1)$; (ii) $u \in U_2(x_2) \Rightarrow \mathcal{F}(\Delta_1(x_1, u)) \subseteq \Delta_2(x_2, u)$, where $U_i(x) := \{u \in U_i : \text{enab}(x) \neq \varnothing\}$.

Our goal is to construct symbolic models for PETC systems such that the approximate input-output simulation relation and feedback refinement relation are satisfied.

III. Construction of Symbolic Models

In this section, we first introduce the time discretization of PETC systems, then approximate the state and input sets, and finally propose an interface to develop symbolic models.

A. Time Discretization of PETC Systems

To develop the symbolic abstraction, we work with the time-discretization of the PETC system Σ. The sampling period $\tau > 0$ is the same as the one in (3). The time discretization of the PETC system Σ is represented as the transition system $T_\tau(\Sigma) := (\mathfrak{X}_1, \mathfrak{X}_1^0, U_1, \Delta_1, Y_1, H_1)$, where,

- the state set is $\mathfrak{X}_1 := X_1 \times \mathbb{R}^m$, where $X_1 = \mathbb{R}^n$;
- the set of the initial states is $\mathfrak{X}_1^0 := \mathbb{R}^n \times \mathbb{R}^m$;
- the input set is $U_1 := \{u \in \mathcal{U} : \mathbf{x}(\tau, x, u)$ is defined for all $x \in X$ under $\mathbb{C}(x, \bar{x}) \geq 0\}$ with $\bar{x}$ defined in (4);
- the transition relation is given as follows: for any $(x, u) \in \mathfrak{X}_1$ and $u \in \text{enab}(x)$, $(x', u') = \Delta_1(x, u)$ if and only if $x' = \mathbf{x}(\tau, x, u)$ and
$$u' = \begin{cases} u, & \text{if } \mathbb{C}_1(x', \bar{x}) < 0, \\ Kx', & \text{if } \mathbb{C}_1(x', \bar{x}) \geq 0, \end{cases}$$
where $\mathbb{C}_1(x, \bar{x}) = \mathbb{C}(x, \bar{x}) = \|x(t) - \bar{x}(t)\| - \sigma\|x(t)\|$ is given in (5), and $\bar{x}$ is defined in (4);
- the output set is $Y_1 := X_1$;
- the output map is defined as $H_1((x, u)) = x$.

The transition system $T_\tau(\Sigma)$ is non-blocking and deterministic. $T_\tau(\Sigma)$ is metric if the output set Y_1 is equipped with the metric $\mathbf{d}(y, \bar{y}) = \|y - \bar{y}\|$ for all $y, \bar{y} \in Y_1$. In $T_\tau(\Sigma)$, the state is augmented to include the original state $x \in \mathbb{R}^n$ and the control input $u \in \mathbb{R}^m$. The event-triggered condition $\mathbb{C}_1(x, \bar{x}) = \mathbb{C}(x, \bar{x})$ is included in the transition relation. The transition system $T_\tau(\Sigma)$ is similar to the time-discretization of switched systems in [5], and the control input here plays a similar role as the switching mode in [5]. However, the switching mode in [5] does not depend on the system state $x \in \mathbb{R}^n$ and is chosen by the self-triggered controller, whereas the event-triggered controller (2) is applied here and thus depends on the system state.

B. Approximation of State and Input Sets

We first approximate the set $\mathbb{R}^n$ by the sequence of the embedded lattices $[\mathbb{R}^n]_\eta$, where $[\mathbb{R}^n]_\eta := \{q \in \mathbb{R}^n : q_i = k_i\eta, k_i \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$, where $\eta \in \mathbb{R}^+$ is the state space sampling parameter. We further associate a quantizer $Q_\eta : \mathbb{R}^n \to [\mathbb{R}^n]_\eta$ such that $Q_\eta(x) = q$ if and only if $x_i \in [q_i - \eta/2, q_i + \eta/2]$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$. Therefore, $\|x - Q_\eta(x)\| \leq \eta/2$ holds from geometrical considerations.

