An improved result for the stability of interconnected systems based on a new Gersgorin-type criterion

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***Abstract*— In this short paper, a new Gersgorin-type criterion for eigenvalue inclusion sets is used to derive a new sufficient condition for the positive definiteness of real symmetric matrices. The result is less conservative than the classic strict diagonal dominance criterion imposed by Gersgorin's Theorem. We show that this result can be applied to various classes of interconnected systems, including systems consisting of subsystems with linear and nonlinear interconnection terms. Numerical examples as well as the intuition behind the derived results are also provided.**

I. Introduction

The study of inclusion sets for the eigenvalues of matrices with real or complex entries has a long history in the area of Lineal Algebra. The most celebrated result is Gersgorin's Theorem [4] which provides specific bounds for the eigenvalues of a matrix based on the difference of its diagonal elements with the corresponding absolute sums of the corresponding off-diagonal elements. Among the various applications of this well-known result, is that it provides a sufficient condition for the positive definiteness of real symmetric matrices. In particular, a straightforward corollary of Gersgorin's Theorem is the fact that strict diagonal dominance (a term that will be defined later) is a sufficient condition for positive definiteness.

A direct application of the aforementioned results to control theory is raised in the area of interconnected systems. For systems consisting of subsystems with linear interconnection terms, the positive definiteness of the matrix encoding these interconnections is a sufficient condition for the asymptotic stability of the whole system. This result has been extensively examined in [10]. Furthermore, for interconnected systems with nonlinear interconnection terms but with linear bounds on these interconnection terms, similar results have been derived, see for example [6],[1]. String stability of interconnected systems with linear bounds on the interconnection terms was also examined in [9],[11]. In addition, properties of positive-*semidefinite* matrices have been used to derive sufficient conditions for the class of interconnected systems that model the multi-agent rendezvous or consensus problem. This problem forces the agents to gather at a common point in the state space in a distributed manner. Results include but are not limited to [8], [7], [5].

The strict diagonal dominance property can be interpreted as follows. The diagonal elements are measures of the stability "degree" of the individual subsystems while the interconnection terms represent the perturbation terms that might be the cause of destabilization. As mentioned in [6], "if the degrees of stability for the isolated subsystems are larger as a whole than the strength of the interconnections, then the interconnected system has an asymptotically stable equilibrium at the origin". It is evident that the diagonal dominance property is only one interpretation of the above conclusion. In this paper, we provide a less conservative result which is based on a new Gersgorin-type theorem for eigenvalue inclusion regions that appeared in [3], [2]. The result is less conservative in the sense that the eigenvalue inclusion sets that are defined in [2], are actually subsets of the classic eigenvalue inclusion sets of Gersgorin's Theorem. A similar interpretation is derived which can be summarized as follows: "if at least one subsystem overcomes the total degree of instability of the other subsystems, then the interconnected system is asymptotically stable". This statement is made clear in the numerical examples section.

The rest of the paper is organized as follows: Section II includes the necessary background on Gersgorin's Theorem as well as the new Gersgorin-type theorem that appeared in [2]. Section III provides a new result on positive definiteness for real symmetric matrices which is used in section IV to derive sufficient condition for the stability of various classes of interconnected systems. Section V provides some numerical examples of the new approach to positive definiteness.

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II. Background

In this section the mathematical preliminaries needed for the subsequent analysis are presented. The following is based on [4].

Theorem 1: (*Gersgorin's Theorem*) Given a matrix $A \in \mathbb{R}^{n\times n}$ then all its eigenvalues lie in the union of n discs:

$$\bigcup_{i=1}^{n}\{z \in \mathbb{C} : |z - a_{ii}| \leq r_i(A)\} \triangleq \bigcup_{i=1}^{n}\Gamma_i(A) \triangleq \Gamma(A)$$

where

$$r_i(A) \triangleq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$$

Each of these discs is called a Gersgorin disc of A.
□
Another version of Gersgorin's Theorem is given by the following Theorem

Theorem 2: Given a matrix $A \in \mathbb{R}^{n\times n}$ and n positive real numbers $p_1, \ldots, p_n$ then all its eigenvalues of A lie in the union of n discs:

$$\bigcup_{i=1}^{n}\left\{z \in \mathbb{C} : |z - a_{ii}| \leq \frac{1}{p_i}\sum_{\substack{j=1\\ j\neq i}}^{n} p_j |a_{ij}|\right\}$$

□

A straightforward corollary of Gersgorin's Theorem involves the positive definiteness of a symmetric matrix:

Theorem 3: Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ then a sufficient condition for A to be positive definite is

$$a_{ii} > r_i(A), \ \forall i \in N$$

In this case the matrix A is called strictly diagonally dominant.
□
In this paper we exploit a less known result on eigenvalue inclusion sets that originally appeared in [3] and was recently given a generalization in [2]. The result of [2] that we use in this paper is summarized in the following Theorem:

Theorem 4: Given a matrix $A \in \mathbb{R}^{n\times n}$ then all its eigenvalues lie in the intersection of n discs:

$$D(A) \triangleq \bigcap_{i=1}^{n} D_i(A)$$

where for each $i = 1, \ldots, n$,

$$D_i(A) \triangleq \bigcup_{\substack{j=1\\ j\neq i}}^{n} V_{ij}(A)$$

where

$$V_{ij}(A) \triangleq \left\{\begin{array}{l} z : |z - a_{ii}| \cdot (|z - a_{jj}| - r_j(A) + |a_{ji}|) \leq \\ \leq r_i(A) \cdot |a_{ji}| \end{array}\right\}$$

□

It is shown in [2] that the new inclusion theorem is actually less conservative than the classic Gersgorin's Theorem, in the sense that

$$D_i(A) \subseteq \Gamma(A)$$

for all $i = 1, \ldots, n$.

The question that naturally arises is if we can draw a conclusion similar to Theorem 3 regarding the positive definiteness of a symmetric matrix in this case as well. This is the topic of the next section.

III. Main result

In this section we present the main result of this paper. In particular, we provide a sufficient condition for positive definiteness of a symmetric matrix based on the extension of Gersgorin's Theorem.

We assume that the matrix A has strictly positive diagonal elements. Excluding the case $z = a_{ii}$, each element z in the set $V_{ij}(A)$ satisfies by definition:

$$|z - a_{ii}| \cdot (|z - a_{jj}| - r_j(A) + |a_{ji}|) \leq r_i(A) \cdot |a_{ji}|$$

$$\overset{z\neq a_{ii}}{\Rightarrow} |z - a_{jj}| \leq \frac{r_i(A) \cdot |a_{ji}|}{|z - a_{ii}|} + r_j(A) - |a_{ji}| \triangleq f_{ij}(z)$$

We then have

$$|z - a_{jj}| \leq f_{ij}(z) \Rightarrow -f_{ij}(z) \leq z - a_{jj} \leq f_{ij}(z)$$

Hence

$$z \geq a_{jj} - f_{ij}(z) = a_{jj} - r_j(A) + |a_{ji}|\left(1 - \frac{r_i(A)}{|z - a_{ii}|}\right)$$

We note that the eigenvalues of a real symmetric matrix are real numbers. Being interested in the positive definiteness of the matrix A we take into account the worst case in which $z < a_{ii}$. For $z < a_{ii}$ we have

$$\begin{array}{l} z \geq a_{jj} - r_j(A) + |a_{ji}|\left(1 + \frac{r_i(A)}{z - a_{ii}}\right) \Rightarrow z(z - a_{ii}) \leq \\ \leq (a_{jj} - r_j(A))(z - a_{ii}) + |a_{ji}|(z - a_{ii} + r_i(A)) \end{array}$$

The last equation yields

$$z^2 - \rho_{ij} z + \pi_{ij} \leq 0$$

where

$$\rho_{ij} \triangleq a_{ii} + a_{jj} - r_j(A) + |a_{ji}|$$

and

$$\pi_{ij} \triangleq a_{ii}(a_{jj} - r_j(A)) + |a_{ji}|(a_{ii} - r_i(A))$$

From elementary calculus we now have

$$z^2 - \rho_{ij} z + \pi_{ij} \leq 0 \Rightarrow$$
$$\Rightarrow z \geq \frac{\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}}}{2}$$

since the term $\rho_{ij}^2 - 4\pi_{ij}$ is always non-negative. Indeed, simple calculations yield

