Lyapunov-based Generic Controller Design for Thrust-Propelled Underactuated Systems

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***Abstract*— We present a controller for an underactuated system which is driven by a one dimensional linear acceleration/thrust along a direction vector, by a time-varying gravity, and by the angular acceleration of the direction vector. We propose state and time-dependent control laws for the linear and angular accelerations that guarantee that the position of the system is steered to the origin. The proposed control law depends on (i) a bounded control law for a double integrator system; and (ii) on a Lyapunov function that guarantees asymptotic stability of the origin for the double integrator system when controlled with the previous bounded control law. As such, the control law forms a family of control laws depending on (i) and (ii). The complete state space of the system, under the proposed control laws, has two equilibria, and by proper control design, a trajectory of the system is guaranteed to converge to only one of those. The overall design provides a common framework for controlling different systems, such as quadrotors and slung load transportation systems.**

I. Introduction

Control of underactuated systems is an active topic of research, with many pratical applications. Vertical take off and landing rotorcrafts, with hover capabilities, form a class of underactuated vehicles for which trajectory tracking controllers are necessary [1], [2], [3]. Slung load transportation by aerial vehicles forms another class of underactuated systems for which trajectory traking is necessary [4]. In essence, the complexity behind an underactuated system lies on the fact that the dynamics of its generalized coordinates cannot be reduced to those of a double integrator system, thus limiting the choice of control techniques.

The system considered in this paper is composed of a three dimensional position and a three dimensional unit vector, corresponding to a five dimensional generalized coordinate. The position acceleration is composed of a one dimensional linear acceleration/thrust along the unit vector, and a time-varying gravity. We take the linear acceleration and the unit vector's angular acceleration as control inputs, thus corresponding to a four dimensional control input. As such, the second time derivative of the generalized coordinate, forming five second order differential equations, depends on a four dimensional input, thus yielding an underactuated system. Moreover, the time-varying gravity introduces an explicit time-dependency in the system's vector field, which is carried to the proposed control laws and to the closed loop system's vector field. This paper presents state and time-dependent control laws for the linear and angular accelerations that guarantee that the position of the system is steered to the origin. The open loop system described above is generic and it provides a common framework for other systems. For example, given an appropriate change of coordinates, the vector field for trajectory tracking with a quadrotor system, and the vector field for trajectory tracking of a slung load transportation system may be transformed into the vector field of the previous system. As such, by designing a control law for the generic system, this applies to those two particular systems, by means of a coordinate change. Many control strategies have been proposed for trajectory tracking of quadrotors [1], [2], [3], [5], [6], [7], [8], [9]. Controllers may be designed by linearizing the system around the hover condition, but these are only stable for small roll and pitch angles [5], [6]. Controllers have also been designed based on an inner attitude control loop and an outer position control loop [7]. Controllers that guarantee trajectory tracking for all initial conditions can also be found [8]. Since the quadrotor dynamics depend on the vehicle's rotation matrix, most control strategies also provide a control law for the space corresponding to the yaw motion. Different parameterizations for the vehicle's rotation matrix have also been used, such as euler angles [7], and unit quaternions [9], [8].

We propose continuous state and time-dependent control laws for the linear and angular accelerations that guarantee that the origin is asymptotically stable for the position. We note that many trajectory tracking problems may be transformed into this stabilization problem. The complete state space of the system, under the proposed control laws, has two equilibria, and by proper control design, a trajectory of the system is guaranteed to converge to only one of those. The existence of two equilibria is expected since unit vectors are part of a non-contractible set, and thus continous global stabilization is not possible [10], [11]. Our main contribution lies in providing a generic control law, in the sense that it does not depend on a specific bounded double integrator control input; instead, the proposed control law is functional for any bounded double integrator controller as long as a Lyapunov function that guarantees asymptotic stability of the origin for the controlled double integrator is available. Also, the proposed control law may be tuned such that the unit vector, where the input linear acceleration is provided, satisfies certain constraints, which may arise from safety constraints. The remainder of this paper is structured as follows. In Section III, we describe the open loop system's vector field. In Section IV, we describe the proposed control laws. And, in Section V, we present illustrative simulations.

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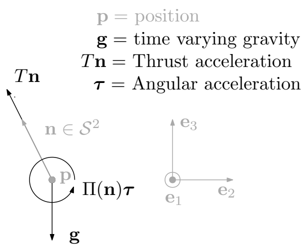


Fig. 1. Vector thrusted system

II. Notation

We denote by $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ the canonical basis in $\mathbb{R}^n$. Given $\mathbf{a} \in \mathbb{R}^3$, the matrix $\mathcal{S}(\mathbf{a})$ is the skew-symmetric matrix that satisfies $\mathcal{S}(\mathbf{a})\ \mathbf{b} = \mathbf{a} \times \mathbf{b}$, for any $\mathbf{b} \in \mathbb{R}^3$. We denote by $\mathcal{S}^2 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{x} = 1\}$ the space of the unit vectors in $\mathbb{R}^3$. The map $\Pi : \mathcal{S}^2 \mapsto \mathbb{R}^{3\times 3}$, defined as $\Pi(\mathbf{x}) = \mathbf{I} - \mathbf{x}\mathbf{x}^T$, yields a matrix that represents the orthogonal projection operator onto the subspace perpendicular to $\mathbf{x}$. Given $\mathcal{X} \subseteq \mathbb{R}^n$ and a function $f : \mathcal{X} \mapsto \mathbb{R}$, we say $f \in \mathcal{C}^k(\mathcal{X}, \mathbb{R})$ when all its partial derivatives, up to order k inclusive, are continuous in the domain $\mathcal{X}$ [12]; we also denote by $f^{(n)}(x) := \frac{d^n f(x)}{dx^n}$ the n^{th} derivative of f at $x \in \mathcal{X}$, for $n \in \{0, 1, \cdots, k\}$, and $\|f(\cdot)\|_\infty := \sup_{x\in\mathcal{X}} |f(x)|$. We say $f \in \mathcal{K}^\infty$ if $f \in \mathcal{C}^0([0,\infty), [0,\infty))$, $f(0) = 0$, $f'(\cdot) > 0$ and $\lim_{x\to\infty} f(x) = \infty$ [13]. Consider a system with state $\mathbf{x} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, a control input $\mathbf{u}_\mathbf{x} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, and denote $\Delta_\mathbf{x} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. We denote by $\mathbf{f}_\mathbf{x} \in \mathcal{C}(\mathbb{R}^{n+m}, \mathbb{R}^n)$ the open-loop vector field, i.e., given $\mathbf{u}_\mathbf{x} \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\dot{\mathbf{x}}(t) = \mathbf{f}_\mathbf{x}(t, \mathbf{x}(t), \mathbf{u}_\mathbf{x}(t))$. Given a control law $\mathbf{u}_\mathbf{x}^{cl} \in \mathcal{C}(\Delta_\mathbf{x}, \mathbb{R}^m)$, we denote by $\mathbf{f}_\mathbf{x}^{cl}(t, \mathbf{x}) := \mathbf{f}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x}^{cl}(t, \mathbf{x}))$ the closed-loop vector field. Moreover, given $V_\mathbf{x} \in \mathcal{C}^1(\Delta_\mathbf{x}, \mathbb{R}_{\geq 0})$, we denote $\tilde{W}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x}) := -\frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial t} - \frac{\partial V_\mathbf{x}(t,\mathbf{x})}{\partial \mathbf{x}}^T \mathbf{f}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x})$ and $W_\mathbf{x}(t, \mathbf{x}) := \tilde{W}_\mathbf{x}(t, \mathbf{x}, \mathbf{u}_\mathbf{x}^{cl}(t, \mathbf{x}))$. Given $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{r} \in \mathbb{R}^n$, we denote $\mathcal{B}(\mathbf{c}, \mathbf{r}) = \{\mathbf{x} \in \mathbb{R}^n : |r_i^{-1}(x_i - c_i)| \leq 1, i \in \{1, \cdots, n\}\}$ as the box centered around $\mathbf{c}$, and whose edges are aligned with the canonical basis vectors of $\mathbb{R}^n$, namely $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, and with length $\{2|r_1|, \cdots, 2|r_n|\}$.