Given a $q \in [\mathbb{R}^n]_\eta$, define the reachable set $\mathfrak{R}(\tau, q) := \{x' \in \mathbb{R}^n : \mathbf{x}(\tau, q, u) = x', u \in U_1\}$, which is well defined due to the input set U_1 and the PETM (5). The reachable set $\mathfrak{R}(\tau, q)$ is approximated below. Given any $\mu \in \mathbb{R}^+$, consider the auxiliary set $\mathcal{Z}_\mu(\tau, q) := \{z \in [\mathbb{R}^n]_\mu : \exists p \in \mathfrak{R}(\tau, q)$ such that $\|z - p\| \leq \mu/2\}$. Here, the choice of μ is not related to η, and limited by the desired precision; see Subsection III-C. We further define the function $\psi : \mathcal{Z}_\mu(\tau, q) \to U_1$ such that for any $z \in \mathcal{Z}_\mu(\tau, q)$, there exists an input $u = \psi(z) \in U_1$ such that $\|z - \mathbf{x}(\tau, q, u)\| \leq \mu/2$. Hence, we define the set $U_2(q) := \psi(\mathcal{Z}_\mu(\tau, q))$, which captures the set of the inputs applied at the state $q \in [\mathcal{S}_i]_{\mu_i}$, and thus the approximation of U_1 is

$$U_2 := \bigcup_{q \in [\mathbb{R}^n]_\eta} U_2(q). \tag{6}$$

That is, the set U_1 is approximated in the following way: given any $q \in [\mathbb{R}^n]_\eta$, for any $u_1 \in U_1$, there exists $u_2 \in U_2(q)$ such that $\|\mathbf{x}(\tau, q, u_1) - \mathbf{x}(\tau, q, u_2)\| \leq \mu$.

With the approximation of the sets $\mathbb{R}^n$ and $\mathbb{R}^m$, we can approximate the state set $\mathfrak{X}_1$ and the input set U_1 in the transition system $T_\tau(\Sigma)$.

C. Symbolic Model

With the approximation of the state and input sets, we construct the symbolic abstraction of $T_\tau(\Sigma)$ as a transition system $T_{\tau,\eta,\mu}(\Sigma) = (\mathfrak{X}_2, \mathfrak{X}_2^0, U_2, \Delta_2, Y_2, H_2)$, where,

- the state set is $\mathfrak{X}_2 := X_2 \times U_2$ with $X_2 := [\mathbb{R}^n]_\eta$ and U_2 defined in (6);
- the set of the initial states is $\mathfrak{X}_2^0 := X_2 \times U_2$;
- the input set is U_2 defined in (6);
- the transition relation is given as follows: for any $(q, v) \in \mathfrak{X}_2$ and $v \in \text{enab}(q)$, $(q', v') = \Delta_2(q, v)$ if and only if $q' = Q_\eta(\mathbf{x}(\tau, q, v))$ and
$$v' = \begin{cases} v, & \text{if } \mathbb{C}_2(q', \bar{q}) < 0, \\ Q_\mu(Kq'), & \text{if } \mathbb{C}_2(q', \bar{q}) \geq 0; \end{cases}$$
where $\mathbb{C}_2 : X_2 \times X_2 \to \mathbb{R}$ is the abstract event-triggered condition and will be discussed later;
- the output set is $Y_2 := \mathbb{R}^n$;
- the output map is defined as $H_2((q, v)) = q$.

The system $T_{\tau,\eta,\mu}(\Sigma)$ is non-blocking and deterministic. $T_{\tau,\eta,\mu}(\Sigma)$ is metric if the output set Y_2 is equipped with the metric $\mathbf{d}(y, \bar{y}) = \|y - \bar{y}\|$ for all $y, \bar{y} \in Y_2$.

In the system $T_{\tau,\eta,\mu}(\Sigma)$, the abstract event-triggered condition $\mathbb{C}_2 : X_2 \times X_2 \to \mathbb{R}$ depends on the abstract state set X_2, and thus is different from the original one. The measurement $\bar{q} \in X_2$ is initialized as the quantized measurement of the initial value of $\bar{x}$, and is updated as follows: for all $t \in (t_k, t_{k+1}]$,

$$\bar{q}(t) = \begin{cases} q(t_k), & \text{if } \mathbb{C}_2(q(t_k), \bar{q}(t_k)) \geq 0, \\ \bar{q}(t_k), & \text{if } \mathbb{C}_2(q(t_k), \bar{q}(t_k)) < 0. \end{cases}$$

On the other hand, due to the difference between the original and abstract states, a possible scenario is that $\mathbb{C}_1(x, \bar{x}) \geq 0$ does not necessarily imply $\mathbb{C}_2(q, \bar{q}) \geq 0$, and *vice versa*. As a result, it is hard to construct an abstract PETM such that it possesses the synchronous triggering mechanism as

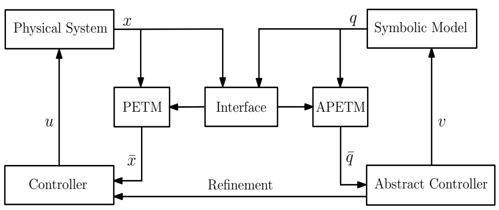

Fig. 1. The synchronization between the original PETM and the APETM.

the original PETM. In addition, different constructions of the abstract event-triggered conditions lead to different symbolic models, which further affects the relation between $T_\tau(\Sigma)$ and $T_{\tau,\eta,\mu}(\Sigma)$. Here, we present two abstract event-triggered conditions using the approximation techniques. Given the original event-triggered condition $\mathbb{C}_1(x,\bar{x}) \geq 0$ as in (5) and a desired precision $\varepsilon \in \mathbb{R}^+$, the abstract event-triggered condition can be constructed as

$$\mathbb{C}_2^a(q,\bar{q}) := \|q-\bar{q}\| - \sigma\|q\| + (2+\sigma)\varepsilon \geq 0; \tag{7}$$

$$\mathbb{C}_2^b(q,\bar{q}) := \|q-\bar{q}\| - \sigma\|q\| - (2+\sigma)\varepsilon \geq 0. \tag{8}$$

Hence, we derive the abstract PETMs (APETMs) and two symbolic abstractions, denoted by $T^a_{\tau,\eta,\mu}(\Sigma)$ and $T^b_{\tau,\eta,\mu}(\Sigma)$, respectively. The abstract event-triggered conditions (7) and (8) will be discussed after Theorem 1 in the next subsection.

D. Interface between PETM and APETM

After constructing the APETMs, we next need to deal with the asynchronization between the original PETM and the APETM. To this end, an interface is introduced between the original system and its symbolic model to coordinate the original PETM and the APETM; see Fig. 1. A similar technique has been applied for the controller design in [23], whereas the interface here just performs a logic operation and does not need to have an explicit form.

The implementation of the interface is presented below. If the original and abstract PETMs are asynchronous, then the interface leads to the synchronization between them based on the APETM. To be specific, if the APETM is triggered whereas the original PETM is not, then the interface leads to the triggering of the original PETM. If the original PETM is triggered whereas the APETM is not, then the interface does not lead to the triggering of the original PETM. The effects of this interface on the controller design will be studied in Section IV. Under the synchronous interface, the following theorem is established to ensure the equivalence relations.

Theorem 1: Consider the PETC system Σ and given any desired precision $(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^2$. Let Assumption 1 hold and the synchronous interface be implemented. Let the time and space sampling parameters be $\tau,\eta,\mu \in \mathbb{R}^+$, and $\sigma \in \mathbb{R}^+$ is a design parameter. If the following holds:

$$\|e^{A_1\tau}\|\varepsilon_1 + \left\|\int_0^\tau e^{A_1 s} B ds\right\|\varepsilon_2 + \eta/2 < \varepsilon_1, \tag{9}$$

$$\|K\|\varepsilon_1 + \mu/2 < \varepsilon_2, \tag{10}$$

where $A_1 := A+BK$, then there exist a feedback refinement relation $\mathcal{R}_a$ from $T_\tau(\Sigma)$ to $T^a_{\tau,\eta,\mu}(\Sigma)$, and a $(\varepsilon_1,\varepsilon_2)$-aIOSR $\mathcal{R}_b$ from $T^b_{\tau,\eta,\mu}(\Sigma)$ to $T_\tau(\Sigma)$.

The proof of Theorem 1 is omitted here due to the space limitation. Under the desired precision, $\mathbb{C}_1(x,\bar{x}) \geq 0$ implies $\mathbb{C}_2^a(q,\bar{q}) \geq 0$ from (7), which shows that $\mathbb{C}_2^a(q,\bar{q}) \geq 0$ is an under-approximation of $\mathbb{C}_1(x,\bar{x}) \geq 0$. In addition, $\mathbb{C}_2^b(q,\bar{q}) \geq 0$ implies $\mathbb{C}_1(x,\bar{x}) \geq 0$ from (8), and thus $\mathbb{C}_2^b(q,\bar{q}) \geq 0$ is an over-approximation of $\mathbb{C}_1(x,\bar{x}) \geq 0$. On the other hand, from the abstract event-triggered conditions (7)-(8), the synchronous interface may impose additional events for the original system due to the under-approximation (7), or forbid some events for the original system due to the over-approximation (8); see Section V. That is, the synchronous interface imposes the event-trigging behaviour of the symbolic model on the original system.