III. Problem Statement

We present a controller for a vector thrusted system, illustrated in Fig 1, and described next. Consider any positive time instant $t \geq 0$. We denote by $\mathbf{p}(t) \in \mathbb{R}^3$ the position; by $\mathbf{v}(t) \in \mathbb{R}^3$ the velocity; by $T(t) \in \mathbb{R}$ the control input acceleration, which is provided along the unit vector $\mathbf{n}(t) \in \mathcal{S}^2$; by $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ the unit vector's angular velocity which is orthogonal to the same unit vector, i.e., $\boldsymbol{\omega}(t)^T\mathbf{n}(t) = 0$; and finally by $\boldsymbol{\tau}(t) \in \mathbb{R}^3$ the control input angular acceleration. Consider then the kinematics $\dot{\mathbf{p}}(t) = \mathbf{v}(t)$ and $\dot{\mathbf{n}}(t) = \mathcal{S}(\boldsymbol{\omega}(t))\,\mathbf{n}(t)$, and the dynamics

$$\dot{\mathbf{v}}(t) = T(t)\mathbf{n}(t) - \mathbf{g}(t), \dot{\boldsymbol{\omega}}(t) = \Pi(\mathbf{n}(t))\,\boldsymbol{\tau}(t), \qquad (1)$$

where $\mathbf{g}(t)$ is a known time-varying gravity term. Notice that if $\mathbf{n}^T(0)\boldsymbol{\omega}(0) = 0$, then $\mathbf{n}^T(t)\boldsymbol{\omega}(t) = 0$ for all $t \geq 0$, since $\frac{d}{dt}(\mathbf{n}^T(t)\boldsymbol{\omega}(t)) = \boldsymbol{\omega}^T(t)\mathcal{S}(\boldsymbol{\omega}(t))\,\mathbf{n}(t) + \mathbf{n}^T(t)\Pi(\mathbf{n}(t))\,\boldsymbol{\tau}(t) = 0$, for all $t \geq 0$. We note that, in general, $\mathbf{n}^T(0)\boldsymbol{\omega}(0) = 0$ does not need to be verified, since $\boldsymbol{\omega}(\cdot)$ is, by construction, orthogonal to $\mathbf{n}(\cdot)$. Given $\mathbf{r} = [r_{xy}\ r_{xy}\ r_z]^T$ with $r_{xy}, r_z > 0$, we assume

$$\mathbf{g} \in \mathcal{C}^2(\mathbb{R}_{\geq 0}, \mathcal{B}(g\mathbf{e}_3, \mathbf{r})) \wedge \max_{i\in\{0,1,2\}} (\|\mathbf{g}^{(i)}(\cdot)\|_\infty) < \infty, \quad (2)$$

i.e., we assume $\mathbf{g}(\cdot)$ is sufficiently smooth and its time derivatives are bounded in norm for all times; and $\mathbf{g}(\cdot)$ is within a box around a constant gravity term $g\mathbf{e}_3$. For reasons that will be apparent later, we require $r_z < g$, which means $\mathbf{0} \notin \mathcal{B}(g\mathbf{e}_3, \mathbf{r})$, where we emphasize that $\mathcal{B}(g\mathbf{e}_3, \mathbf{r})$ is a closed set, and therefore $\mathbf{0}$ is not in the boundary of $\mathcal{B}(g\mathbf{e}_3, \mathbf{r})$.

Problem 1: Given the system with dynamics (1), design control inputs $T : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ and $\boldsymbol{\tau} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^3$ such that $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$.

Remark 1: Several trajectory tracking problems can be converted into the form (1) and restated as in Problem 1 [14]. Our approach is particularly motivated by quadrotor applications.

IV. Controller Design

In this section, we construct controllers $T(\cdot)$ and $\boldsymbol{\tau}(\cdot)$ for the system (1) described in previous section. We denote the state of the system as

$$\mathbf{x}^T := \begin{bmatrix}\bar{\mathbf{x}}^T & \boldsymbol{\omega}^T\end{bmatrix} := \begin{bmatrix}\begin{bmatrix}\mathbf{p}^T & \mathbf{v}^T & \mathbf{n}^T\end{bmatrix} & \boldsymbol{\omega}^T\end{bmatrix},$$

and from (1) it follows that $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), T(t), \boldsymbol{\tau}(t))$ with

$$\mathbf{f}_\mathbf{x}(t, \mathbf{x}, T, \boldsymbol{\tau}) := \begin{bmatrix}\mathbf{f}_{\bar{\mathbf{x}}}(t, \bar{\mathbf{x}}, \boldsymbol{\omega}, T) \\ \mathbf{f}_\omega(\mathbf{n}, \boldsymbol{\tau})\end{bmatrix} := \begin{bmatrix}\begin{bmatrix}\mathbf{f}_{pv}(t, \bar{\mathbf{x}}, T) \\ \mathbf{f}_n(\mathbf{n}, \boldsymbol{\omega})\end{bmatrix} \\ \mathbf{f}_\omega(\mathbf{n}, \boldsymbol{\tau})\end{bmatrix}, (3)$$

where

$$\begin{bmatrix}\mathbf{f}_{pv}(t, \bar{\mathbf{x}}, T) \\ \mathbf{f}_n(\mathbf{n}, \boldsymbol{\omega}) \\ \mathbf{f}_\omega(\mathbf{n}, \boldsymbol{\tau})\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\mathbf{v} \\ T\mathbf{n} - \mathbf{g}(t)\end{bmatrix} \\ \mathcal{S}(\boldsymbol{\omega})\,\mathbf{n} \\ \Pi(\mathbf{n})\,\boldsymbol{\tau}\end{bmatrix}. \qquad (4)$$

We design the controller in three steps. First, we construct a thrust $T^{cl}(t, \bar{\mathbf{x}})$ such that the vector field

$$\mathbf{f}_{pv}^{cl}(t, \bar{\mathbf{x}}) := \mathbf{f}_{pv}(t, \bar{\mathbf{x}}, T^{cl}(t, \bar{\mathbf{x}})) \qquad (5)$$

approximates that of a double integrator, up to an error. Regarding the control of a double integrator we refer to [15], [16], [17], [14].

In the second step, we construct an angular velocity that guarantees that the previous error is steered to $\mathbf{0}$; more precisely, we construct a desired angular velocity $\boldsymbol{\omega}^{cl}(t, \bar{\mathbf{x}})$ such that $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$ for $\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}_{\bar{\mathbf{x}}}^{cl}(t, \bar{\mathbf{x}}(t))$, where

$$\mathbf{f}_{\bar{\mathbf{x}}}^{cl}(t, \bar{\mathbf{x}}) := \mathbf{f}_{\bar{\mathbf{x}}}(t, \bar{\mathbf{x}}, \boldsymbol{\omega}^{cl}(t, \bar{\mathbf{x}}), T^{cl}(t, \bar{\mathbf{x}})) = \begin{bmatrix}\mathbf{f}_{pv}^{cl}(t, \bar{\mathbf{x}}) \\ \mathbf{f}_n(\mathbf{n}, \boldsymbol{\omega}^{cl}(t, \bar{\mathbf{x}}))\end{bmatrix}, (6)$$

and where $\mathbf{f}_{pv}^{cl}(t, \bar{\mathbf{x}})$ is given from the first step, in (5).

In the final step, we compensate for the error $\mathcal{S}(\mathbf{n})\,(\boldsymbol{\omega} - \boldsymbol{\omega}^{cl}(t, \bar{\mathbf{x}}))$ as part of a backstepping procedure, and from which we construct $\boldsymbol{\tau}^{cl}(t, \mathbf{x})$ which guarantees $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$ for $\dot{\mathbf{x}}(t) = \mathbf{f}_\mathbf{x}^{cl}(t, \mathbf{x}(t))$, where

$$\mathbf{f}_\mathbf{x}^{cl}(t, \mathbf{x}) := \mathbf{f}_\mathbf{x}(t, \mathbf{x}, T^{cl}(t, \bar{\mathbf{x}}), \boldsymbol{\tau}^{cl}(t, \mathbf{x})). \qquad (7)$$

In the next sections, given a function $\mathbf{f} : \mathcal{X} \mapsto \mathbb{R}^m$, we denote $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = [\frac{\partial \mathbf{e}_1^T \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \cdots \frac{\partial \mathbf{e}_m^T \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}]$ for $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$.

1) First Step: Consider the double integrator system

$$\begin{bmatrix}\dot{\mathbf{p}}(t)\\ \dot{\mathbf{v}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{v}(t)\\ \mathbf{u}(\mathbf{p}(t),\mathbf{v}(t))\end{bmatrix} =: \mathbf{f}_{di}(\mathbf{p}(t),\mathbf{v}(t)) \tag{8}$$

where $\mathbf{u}(\mathbf{0},\mathbf{0}) = \mathbf{0}$, and where $\lim_{t\to\infty}\mathbf{p}(t) = 0$ for $\ddot{\mathbf{p}}(t) = \mathbf{u}(\mathbf{p}(t),\dot{\mathbf{p}}(t))$. We assume

$$\mathbf{u} \in \mathcal{C}^2(\mathbb{R}^3\times\mathbb{R}^3, \{\tilde{\mathbf{u}}\in\mathbb{R}^3 : |\tilde{\mathbf{u}}_x|, |\tilde{\mathbf{u}}_y| \le u^\infty_{xy}, |\tilde{\mathbf{u}}_z| \le u^\infty_z\}) \tag{9}$$

where $u^\infty_{xy} \in (0,+\infty]$ and $u^\infty_z \in (0, g - r_z)$ – recall that $g - r_z > 0$ since $\mathbf{0} \notin \mathcal{B}(g\mathbf{e}_3, \mathbf{r})$. We emphasize that $u^\infty_{xy} \le \infty$, i.e., the control of the double integrator systems along the x and y components does not need to be bounded in norm. For convenience, we denote $\mathbf{u}^\infty := [u^\infty_{xy}\, u^\infty_{xy}\, u^\infty_z]^T$. Furthermore, we assume the existence of a positive definite function

$$V_{di} \in \mathcal{C}^2(\mathbb{R}^3\times\mathbb{R}^3, \mathbb{R}_{\ge 0}), V_{di}(\mathbf{p},\mathbf{v}) \ge \alpha_1(\|[\mathbf{p}^T\,\mathbf{v}^T]^T\|), \tag{10}$$

for some $\alpha_1 \in \mathcal{K}^\infty$, and such that

$$W_{di}(\mathbf{p},\mathbf{v}) := -\frac{\partial V_{di}(\mathbf{p},\mathbf{v})}{\partial \mathbf{p}}^T\mathbf{v} - \frac{\partial V_{di}(\mathbf{p},\mathbf{v})}{\partial \mathbf{v}}^T\mathbf{u}(\mathbf{p},\mathbf{v}) \tag{11}$$

is positive definite; moreover, $W_{di} \in \mathcal{C}^1(\mathbb{R}^3\times\mathbb{R}^3, \mathbb{R}_{\ge 0})$, whose smoothness properties follow from (9) and (10), If $\sup_{t\ge 0}\|[\mathbf{p}^T(t)\ \mathbf{v}^T(t)]^T\| < \infty$, then, from positive definiteness of $W_{di}(\cdot,\cdot)$, it follows that

$$\lim_{t\to\infty} W_{di}(\mathbf{p}(t),\mathbf{v}(t)) = 0 \Rightarrow \lim_{t\to\infty}\|[\mathbf{p}^T(t)\ \mathbf{v}^T(t)]^T\| = 0. \tag{12}$$

Consider now $\mathbf{T}^{cl} \in \mathcal{C}^2(\mathbb{R}_{\ge 0}\times\mathbb{R}^3\times\mathbb{R}^3, \mathcal{B}(g\mathbf{e}_3, \mathbf{r}+\mathbf{u}^\infty))$ as

$$\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v}) := \mathbf{g}(t) + \mathbf{u}(\mathbf{p},\mathbf{v}), \tag{13}$$

which is the force that one would choose if $T(\cdot)\mathbf{n}(\cdot)$ in (1) were a control input. Notice the codomain and smoothness properties of (13) follows from (2) and (9). Furthermore, (9) implies that $\mathbf{e}_3^T(\mathbf{g}(\cdot) + \mathbf{u}(\cdot,\cdot)) \ge g - r_z - u^\infty_z > 0 \Rightarrow \|\mathbf{T}^{cl}(\cdot,\cdot,\cdot)\| \ge g - r_z - u^\infty_z > 0$. We can then define the unit vector associated to (13) $\mathbf{n}^{cl} \in \mathcal{C}^2(\mathbb{R}_{\ge 0}\times\mathbb{R}^3\times\mathbb{R}^3, \mathcal{S}^2)$ as

$$\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}) := \mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|^{-1}, \tag{14}$$

$$\mathbf{n}^\star(t) := \mathbf{n}^{cl}(t,\mathbf{0},\mathbf{0}) = \mathbf{g}(t)\|\mathbf{g}(t)\|^{-1}, \tag{15}$$

whose smoothness properties follow from (13). Additionally notice that if $\lim_{t\to\infty}\|[\mathbf{p}^T(t)\ \mathbf{v}^T(t)]^T\| = 0$ then

$$\lim_{t\to\infty}\|\mathcal{S}(\mathbf{n}(t))\,\mathbf{n}^{cl}(t,\mathbf{p}(t),\mathbf{v}(t))\| = 0 \Rightarrow \lim_{t\to\infty}(\mathbf{n}(t) \pm \mathbf{n}^\star(t)) = 0, \tag{16}$$

i.e., $\pm\mathbf{n}^\star(\cdot)$ represents the equilibrium trajectories $\mathbf{n}(\cdot)$ converges to, provided that the conditions in (16) are satisfied. Later we show that, under certain constraints, $\mathbf{n}^\star(\cdot)$ is the solution $\mathbf{n}(\cdot)$ converges to. Consider also $\boldsymbol{\omega}^{\mathbf{n}^{cl}} \in \mathcal{C}^1(\mathbb{R}_{\ge 0}\times\mathbb{R}^6\times\mathcal{S}^2, \mathbb{R}^3)$ defined as

$$\boldsymbol{\omega}^{\mathbf{n}^{cl}}(t,\bar{\mathbf{x}}) = \mathcal{S}(\mathbf{n}^{cl})\,\frac{\dot{\mathbf{g}} + \frac{\partial\mathbf{u}}{\partial\mathbf{p}}^T\mathbf{v} + \frac{\partial\mathbf{u}}{\partial\mathbf{v}}^T\left(\mathbf{u} - \Pi(\mathbf{n})\mathbf{n}^{cl}\|\mathbf{T}^{cl}\|\right)}{\|\mathbf{T}^{cl}\|}, \tag{17}$$

which is the angular velocity of $\mathbf{n}^{cl}(\cdot,\mathbf{p}(\cdot),\mathbf{v}(\cdot))$, i.e., $\dot{\mathbf{n}}^{cl}(t,\mathbf{p}(t),\mathbf{v}(t)) = \mathcal{S}\left(\boldsymbol{\omega}^{\mathbf{n}^{cl}}(t,\bar{\mathbf{x}}(t))\right)\mathbf{n}^{cl}(t,\mathbf{p}(t),\mathbf{v}(t))$, and whose smoothness properties follow from (2) and (14) (in (17), for brevity, we omitted the dependencies). And finally, denote

$$\boldsymbol{\omega}^\star(t) = \boldsymbol{\omega}^{\mathbf{n}^{cl}}(t,[\mathbf{0}_6^T\ \pm\mathbf{n}^{\star T}(t)]^T) \overset{(17)}{=} \mathcal{S}\left(\frac{\mathbf{g}(t)}{\|\mathbf{g}(t)\|}\right)\frac{\dot{\mathbf{g}}(t)}{\|\mathbf{g}(t)\|}, \tag{18}$$

where $\pm\dot{\mathbf{n}}^\star(t) = \pm\mathcal{S}(\boldsymbol{\omega}^\star(t))\,\mathbf{n}^\star(t)$, i.e., $\boldsymbol{\omega}^\star(\cdot)$ is the angular velocity $\boldsymbol{\omega}^{\mathbf{n}^{cl}}(\cdot,\bar{\mathbf{x}}(\cdot))$ converges to when $\lim_{t\to\infty}\{\mathbf{p}(t),\mathbf{v}(t),(\mathbf{n}(t)\pm\mathbf{n}^\star(t))\} = 0$. Note that $\boldsymbol{\omega}^\star(\cdot)$ is bounded in norm owing to (2), i.e. $\sup_{t\ge 0}\|\boldsymbol{\omega}^\star(t)\| < \infty$.

Let us now present a definition and some results that are useful in later sections.

Definition 1: Given two unit vectors $\mathbf{n},\boldsymbol{\nu} \in \mathcal{S}^2$ and $\alpha \in [0,\pi]$, we say $\mathbf{n} \in \mathcal{C}(\alpha,\boldsymbol{\nu})$ ($\mathbf{n} \in \bar{\mathcal{C}}(\alpha,\boldsymbol{\nu})$), if $\mathbf{n}^T\boldsymbol{\nu} > \cos(\alpha)$ ($\mathbf{n}^T\boldsymbol{\nu} \ge \cos(\alpha)$).

Proposition 2: Consider (14). It follows that, for all $(t,\mathbf{p},\mathbf{v}) \in \mathbb{R}_{\ge 0}\times\mathbb{R}^6$, $\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}) \in \mathcal{C}\left(\frac{\pi}{2},\mathbf{e}_3\right)$; moreover, if $u^\infty_{xy} < \infty$ in (9), then $\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}) \in \bar{\mathcal{C}}(\alpha^\star,\mathbf{e}_3)$, with $\alpha^\star := \arccos\left(\frac{g - r_z - u^\infty_z}{\sqrt{(g-r_z-u^\infty_z)^2 + 2(r_{xy}+u^\infty_{xy})^2}}\right)$.