Remark 1: Comparing with [6], where a power quotient system was constructed as the symbolic model of ETC systems, we propose a novel construction approach for symbolic models for PETC systems. In terms of system models, we focus on PETC systems here instead of linear ETC systems as in [6]. In terms of the abstraction construction, the symbolic model in [6] is of the power quotient system form, and thus the lower and upper bounds are determined for the event-triggering time intervals. However, the symbolic model here is of the transition system form, and the lower and upper bounds are established for the event-triggered condition. □

IV. Controller Design

Since the event-triggered condition is approximated and the existing results in [6], [17] on the controller design cannot be applied here directly, we need to reconsider the controller design problem, which is the topic of this section.

We first recall some formal language concepts. Given a set X, X^* denotes the set of all finite strings by concatenating elements in X. An element s of X^* is defined as $s = s_1 \ldots s_n$ with $s_i \in X$ and $i \in \{1,\ldots,n\}$. The empty string $\epsilon \in X^*$ is a string satisfying $s\epsilon = \epsilon s = s$ for any $s \in S$. Given a string $s \in X^*$, $s(i)$ denotes the i-th element of s. The length of a string s is denoted by $|s|$, and a subset of X^* is called a language.

Definition 5 ([24]): A *run* of a transition system $T = (X, X^0, U, \Delta, Y, H)$ is a string $r \in X^*$, if there exists $u \in U$ satisfying $r(i+1) = \Delta(r(i),u)$ for all $i \in \{1,\ldots,|r|-1\}$. The *language* of the system T, denoted by $\mathfrak{L}(T)$, is the set of all the output runs of the system T.

Definition 6 ([24]): Given two transition systems $T_i = (X_i, X_i^0, U_i, \Delta_i, Y, H_i)$, $i = 1,2$, with the same output set Y and the metric $\mathbf{d}$. The *ε-approximate parallel composition of T_1 and T_2*, denoted by $T_1\|_\varepsilon T_2$, is a transition system $T = (X, X^0, U, \Delta, Y, H)$, with

- the state set $X = \{(x_1,x_2) \in X_1 \times X_2 : \mathbf{d}(y_1,y_2) \leq \varepsilon\}$;
- the set of initial states $X^0 = (X_1^0 \times X_2^0) \cap X$;
- the input set $U = U_1 \times U_2$;
- the transition relation given by: for any $x = (x_1,x_2) \in X$ and $u = (u_1,u_2) \in \text{enab}(x)$, $(x_1',x_2') = \Delta(x,u)$ if

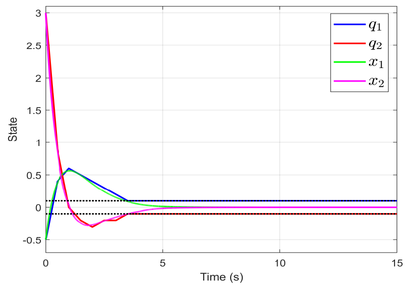

Fig. 2. State responses of the symbolic model $T^a_{0.5,0.1,0.2}(\Sigma)$ under the abstract event-triggered condition (13) and the system $T_{0.5}(\Sigma)$ under the designed event-triggered condition (15).

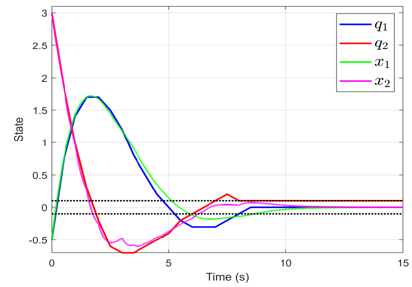

Fig. 3. State responses of the symbolic model $T^b_{0.5,0.1,0.2}(\Sigma)$ under the abstract event-triggered condition (14) and the system $T_{0.5}(\Sigma)$ under the designed event-triggered condition (16).

and only if $x'_1 = \Delta_1(x_1, u_1)$ and $x'_2 = \Delta_2(x_2, u_2)$;

- the output set $Y = X_1$;
- the output function $H : X \to X_1$.