The proof is omitted due to space limitations. In brief, it suffices to check that $\mathbf{e}_3^T\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}) \ge \cos(\alpha^\star)$ for all $(t,\mathbf{p},\mathbf{v}) \in \mathbb{R}_{\ge 0}\times\mathbb{R}^6$. Also, given a time instant $t \in \mathbb{R}_{\ge 0}$, if $\mathbf{n}(t) = \mathbf{n}^{cl}(t,\mathbf{p}(t),\mathbf{v}(t))$, then Proposition 2 guarantees that $\mathbf{n}(t)$ is pointing upwards (i.e., $\mathbf{e}_3^T\mathbf{n}(t) > 0$), which may be a safety condition the system in (1) might have to respect. For example, in quadrotor systems, $\mathbf{e}_3^T\mathbf{n}(t) < 0$ implies a pitch angle greater than 90° which may violate safety constraints.

Proposition 3: Consider (14) and $\alpha^\star$ as in Proposition 2. If $u^\infty_{xy} < \infty$ and given an $\boldsymbol{\nu} \in \mathcal{C}(\delta,\mathbf{e}_3)$ for some $\delta \in \left[0,\frac{\pi}{2}\right]$, it follows that $\boldsymbol{\nu} \in \mathcal{C}(\delta + \alpha^\star, \mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}))$ and $1 - \boldsymbol{\nu}^T\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}) < 1 - \cos(\delta + \alpha^\star)$ for all $(t,\mathbf{p},\mathbf{v}) \in \mathbb{R}_{\ge 0}\times\mathbb{R}^3\times\mathbb{R}^3$. For $u^\infty_{xy} = \infty$, the same conclusion holds if $\alpha^\star$ is replaced by $\frac{\pi}{2}$.

The proof is omitted due to space limitations. In brief, it amounts to deriving the Proposition's conclusions by means of a triangular inequality.

For the thrust controller, we propose the control law $T^{cl} \in \mathcal{C}^2(\mathbb{R}_{\ge 0}\times\mathbb{R}^6\times\mathcal{S}^2, \mathcal{B}(\mathbf{0}, g\mathbf{e}_3 + \mathbf{r} + \mathbf{u}^\infty))$ defined as

$$T^{cl}(t,\bar{\mathbf{x}}) = \mathbf{n}^T\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v}), \tag{19}$$

which is the projection of the desired force in (13) onto the direction where thrust can be provided, and whose smoothness properties follow from (13). Note that $T^{cl}(t,\bar{\mathbf{x}}) = \arg\min_{T\in\mathbb{R}}\|T\mathbf{n} - \mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|$, i.e., (19) minimizes the error between the provided force and the desired force. Denote for convenience $\mathbf{f}^{cl}_{pv}(t,\bar{\mathbf{x}}) = \mathbf{f}_{pv}(t,\bar{\mathbf{x}},T^{cl}(t,\bar{\mathbf{x}}))$. Taking (19) and $\mathbf{f}_{pv}(t,\bar{\mathbf{x}},T)$ in (4), and with the help of (13), (14) and (8), it follows that

$$\mathbf{f}^{cl}_{pv}(t,\bar{\mathbf{x}}) = \mathbf{f}_{di}(\mathbf{p},\mathbf{v}) - \begin{bmatrix}\mathbf{0}\\ \Pi(\mathbf{n})\,\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v})\|\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|\end{bmatrix},$$

which means the vector field $\mathbf{f}^{cl}_{pv}(t,\bar{\mathbf{x}})$ approximates the double integrator vector field $\mathbf{f}_{di}(\mathbf{p},\mathbf{v})$ in (8), up to an error, namely $[\mathbf{0}^T\ (\Pi(\mathbf{n})\,\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v}))^T]^T$. Thus, if we denote $W_{pv}(\mathbf{p},\mathbf{v}) := -\frac{\partial V_{di}^T(\mathbf{p},\mathbf{v})}{\partial(\mathbf{p},\mathbf{v})}\mathbf{f}^{cl}_{pv}(t,\mathbf{p},\mathbf{v})$ it follows that

$$W_{pv}(\mathbf{p},\mathbf{v}) \overset{(11)}{=} W_{di}(\mathbf{p},\mathbf{v}) + \frac{\partial V_{di}(\mathbf{p},\mathbf{v})}{\partial\mathbf{v}}^T\Pi(\mathbf{n})\,\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v}), \tag{20}$$

where $\dot{V}_{di}(\mathbf{p}(t),\mathbf{v}(t)) = W_{pv}(\mathbf{p}(t),\mathbf{v}(t))$ for $[\dot{\mathbf{p}}^T(t)\,\dot{\mathbf{v}}^T(t)]^T = \mathbf{f}^{cl}_{pv}(t,\bar{\mathbf{x}}(t)) = \mathbf{f}_{pv}(t,\bar{\mathbf{x}}(t),T^{cl}(t,\bar{\mathbf{x}}(t)))$.

2) Second Step: We now construct an angular velocity $\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})$ such that $\lim_{t\to\infty}\mathbf{p}(t) = \mathbf{0}$ for $\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}^{cl}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}(t))$, with the vector field as in (6). In order to construct such angular velocity, consider $\epsilon > 0$ and a positive definite function

$$V_\theta \in \mathcal{C}^2([0,\epsilon),\mathbb{R}_{\geq 0}), \text{ and, if } \epsilon \leq 2, \lim_{s\to\epsilon^-} V_\theta(s) = \infty, \quad (21)$$

satisfying $V_\theta(0) = 0$ and $V'_\theta(s) > 0, \forall s \in [0,\epsilon)$. It follows that $V_\theta(\cdot)$ is invertible, and for $\epsilon \leq 2$, the codomain of $V_\theta^{-1}(\cdot)$ is $[0,\infty)$. Examples of such functions, for $\epsilon \in (0,2]$, are $V_\theta(s) = ks(\epsilon^\alpha - s^\alpha)^{-\frac{1}{\alpha}}$ with $k > 0$ and $\alpha \geq 1$; and for $\epsilon > 2$, examples are $V_\theta(s) = ks$ with $k > 0$. The idea behind the choice of ϵ is explained in Remark 9. For convenience, denote $\xi \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^6 \times \mathcal{S}^2, [0,2])$

$$\xi(t,\bar{\mathbf{x}}) = 1 - \mathbf{n}^T\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}), \quad (22)$$

whose smoothness properties follow from (14). Denote also $\Omega_{\bar{\mathbf{x}}}(t) = \{(\mathbf{p},\mathbf{v},\mathbf{n}) \in \mathbb{R}^6 \times \mathcal{S}^2 : \xi(t,\bar{\mathbf{x}}) < \epsilon\}$ and

$$\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma) = \{(\mathbf{p},\mathbf{v},\mathbf{n}) \in \mathbb{R}^6 \times \mathcal{S}^2 : \|[\mathbf{p}^T\, \mathbf{v}^T]\| \leq r, \xi(t,\bar{\mathbf{x}}) \leq \gamma\},$$

where we emphasize that, since $\epsilon > 0$, $\Omega_{\bar{\mathbf{x}}}(t) \neq \emptyset$ for all $t \geq 0$; and that, for $\gamma \in [0,\epsilon)$, $\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma)$ is closed and $\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma) \subset \Omega_{\bar{\mathbf{x}}}(t)$ for all $t \geq 0$ and for all $r \geq 0$. In other words, $\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma)$ is compact which is of importance later, particularly for functions that are continuous on $\Omega_{\bar{\mathbf{x}}}(t)$: specifically, we make use of the fact that if $\mathbf{f}(t,\cdot) \in \mathcal{C}^0(\Omega_{\bar{\mathbf{x}}}(t),\mathbb{R}^m) \forall t \geq 0$, then $\max_{\bar{\mathbf{x}}\in\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma)} \|\mathbf{f}(t,\bar{\mathbf{x}})\| < \infty$ for all $t \geq 0$. Note that for $\epsilon > 2$, $\Omega_{\bar{\mathbf{x}}}(t) = \mathbb{R}^6 \times \mathcal{S}^2$ for all $t \geq 0$, where $\mathcal{S}^2$ is a compact set. This means that, when $\epsilon > 2$ and $\mathbf{f}(t,\cdot) \in \mathcal{C}^0(\Omega_{\bar{\mathbf{x}}}(t),\mathbb{R}^m)$, then $\max_{\bar{\mathbf{x}}\in\bar{\Omega}_{\bar{\mathbf{x}}}(t,R,\epsilon)} \|\mathbf{f}(t,\bar{\mathbf{x}})\| < \infty$ for all $t \geq 0$ and for any $R \geq 0$. Note that for a trajectory of (6), $\bar{\mathbf{x}}(0) \in \bar{\Omega}_{\bar{\mathbf{x}}}(t,0,0) \Rightarrow \bar{\mathbf{x}}(t) \in \bar{\Omega}_{\bar{\mathbf{x}}}(t,0,0)$ for all $t \geq 0$, which means $\bar{\Omega}_{\bar{\mathbf{x}}}(t,0,0) = [\mathbf{0}_3^T\, \mathbf{0}_3^T\, \mathbf{n}^{\star T}(t)]^T$ corresponds to an equilibrium trajectory, and, on the other hand, it means that $\bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma)$ quantifies a region around the equilibrium trajectory, for every $t \geq 0$. Now, denote $\mathit{\Delta}_{\bar{\mathbf{x}}} = \{(t,\bar{\mathbf{x}}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^6 \times \mathcal{S}^2 : \bar{\mathbf{x}} \in \Omega_{\bar{\mathbf{x}}}(t)\}$, and note that (19) and (22) are $\mathcal{C}^2$ on $\mathit{\Delta}_{\bar{\mathbf{x}}}$ (and (17) is $\mathcal{C}^1$).

We now proceed to the design of $\boldsymbol{\omega}^{cl}(\cdot,\cdot)$. Consider the function $V_{\bar{\mathbf{x}}} \in \mathcal{C}^2(\mathit{\Delta}_{\bar{\mathbf{x}}},\mathbb{R}_{\geq 0})$ defined as

$$V_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) = V_{di}(\mathbf{p},\mathbf{v}) + V_\theta(\xi(t,\bar{\mathbf{x}})), \quad (23)$$

whose smoothness properties follow from (10), (21) and (22). Denote $\tilde{W}_{\bar{\mathbf{x}}}(t,\mathbf{x},\boldsymbol{\omega}) = -\frac{\partial V_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}})}{\partial t} - \frac{\partial V_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\mathbf{f}^{cl}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\boldsymbol{\omega},T^{cl}(t,\bar{\mathbf{x}}))$, and it follows from (6), (17), (20) and (23) that

$$\tilde{W}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\boldsymbol{\omega}) = W_{di}(\mathbf{p},\mathbf{v}) - V'_\theta(\xi(t,\bar{\mathbf{x}}))(\mathcal{S}(\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}))\mathbf{n})^T \times$$
$$\times \left(\boldsymbol{\omega} - \boldsymbol{\omega}^{\mathbf{n}^{cl}}(t,\bar{\mathbf{x}}) + \frac{\|\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|}{V'_\theta(\xi(t,\bar{\mathbf{x}}))}\mathcal{S}(\mathbf{n})\frac{\partial V_{di}}{\partial \mathbf{v}}\right).$$

We construct $\boldsymbol{\omega}^{cl}(\cdot,\cdot)$ by enforcing the equality $W_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) = \tilde{W}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}))$ with

$$W_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) = W_{di}(\mathbf{p},\mathbf{v}) + k_\theta V'_\theta(\xi(t,\bar{\mathbf{x}}))\|\mathcal{S}(\mathbf{n})\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v})\|^2, \quad (24)$$

where $k_\theta \geq 0$, and where, by construction, $W_{\bar{\mathbf{x}}} \in \mathcal{C}^1(\mathit{\Delta}_{\bar{\mathbf{x}}},\mathbb{R}_{\geq 0})$. The previous condition (24) is satisfied for $\boldsymbol{\omega}^{cl} \in \mathcal{C}^1(\mathit{\Delta}_{\bar{\mathbf{x}}},\mathbb{R}_{\geq 0})$ defined as

$$\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}) = \boldsymbol{\omega}^{\mathbf{n}^{cl}}(t,\bar{\mathbf{x}}) - k_\theta\mathcal{S}(\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}))\mathbf{n} - \frac{\|\mathbf{T}^{cl}(t,\mathbf{p},\mathbf{v})\|}{V'_\theta(\xi(t,\bar{\mathbf{x}}))}\mathcal{S}(\mathbf{n})\frac{\partial V_{di}(\mathbf{p},\mathbf{v})}{\partial \mathbf{v}}, \quad (25)$$

whose smoothness properties follow from (10), (13), (14), (17) and (21). Note that, if $\epsilon > 2$, then $\min_{s\in[0,2]} V'_\theta(s) > 0$; and if $\epsilon \leq 2$, then $\min_{s\in[0,\epsilon)} V'_\theta(s) > 0$; therefore, in either case, (25) is well defined. Given (19) and (25) it follows that for the vector field in (6), in fact, $\mathbf{f}^{cl}_{\bar{\mathbf{x}}} \in \mathcal{C}^1(\mathit{\Delta}_{\bar{\mathbf{x}}},\mathbb{R}^9)$.

Proposition 4: If $\forall t \geq 0 \exists R \geq 0, \Gamma \in [0,\epsilon)$: $\bar{\mathbf{x}}(t) \in \bar{\Omega}_{\bar{\mathbf{x}}}(t,R,\Gamma) \subset \Omega_{\bar{\mathbf{x}}}(t)$, then $\|\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}(t))\| \leq \sup_{t\geq 0}\max_{\bar{\mathbf{x}}\in\Omega_{\bar{\mathbf{x}}}(t,R,\Gamma)} \|\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})\| =: \bar{\omega}^d(R,\Gamma) < \infty$, for all $t \geq 0$.

Proof: Since, for every $t \geq 0$, $\boldsymbol{\omega}^{cl}(t,\cdot) \in \mathcal{C}^1(\Omega_{\bar{\mathbf{x}}}(t),\mathbb{R}^3)$, and since $\Omega_{\bar{\mathbf{x}}}(t,R,\Gamma)$ is a compact subset of $\Omega_{\bar{\mathbf{x}}}(t)$, it follows that $F(t) := \max_{\bar{\mathbf{x}}\in\Omega_{\bar{\mathbf{x}}}(t,R,\Gamma)} \|\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})\| < \infty$. Moreover, since $F(t)$ is a function of $\mathbf{g}^{(0)}(t)$ and $\mathbf{g}^{(1)}(t)$, $\sup_{t\geq 0} F(t) < \infty$ follows from (2) and (13). ∎

Remark 5: Note that $\boldsymbol{\omega}^{cl}(t,[\mathbf{0}_3^T\,\mathbf{0}_3^T\,\mathbf{n}^{\star T}(t)]^T) = \boldsymbol{\omega}^\star(t)$, thus it follows that $\sup_{t\geq 0}\|\boldsymbol{\omega}^\star(t)\| = \bar{\omega}^d(0,0) \leq \bar{\omega}^d(R,\Gamma)$, for any $R \geq 0, \Gamma \in [0,\epsilon)$.

3) Third Step: We now perform the final step, where we construct $\boldsymbol{\tau}^{cl}(t,\mathbf{x})$ which guarantees $\lim_{t\to\infty}\mathbf{p}(t) = \mathbf{0}$ for $\dot{\mathbf{x}}(t) = \mathbf{f}^{cl}_{\mathbf{x}}(t,\mathbf{x}(t))$, where $\mathbf{f}^{cl}_{\mathbf{x}}(t,\mathbf{x})$ is given in (7). For brevity, and similarly to what was done in the second step, denote $\Omega_{\mathbf{x}}(t) = \Omega_{\bar{\mathbf{x}}}(t) \times \mathbb{R}^3$, $\mathit{\Delta}_{\mathbf{x}} = \mathit{\Delta}_{\bar{\mathbf{x}}} \times \mathbb{R}^3$ and

$$\bar{\Omega}_{\mathbf{x}}(t,r,\gamma) = \{(\bar{\mathbf{x}},\boldsymbol{\omega}) \in \bar{\Omega}_{\bar{\mathbf{x}}}(t,r,\gamma) \times \mathbb{R}^3 : \|\boldsymbol{\omega} - \boldsymbol{\omega}^\star(t)\| \leq r\}, \quad (26)$$

where we emphasize, once again, that $\Omega_{\mathbf{x}}(t) \neq \emptyset$ for all $t \geq 0$, and that $\bar{\Omega}_{\mathbf{x}}(t,r,\gamma) \subset \Omega_{\mathbf{x}}(t)$ is a compact set for all $t \geq 0$. Also, we prove later that $\mathbf{x}(0) \in \bar{\Omega}_{\mathbf{x}}(0,0,0) \Rightarrow \mathbf{x}(t) \in \bar{\Omega}_{\mathbf{x}}(t,0,0)$ for all $t \geq 0$, and therefore $\bar{\Omega}_{\mathbf{x}}(t,0,0) = [\mathbf{0}_6^T\,\mathbf{n}^{\star T}(t)\,\boldsymbol{\omega}^\star(t)]^T$ corresponds to an equilibrium trajectory, while $\bar{\Omega}_{\mathbf{x}}(t,r,\gamma)$ quantifies closeness to the equilibrium trajectory. We also prove later that if $\exists r \geq 0, \gamma \in [0,\epsilon)$: $\bar{\mathbf{x}}(0) \in \bar{\Omega}_{\mathbf{x}}(0,r,\gamma) \subset \Omega_{\mathbf{x}}(0)$, then

$$\forall t \geq 0 \exists R \geq 0, \Gamma \in [0,\epsilon) : \mathbf{x}(t) \in \bar{\Omega}_{\mathbf{x}}(t,R,\Gamma) \subset \Omega_{\mathbf{x}}(t). \quad (27)$$