The following proposition shows the controller design based on the symbolic model $T^b_{\tau,\eta,\mu}(\Sigma)$, and is an extension of Theorem 5.1 in [24] to the case of PETC systems.

Proposition 1: Consider the PETC system Σ, and assume there exists a $(\varepsilon_1, \varepsilon_2)$-aIOSR from $T^b_{\tau,\eta,\mu}(\Sigma)$ to $T_\tau(\Sigma)$. If there exists a controller C_b such that $\mathfrak{L}(T^b_{\tau,\eta,\mu}(\Sigma)\|_0 C_b) \subset \mathcal{S}$ for some specification $\mathcal{S}$, then the controller designed as $C := T^b_{\tau,\eta,\mu}(\Sigma)\|_0 C_b$ is such that $\mathfrak{L}(T_\tau(\Sigma)\|_{\varepsilon_1} C) \subset \mathbf{B}_{\varepsilon_1}(\mathcal{S})$.

If the abstract controller for $T^b_{\tau,\eta,\mu}(\Sigma)$ does not exist, then we can refine the abstract controller for $T^a_{\tau,\eta,\mu}(\Sigma)$ based on the feedback refinement relation in [22].

Proposition 2: Consider the PETC system Σ, and assume there exists a feedback refinement relation from $T_\tau(\Sigma)$ to $T^b_{\tau,\eta,\mu}(\Sigma)$. If there exists a controller C_a for $T^a_{\tau,\eta,\mu}(\Sigma)$ such that the specification $\mathcal{S}$ is satisfied, then there exists a controller $C(x) := C_a(\mathcal{F}(x))$ for $T_\tau(\Sigma)$ such that the specification $\mathbf{B}_{\varepsilon_1}(\mathcal{S})$ is satisfied.

From Propositions 1 and 2, the abstract controller can be refined as the controller for the original system. However, two controller refinement techniques have different effects on the satisfaction of the specification. According to the APETM based on the over-approximation in (8), the number of the event-triggering times is reduced to guarantee the satisfaction of the specification. For the APETM based on the under-approximation in (7), the number of the event-triggering times is increased to guarantee the satisfaction of the specification. Therefore, the existence of the controller can be deduced from the symbolic models. If the abstract controllers C_a and C_b do not exist, then the controller for the original system does not exist. If C_b exists, then a coarser PETM can be designed such that the specification is satisfied. If C_a exists, then a finer PETM guarantees the satisfaction of the specification whereas the coarse PETM cannot.

V. Numerical Example

In this section, we borrow the example in [14] to show the obtained results. Consider the following plant

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} u, \tag{11}$$

and the state-feedback controller of the form (2) with $K = \begin{bmatrix} 1 & 2 \end{bmatrix}$. The event-triggering condition is given by

$$\mathbb{C}(x, \bar{x}) := \|x - \bar{x}\| - \sigma\|x\| \geq 0, \tag{12}$$

which is verified periodically. Assume that the period is τ. In addition, the state set is $[-1, 2] \times [-1, 3.5]$, and thus the input set is $[-3, 9]$. For the PETC system (11)-(12), we first construct the symbolic models, and then study the controller design for the consensus problem.

Given $(\varepsilon_1, \varepsilon_2) = (0.4, 0.2)$, let $\tau = 0.5$, $\eta = 0.1$, and $\mu = 0.2$. The resulting transition system $T_{0.5,0.1,0.2}(\Sigma) = (\mathfrak{X}_2, \mathfrak{X}^0_2, U_2, \Delta_2, Y_2, H_2)$ is given by:

- $\mathfrak{X}_2 = [\mathbb{R}^2]_{0.1} \times \bigcup_{q \in [\mathbb{R}^2]_{0.1}} U_2(q)$ and $\mathfrak{X}^0_2 = \mathfrak{X}_2$;
- $U_2 = \bigcup_{q \in [\mathbb{R}^2]_{0.1}} U_2(q)$;
- Δ_2 is obtained below: for any $(q, v) \in \mathfrak{X}_2$ and $v \in \text{enab}(q)$, $(q', v') = \Delta_2(q, v)$ if and only if $q' = Q_\eta(\mathbf{x}(\tau, q, v))$ and
$$v' = \begin{cases} v, & \mathbb{C}_2(q', \bar{q}) < 0, \\ Q_\mu(Kq') & \mathbb{C}_2(q', \bar{q}) \geq 0; \end{cases}$$
- $Y_2 = \mathbb{R}^2$ and $H_2 : \mathfrak{X}_2 \to \mathbb{R}^2$.