Recall that $\boldsymbol{\omega}^\star(\cdot)$ is bounded, and therefore if (27) is satisfied then $\boldsymbol{\omega}(\cdot)$ is bounded in norm due to the definition of $\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma)$ in (26). Recall that we do not have control over the angular velocity $\boldsymbol{\omega}(t)$, otherwise $\boldsymbol{\omega}(t) = \boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}(t))$ would suffice to accomplish the goal of Problem 1. Denote then $\mathbf{e}_\omega \in \mathcal{C}^1(\mathit{\Delta}_{\mathbf{x}},\mathbb{R}^3)$ defined as

$$\mathbf{e}_\omega(t,\mathbf{x}) := \mathcal{S}(\mathbf{n})(\boldsymbol{\omega} - \boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})), \quad (28)$$

corresponding to an error that we shall use in a backstepping procedure, and whose smoothness properties follow from (25). Let us also define $\tilde{\mathbf{f}}_{\bar{\mathbf{x}}} \in \mathcal{C}^1(\mathit{\Delta}_{\mathbf{x}},\mathbb{R}^9)$ as

$$\tilde{\mathbf{f}}_{\bar{\mathbf{x}}}(t,\mathbf{x}) = \mathbf{f}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\boldsymbol{\omega},T^{cl}(t,\bar{\mathbf{x}})) \overset{(28)}{=} \mathbf{f}^{cl}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) - \begin{bmatrix}\mathbf{0}_6^T & \mathbf{e}_\omega^T(t,\mathbf{x})\end{bmatrix}^T \quad (29)$$

whose smoothness properties follow from (28) and smoothness of $\mathbf{f}^{cl}_{\bar{\mathbf{x}}}(\cdot,\cdot)$; intuitively, the vector field $\tilde{\mathbf{f}}_{\bar{\mathbf{x}}}(t,\mathbf{x})$ approximates the vector field $\mathbf{f}^{cl}_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}})$, designed in the previous step, up to an error that we compensate for later. For convenience, let us also define $\boldsymbol{\tau}^{\omega^{cl}} \in \mathcal{C}^0(\mathit{\Delta}_{\mathbf{x}},\mathbb{R}^3)$ as

$$\boldsymbol{\tau}^{\omega^{cl}}(t,\mathbf{x}) = \frac{\partial \boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})}{\partial t} + \frac{\partial \boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}^T \tilde{\mathbf{f}}_{\bar{\mathbf{x}}}(t,\mathbf{x}), \quad (30)$$

which provides the time derivative of $\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}(t))$, i.e., $\dot{\boldsymbol{\omega}}^{cl}(t,\bar{\mathbf{x}}(t)) = \boldsymbol{\tau}^{\omega^{cl}}(t,\mathbf{x}(t))$ for $\dot{\bar{\mathbf{x}}}(t) = \tilde{\mathbf{f}}_{\bar{\mathbf{x}}}(t,\mathbf{x}(t))$; and whose smoothness properties follow from (25) and (29).

Consider now the function $V_\omega(\mathbf{z}) = \frac{1}{2}v_\omega \mathbf{z}^T\mathbf{z}$ with $v_\omega > 0$, and consider $V_{\mathbf{x}} \in \mathcal{C}^1(\Delta_{\mathbf{x}}, \mathbb{R}_{\geq 0})$ defined as

$$V_{\mathbf{x}}(t,\mathbf{x}) = V_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) + V_\omega(\mathbf{e}_\omega(t,\mathbf{x})),$$

whose smoothness properties follow from (23), (28) and smoothness of $V_\omega(\cdot)$. If we denote $\tilde{W}_{\mathbf{x}}(t,\mathbf{x},\boldsymbol{\tau}) = -\frac{\partial V_{\mathbf{x}}(t,\mathbf{x})}{\partial t} - \frac{\partial V_{\mathbf{x}}(t,\mathbf{x})}{\partial \bar{\mathbf{x}}}^T \mathbf{f}_{\mathbf{x}}(t,\mathbf{x},T^{cl}(t,\bar{\mathbf{x}}),\boldsymbol{\tau})$ and design $\boldsymbol{\tau}^{cl} \in \mathcal{C}^0(\Delta_{\mathbf{x}}, \mathbb{R}^3)$ defined as

$$\begin{aligned}\boldsymbol{\tau}^{cl}(t,\mathbf{x}) =& \Pi(\mathbf{n})\,\boldsymbol{\tau}^{\boldsymbol{\omega}^{cl}}(t,\mathbf{x}) + \mathcal{S}(\mathbf{n})\,\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}})\mathbf{n}^T\boldsymbol{\omega}^{cl}(t,\bar{\mathbf{x}}) + \\ & k_\omega \mathcal{S}(\mathbf{n})\,\mathbf{e}_\omega(t,\mathbf{x}) + \tfrac{V'_\theta(\cdot)}{v_\omega}\mathcal{S}(\mathbf{n})\,\mathbf{n}^{cl}(t,\mathbf{p},\mathbf{v}), \quad (31)\end{aligned}$$

for some $k_\omega \geq 0$, and whose smoothness properties follow from (14), (21), (28) and (30). For this choice, it follows that $W_{\mathbf{x}}(t,\mathbf{x}) = \tilde{W}_{\mathbf{x}}(t,\mathbf{x},\boldsymbol{\tau}^{cl}(t,\mathbf{x}))$ where

$$W_{\mathbf{x}}(t,\mathbf{x}) = W_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}) + 2k_\omega V_\omega(\mathbf{e}_\omega(t,\mathbf{x})) \qquad (32)$$

with $W_{\mathbf{x}} \in \mathcal{C}^1(\Delta_{\mathbf{x}}, \mathbb{R}_{\geq 0})$ and whose smoothness properties follow from (24), (28) and smoothness of $V_\omega(\cdot)$; moreover, along a trajectory $\mathbf{x}(\cdot)$ of $\dot{\mathbf{x}}(t) = \mathbf{f}^{cl}_{\mathbf{x}}(t,\mathbf{x}(t))$, $\dot{V}_{\mathbf{x}}(t,\mathbf{x}(t)) = -W_{\mathbf{x}}(t,\mathbf{x}(t)) \leq 0$. Moreover, it follows that $\mathbf{f}^{cl}_{\mathbf{x}} \in \mathcal{C}^0(\Delta_{\mathbf{x}}, \mathbb{R}^{12})$, owing to (2), (19) and (31). It follows that if $\mathbf{x}(0) \in \bar{\Omega}_{\mathbf{x}}(0,0,0)$, then $\dot{V}_{\mathbf{x}}(t,\mathbf{x}(t)) = -W_{\mathbf{x}}(t,\mathbf{x}(t)) \leq 0 \Rightarrow 0 \leq V_{\mathbf{x}}(t,\mathbf{x}(t)) \leq V_{\mathbf{x}}(0,\mathbf{x}(0)) = 0 \Rightarrow \mathbf{x}(t) \in \bar{\Omega}_{\mathbf{x}}(t,0,0)$ for all $t \geq 0$, which implies that $\bar{\Omega}_{\mathbf{x}}(t,0,0)$ defines an equilibrium trajectory.

Finally, denote $dW_{\mathbf{x}}(t,\mathbf{x}) = \frac{\partial W_{\mathbf{x}}(t,\mathbf{x})}{\partial t} + \frac{\partial W_{\mathbf{x}}(t,\mathbf{x})}{\partial \mathbf{x}}^T \mathbf{f}^{cl}_{\mathbf{x}}(t,\mathbf{x})$ where $dW_{\mathbf{x}} \in \mathcal{C}^0(\Delta_{\mathbf{x}}, \mathbb{R})$, whose smoothness properties follow from (32) and from smoothness of $\mathbf{f}^{cl}_{\mathbf{x}}(\cdot,\cdot)$. Since $dW_{\mathbf{x}} \in \mathcal{C}^0(\Delta_{\mathbf{x}}, \mathbb{R})$, it follows that, if (2) and (27) are satisfied, then, along a trajectory $\mathbf{x}(\cdot)$ of (7),

$$\sup_{t\geq 0} |dW_{\mathbf{x}}(t,\mathbf{x}(t))| \overset{(27)}{\leq} \sup_{t\geq 0} \max_{\mathbf{x}\in\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma)} |dW_{\mathbf{x}}(t,\mathbf{x})| < \infty, \quad (33)$$

where $F(t) := \max_{\mathbf{x}\in\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma)} |dW_{\mathbf{x}}(t,\mathbf{x})|$ is bounded for every $t \geq 0$, since $dW_{\mathbf{x}}(t,\cdot) \in \mathcal{C}^0(\bar{\Omega}_{\mathbf{x}}(t), \mathbb{R})$ for all $t \geq 0$, and $\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma)$ is a compact subset of $\bar{\Omega}_{\mathbf{x}}(t)$ for every $t \geq 0$, i.e., $\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma)$ is closed and $\bar{\Omega}_{\mathbf{x}}(t,R,\Gamma) \subset \bar{\Omega}_{\mathbf{x}}(t) \forall t \geq 0$; and $F(t)$ is also bounded, i.e., $\sup_{t\geq 0} F(t) < \infty$, since it is a function of $\mathbf{g}^{(0)}(t)$, $\mathbf{g}^{(1)}(t)$ and $\mathbf{g}^{(2)}(t)$ and these satisfy (2) (as well as (9)).

Proposition 6: If, for some $R_1, R_2 \geq 0, \Gamma \in [0,\epsilon)$, $\bar{\mathbf{x}}(t) \in \Omega_{\bar{\mathbf{x}}}(t,R_1,\Gamma)$ and $\|\mathbf{e}_\omega(t,\mathbf{x}(t))\| \leq R_2$, for all $t \geq 0$, then $\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}^\star(t)\| \leq 2\bar{\omega}^d(R_1,\Gamma) + R_2 =: R_\omega(R_1,R_2,\Gamma) < \infty$, for all $t \geq 0$.

We omit the proof due to paper length constraints. In brief, it suffices to explore a triangular inequality and Proposition 4 to derive a bound on $\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}^\star(t)\|$.

Theorem 7: Consider the system with vector field (3), and the control laws (19) and (31). If $T(t) = T^{cl}(t,\bar{\mathbf{x}}(t))$ and $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^{cl}(t,\mathbf{x}(t))$, then $\forall \mathbf{x}(0) \in \bar{\Omega}_{\mathbf{x}}(0)$, it follows that $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$. Moreover, when $\epsilon \leq 2$, $\lim_{t\to\infty} (\mathbf{n}(t) - \mathbf{n}^\star(t)) = \mathbf{0}$, with $\mathbf{n}^\star(\cdot)$ in (15).

Proof: First note that, when $\epsilon > 2$, $\bar{\Omega}_{\mathbf{x}}(0) = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}^2 \times \mathbb{R}^3$, which corresponds to the complete state space. When $\epsilon \leq 2$, $\bar{\Omega}_{\mathbf{x}}(0) \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}^2 \times \mathbb{R}^3$, which corresponds to a subset of the complete state space. When $\epsilon \leq 2$, since $\bar{\Omega}_{\mathbf{x}}(0)$ is an open subset, it follows that $\exists r \geq 0, \gamma \in [0,\epsilon) : \bar{\mathbf{x}}(0) \in \bar{\Omega}_{\bar{\mathbf{x}}}(0,r,\gamma) \subset \Omega_{\bar{\mathbf{x}}}(0)$, which in turn implies that $V_{\mathbf{x}}(0,\mathbf{x}(0)) < \infty$. Consider a trajectory $\mathbf{x}(\cdot)$ of (7), for which $\dot{V}_{\mathbf{x}}(t,\mathbf{x}(t)) = -W_{\mathbf{x}}(t,\mathbf{x}(t)) \leq 0$, and consequently $\max_t(V_{di}(\mathbf{p}(t),\mathbf{v}(t)), V_\theta(\xi(t,\bar{\mathbf{x}}(t))), V_\omega(\mathbf{e}(t,\mathbf{x}(t)))) \leq V_{\mathbf{x}}(t,\mathbf{x}(t)) \leq V_{\mathbf{x}}(0,\mathbf{x}(0)) =: V_0 < \infty$ for all $t \geq 0$. Consider first the case when $\epsilon \leq 2$. Since $V_{\mathbf{x}}(\cdot,\mathbf{x}(\cdot))$ is upper bounded by its initial condition, it follows that, for all $t \geq 0$, $\xi(t,\bar{\mathbf{x}}(t)) \in [0, V_\theta^{-1}(V_0)] \overset{(21)}{\subset} [0,\epsilon)$, $\|[\mathbf{p}^T(t)\ \mathbf{v}^T(t)]^T\| \overset{(10)}{\leq} \alpha_1^{-1}(V_0)$ and $\|\mathbf{e}_\omega(t,\mathbf{x}(t))\| \leq \sqrt{\frac{V_0}{v_\omega}}$. Thus $\bar{\mathbf{x}}(t) \in \Omega_{\bar{\mathbf{x}}}(t,R_1,\Gamma)$, for $R_1 := \alpha_1^{-1}(V_0)$ and $\Gamma := V_\theta^{-1}(V_0) \overset{(21)}{<} \epsilon$, and Proposition 4 applies. Additionally, Proposition 6 also applies for $R_2 := \sqrt{\frac{V_0}{v_\omega}}$ and therefore $\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}^\star(t)\| \leq R_\omega(R_1,R_2,\Gamma) =: R_3$.

This implies that (27) is satisfied for $R = \max(R_1,R_3)$ and $\Gamma := V_\theta^{-1}(V_0)$, and consequently (33) also follows. Therefore, along a trajectory $\mathbf{x}(\cdot)$, it follows that $|\dot{W}_{\mathbf{x}}(t,\mathbf{x}(t))| = |dW_{\mathbf{x}}(t,\mathbf{x}(t))|$ is bounded in norm, and therefore $W_{\mathbf{x}}(t,\mathbf{x}(t))$ is uniformly continuous; moreover, since $\dot{V}_{\mathbf{x}}(t,\mathbf{x}(t)) = -W_{\mathbf{x}}(t,\mathbf{x}(t)) \leq 0$ and $V_{\mathbf{x}}(t,\mathbf{x}(t)) \geq 0$, then $\lim_{t\to\infty} V_{\mathbf{x}}(t,\mathbf{x}(t))$ exists. Barbalat's lemma then guarantees that $\lim_{t\to\infty} W_{\mathbf{x}}(t,\mathbf{x}(t)) = 0 \overset{(32),(24)}{\Rightarrow} \lim_{t\to\infty} W_{di}(\mathbf{p}(t),\mathbf{v}(t)) = 0 \overset{(12)}{\Rightarrow} \lim_{t\to\infty} \mathbf{p}(t) = 0$. For the case $\epsilon > 2$, it suffices to replace Γ by ϵ in the results above, and the same conclusion follows.

For the later statement in the Theorem and for $k_\theta > 0$, notice that $\lim_{t\to\infty} W_{\mathbf{x}}(t,\mathbf{x}(t)) = 0 \overset{(32)}{\Rightarrow} \lim_{t\to\infty} W_{\bar{\mathbf{x}}}(t,\bar{\mathbf{x}}(t)) = 0 \overset{(24)}{\Rightarrow} \lim_{t\to\infty} \|\mathcal{S}(\mathbf{n}(t))\,\mathbf{n}^{cl}(t,\mathbf{p}(t),\mathbf{v}(t))\|^2 = 0 \Leftrightarrow \lim_{t\to\infty} \xi(t,\bar{\mathbf{x}}(t))(2 - \xi(t,\bar{\mathbf{x}}(t))) = 0$. Since $\xi(t,\bar{\mathbf{x}}(t)) \in [0, V_\theta^{-1}(V_0)] \subset [0,\epsilon) \subset [0,2]$, it follows that $\lim_{t\to\infty} \xi(t,\bar{\mathbf{x}}(t)) = 0$. For $k_\theta = 0$, the same conclusion follows if we notice that (i) $\lim_{t\to\infty} W_{di}(\mathbf{p}(t),\mathbf{v}(t)) = 0 \overset{(12)}{\Rightarrow} \lim_{t\to\infty} \mathbf{v}(t) = 0$; (ii) $\ddot{\mathbf{v}}(t)$ is bounded in norm along a trajectory $\mathbf{x}(\cdot)$, which is supported by similar arguments as those used in the proof of boundedness of $|\ddot{V}_{\mathbf{x}}(t,\mathbf{x}(t))| = |dW_{\mathbf{x}}(t,\mathbf{x}(t))|$. Conditions (i) and (ii) then imply that, by Barbalat's lemma, $\lim_{t\to\infty} \dot{\mathbf{v}}(t) = 0$, and consequently $\lim_{t\to\infty} \xi(t,\bar{\mathbf{x}}(t)) = 0$, due to the fact that $\xi(t,\bar{\mathbf{x}}(t)) \in [0,\epsilon) \subset [0,2], \forall t \geq 0$. ■

Corollary 8: Consider the system with vector field (3), and the control laws (19) and (31), and that $T(t) = T^{cl}(t,\bar{\mathbf{x}}(t))$ and $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^{cl}(t,\mathbf{x}(t))$. If $\epsilon \in (1,2]$, then $\forall \mathbf{x}(0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{C}(\arccos(1-\epsilon) - \alpha^\star, \mathbf{e}_3) \times \mathbb{R}^3 \subset \bar{\Omega}_{\mathbf{x}}(0)$ with $\alpha^\star$ as in Proposition 2, it follows that $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$. Moreover, $\lim_{t\to\infty} (\mathbf{n}(t) - \mathbf{n}^\star(t)) = \mathbf{0}$.