In the above symbolic model, there are 1426 abstract states, 61 abstract inputs, and 2397666 transitions (using MATLAB on a laptop with 16 GB of RAM and a 1.9 GHz Intel Core i7 processor). The computation time is 128.477 seconds.

We consider the symbolic model $T^b_{0.5,0.1,0.2}(\Sigma)$, where

$$\mathbb{C}^b_2(q, \bar{q}) := \|q - \bar{q}\| - \sigma\|q\| - (2 + \sigma)\varepsilon \geq 0, \tag{13}$$

and the symbolic model $T^a_{0.5,0.1,0.2}(\Sigma)$ with

$$\mathbb{C}^a_2(q, \bar{q}) := \|q - \bar{q}\| - \sigma\|q\| + (2 + \sigma)\varepsilon \geq 0, \tag{14}$$

By Theorem 1, we can establish the (0.4, 0.2)-aIOSR from $T^b_{0.5,0.1,0.2}(\Sigma)$ to $T_{0.5}(\Sigma)$, and the feedback refinement relation from $T_{0.5}(\Sigma)$ to $T^a_{0.5,0.1,0.2}(\Sigma)$.

Next, assume that the specification is $\mathcal{S} = \{x \in \mathbb{R}^2 : \|x\| \leq 0.1\}$ (see the dashed black line), and then $\mathbf{B}_{\varepsilon_1}(\mathcal{S}) = \{x \in \mathbb{R}^2 : \|x\| \leq 0.5\}$. From (13), we design the following event-triggered condition

$$\begin{aligned} \mathbb{C}^b_1(x, \bar{x}) := &\|\mathcal{R}^{-b}_X(x) - \mathcal{R}^{-b}_X(\bar{x})\| - \sigma\|\mathcal{R}^{-b}_X(x)\| \\ &- 0.05(2 + \sigma)\varepsilon \geq 0, \end{aligned} \tag{15}$$

TABLE I
COMPARISON OF THE NUMBERS OF EVENT-TRIGGERING TIMES FOR DIFFERENT PETMS

PETMs	(12)	(13)	(15)	(14)	(16)
The event-triggering times	26	2	7	40	37

where the constant σ is set as 0.2. Under the event-triggered conditions (13) and (15), the state response is given in Fig. 3. For (14), the event-triggered condition is designed as

$$\mathbb{C}_1^a(x,\bar{x}) := \|\mathcal{R}_X^a(x) - \mathcal{R}_X^a(\bar{x})\| - 0.26\|\mathcal{R}_X^a(x)\| \geq 0. \quad (16)$$

Under the event-triggered conditions (14) and (16), the response of the system state is given in Fig. 2.

According to different PETMs, the numbers of the event-triggering times in 20 seconds are presented in Table I. Since the APETMs are the under-approximation and over-approximation of the original PETM, we have from Table I that, the number of the event-triggering times for $T^b_{0.5,0.1,0.2}(\Sigma)$ is smaller than 26, which is obtained from the continuous-time PETM with (12), whereas the number of the event-triggering times for $T^a_{0.5,0.1,0.2}(\Sigma)$ is larger than the one from (12).

Observe from Figs. 3-2 that the specification $\mathcal{S}$ is satisfied by the symbolic models, and the specification $\mathbf{B}_{\varepsilon_1}(\mathcal{S})$ is satisfied by the original system. On the one hand, from Figs. 3-2, different approximations lead to different satisfaction times for $\mathbf{B}_{\varepsilon_1}(\mathcal{S})$. In this example, the smaller the number of the event-triggering times is, the shorter the satisfaction time is. On the other hand, different refinement techniques have different effects on the satisfaction of the specification. The APETM with (13) is refined as the PETM with (15) by reducing the last item; whereas the APETM with (14) is refined as the PETM with (16) by increasing the parameter σ. Both refinement techniques lead to the satisfaction of the desired specification.

VI. CONCLUSIONS

In this paper, we constructed symbolic models for periodic event-triggered control systems. Since the event-triggered mechanism is related to the state, we proposed two abstract event-triggered mechanisms, developed two symbolic models, and established the equivalence relations. Furthermore, we discussed the controller design based on the constructed symbolic models, and a numerical example was given to compare the controllers designed from the symbolic models. Future work will focus on symbolic abstractions of distributed event-triggered control systems.

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