Proof: Recall Proposition 3, and denote $\delta = \arccos(1-\epsilon) - \alpha^\star > 0$, where the inequality follows since $\epsilon \in (1,2]$ and $\alpha^\star \leq \frac{\pi}{2}$. If $\mathbf{n}(0) \in \mathcal{C}(\delta,\mathbf{e}_3)$, then $1 - \mathbf{n}^T(0)\mathbf{n}^{cl}(0,\mathbf{p},\mathbf{v}) = \xi(0,[\mathbf{p}^T\,\mathbf{v}^T\,\mathbf{n}^T(0)]^T) < \epsilon\,\forall(\mathbf{p},\mathbf{v}) \in \mathbb{R}^6$ and therefore $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{C}(\arccos(1-\epsilon) - \alpha^\star, \mathbf{e}_3) \times \mathbb{R}^3 \overset{(26)}{\subset} \bar{\Omega}_{\mathbf{x}}(0)$. Thus, Theorem 7 may be invoked. ■

In Corollary 8, the set $\Theta_0 := \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{C}(\arccos(1-\epsilon) - \alpha^\star, \mathbf{e}_3) \times \mathbb{R}^3$ does not depend on the initial time instant, and consequently it is straightforward to verify if $\mathbf{x}(0) \in \Theta_0$. Also note that $\mathbf{e}_3 \in \Theta_0$, and therefore, if $\mathbf{n}(0)$ is in a neighborhood of $\mathbf{e}_3$ then Corrolary's 8 conclusions follow.

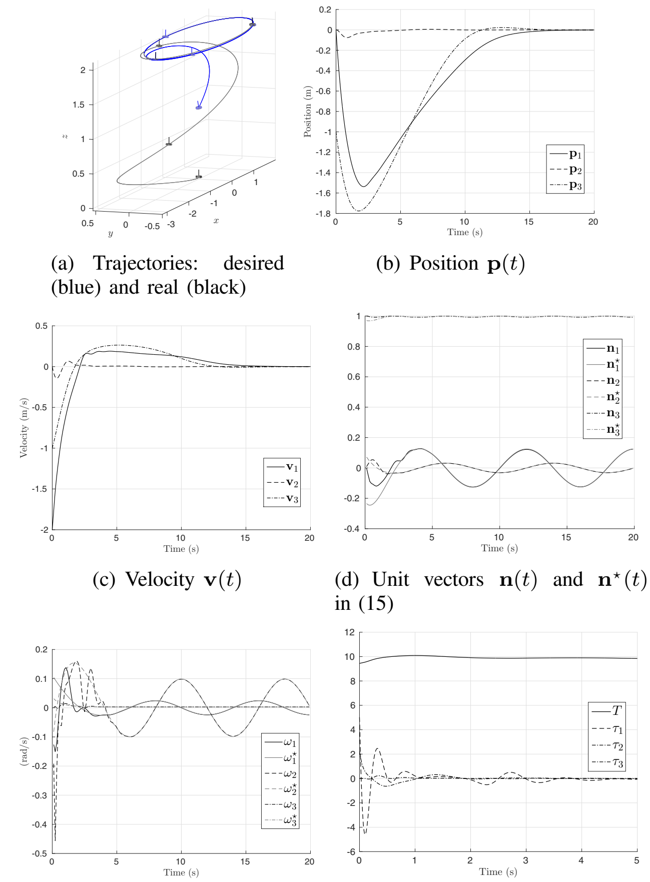
Remark 9: The idea behind the choice of ϵ is the following: if $\epsilon > 2$, $\mathbf{n}(\cdot)$ may converge to either $\mathbf{n}^\star(\cdot)$ or $-\mathbf{n}^\star(\cdot)$; on the other hand, if $\epsilon \leq 2$, $\mathbf{n}(\cdot)$ converges to $\mathbf{n}^\star(\cdot)$ (assuming that $\mathbf{x}(0) \in \Omega_{\mathbf{x}}(0)$). Moreover, if $0 < \epsilon \leq 1$, and along a trajectory $\mathbf{x}(\cdot)$ of (7) for $\mathbf{x}(0) \in \bar{\Omega}_{\mathbf{x}}(0)$, it follows from Theorem 7 that $\xi(t, \bar{\mathbf{x}}(t)) \in [0, \epsilon) \Rightarrow \mathbf{n}(t) \in \mathcal{C}(\arccos(1 - \epsilon), \mathbf{n}^{cl}(t, \mathbf{p}(t), \mathbf{v}(t)))$ for all $t \geq 0$, and therefore, from Proposition 3, $\mathbf{n}(t) \in \mathcal{C}(\arccos(1 - \epsilon) + \alpha^\star, \mathbf{e}_3)$ for all $t \geq 0$. This means that ϵ, in conjunction with $\alpha^\star$, may be chosen such that, for example, $\mathbf{n}(\cdot)$ points upwards at all times (provided, obviously, that $\mathbf{n}(0)$ points upwards).

V. Simulations

Here we present a simulation for a quadrotor system whose objective is to track the trajectory $\mathbf{p}_d(t) = [2(1 - \exp^{-t})\cos(\frac{2\pi}{8}t)\ 0.5(1-\exp^{-t})\sin(\frac{2\pi}{8}t)\ 1]^T$. The details on how to transform a trajectory tracking problem for a quadrotor into (1) are found in [14]. In brief, $\mathbf{p}(t) = \mathbf{p}_Q(t) - \mathbf{p}_d(t)$, where $\mathbf{p}_Q(t)$ is the quadrotor's position; $\mathbf{n}(t) = \mathcal{R}_Q(t)\mathbf{e}_3$, where $\mathcal{R}_Q(t)$ is the quadrotor's rotation matrix (i.e., $\mathbf{n}(t)$ is the third body axis, along which a thrust force is provided to the quadrotor); and $\mathbf{g}(t) = g\mathbf{e}_3 + \ddot{\mathbf{p}}_d(t)$. For the chosen trajectory $\mathbf{g}(\cdot) \in \mathcal{B}(g\mathbf{e}_3, [2.5\ 2.5\ 0]^T)$, and $\|\dot{\mathbf{g}}(\cdot)\|_\infty \leq 6$ and $\|\ddot{\mathbf{g}}(\cdot)\|_\infty \leq 7$. The chosen control law $\mathbf{u}(\mathbf{p}, \mathbf{v})$ in (9), and the associated function $V_1(\mathbf{p}, \mathbf{v})$, are those proposed in [14], where $\mathbf{u}^\infty = [0.5\ 0.5\ 0.5]^T$; therefore, $\alpha^\star \leq 25°$. Also, $V_\theta(s) = 5s$, $k_\theta = 2$, $v_\omega = 10$ and $k_\omega = 2$. In Fig. 2, a simulation is presented where the quadrotor's reference frame starts at the inertial reference frame, with zero linear velocity and zero angular velocity. In Fig. 2(a), we see that trajectory tracking is accomplished, corresponding to $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{0}$, and verified in Fig. 2(b). In Figs. 2(d) and 2(e), we see that $\mathbf{n}(\cdot)$ and $\boldsymbol{\omega}(\cdot)$ converge asymptotically to $\mathbf{n}^\star(\cdot)$ and $\boldsymbol{\omega}^\star(\cdot)$ (see (15) and (18)). Finally, in Fig. 2(f), the control laws (19) and (31), along the trajectoty, are presented. Intuitively, $T(\cdot)$ changes slightly aroung g, so as to cancel gravity, and $\boldsymbol{\tau}(\cdot)$ changes slightly aroung $\mathbf{0}$, so as to guarantee that $\mathbf{n}(\cdot)$ tracks $\mathbf{n}^\star(\cdot)$, which is time-varying.

VI. Conclusion

In this paper, a controller is contructed for an underactuated system which is driven by a one dimensional acceleration/thrust along a direction vector, by a time-varying gravity, and by the angular acceleration of the direction vector. State and time-dependent control laws are construted for the thrust and the angular acceleration that guarantee that the position of the system is steered to the origin. This system provides a common framework for controlling many systems, such as quadrotors and slung load transportation systems. The proposed control laws can be applied to those systems given an appropriate change of coordinates.



(a) Trajectories: desired (blue) and real (black)

(b) Position $\mathbf{p}(t)$

(c) Velocity $\mathbf{v}(t)$

(d) Unit vectors $\mathbf{n}(t)$ and $\mathbf{n}^\star(t)$ in (15)

(e) Angular velocities $\boldsymbol{\omega}(t)$ and $\boldsymbol{\omega}^\star(t)$ in (18)

(f) Inputs $T(t) = T^{cl}(t, \bar{\mathbf{x}}(t))$ and $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^{cl}(t, \mathbf{x}(t))$

Fig. 2. Quadrotor tracking the trajectory $\mathbf{p}_d(t)$.

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