$$\begin{aligned}
&\rho_{ij}^2 - 4\pi_{ij} = (a_{ii} + a_{jj})^2 + (r_j(A) - |a_{ji}|)^2 + \\
&+2(a_{ii} + a_{jj}) \cdot (|a_{ji}| - r_j(A)) - 4a_{ii}(a_{jj} - r_j(A)) \\
&-4|a_{ji}|(a_{ii} - r_i(A)) = \\
&= a_{ii}^2 + a_{jj}^2 - 2a_{ii}a_{jj} + |a_{ji}|^2 + r_j^2(A) + \\
&+2r_j(A) \cdot (a_{ii} - a_{jj}) - 2|a_{ji}| \cdot (a_{ii} - a_{jj}) = \\
&= (a_{ii} - a_{jj})^2 + r_j^2(A) + |a_{ji}|^2 + \\
&+2(r_j(A) - |a_{ji}|) \cdot (a_{ii} - a_{jj}) = \\
&= (a_{ii} - a_{jj})^2 + (r_j(A) - |a_{ji}|)^2 + \\
&+2(r_j(A) - |a_{ji}|) \cdot (a_{ii} - a_{jj}) + 2|a_{ji}| \cdot r_j(A) = \\
&= (a_{ii} - a_{jj} + r_j(A) - |a_{ji}|)^2 + 2|a_{ji}| \cdot r_j(A) \geq 0
\end{aligned}$$

From Theorem 4, we have that

$$\sigma(A) \subseteq \bigcap_{i=1}^{n} \bigcup_{\substack{j=1 \\ j \neq i}}^{n} V_{ij}(A)$$

where $\sigma(A)$ is the spectrum of A. Hence a sufficient condition for the positive definiteness of the matrix A is given by

$$\max_{i=1,\ldots,n} \left\{ \min_{\substack{j=1,\ldots,n \\ j \neq i}} \left\{ \frac{\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}}}{2} \right\} \right\} > 0$$

The result is summarized in the following Theorem:

Theorem 5: Suppose that the elements of a symmetric matrix $A \in \mathbb{R}^{n \times n}$ satisfy the following conditions:

$$a_{ij} = \begin{cases} > 0, & \text{if} i = j \\ a_{ji}, & \text{if} i \neq j \end{cases}$$

Then a sufficient condition for the positive definiteness of A is given by

$$\max_{i=1,\ldots,n} \left\{ \min_{\substack{j=1,\ldots,n \\ j \neq i}} \left\{ \frac{\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}}}{2} \right\} \right\} > 0$$

where

$$\rho_{ij} \triangleq a_{ii} + a_{jj} - r_j(A) + |a_{ji}|$$

and

$$\pi_{ij} \triangleq a_{ii}(a_{jj} - r_j(A)) + |a_{ji}|(a_{ii} - r_i(A))$$

□

The result can be also expressed as follows: there exists *at least one* $i = 1, \ldots, n$ for which the condition

$$\min_{\substack{j=1,\ldots,n \\ j \neq i}} \left\{ \frac{\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}}}{2} \right\} > 0$$

holds. It is clear that this result is less conservative than the classic Gersgorin's Theorem, something expected, since each subset $D_i(A)$ is a subset of the Gersgorin region $\Gamma(A)$.

IV. Application to interconnected systems

In the next paragraphs, we present two classes of interconnected systems in which the new results presented previously can be applied to ensure asymptotic stability.

A. Subsystems with linear interconnection terms

We first consider interconnected systems of the form

$$\dot{x}_i = -a_{ii}x_i - \sum_{j \neq i} a_{ij}x_j, \; i = 1, \ldots, N \tag{1}$$

where, without loss of generality, we assume that $x_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. The coefficients a_{ij} are assumed to satisfy the following properties:

$$a_{ii} > 0, a_{ij} = a_{ji}$$

for all $i, j = 1, \ldots, n$ with $j \neq i$. The stack vector of the subsystems' state is denoted by

$$x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T$$

Equation (1) is now written as

$$\dot{x} = -Ax \tag{2}$$

where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$.

Elementary linear control theory states that the interconnected system (2) is asymptotically stable, provided that the matrix A is positive definite. A test for the positive definiteness of A can be provided by Theorem 5.

B. Linearly bounded nonlinear interconnection terms

Consider now interconnected systems of the form

$$\dot{x}_i = f_i(x_i) + \sum_{j \neq i} g_{ij}(x_j),\ i = 1, \ldots, N \quad (3)$$

where for each $i, j = 1, \ldots, N$, f_i and g_{ij} are sufficiently smooth to assure local existence and uniqueness of solution in the domain of interest, and $f_i(0) = 0, g_{ij}(0) = 0$, so that 0 is an equilibrium point of the interconnected system. Assuming that the "isolated" subsystem, i.e. the system

$$\dot{x}_i = f_i(x_i)$$

is asymptotically stable to the origin for each i, the terms g_{ij} can be interpreted as disturbance elements on the i-th subsystem, induced by the existence of the other subsystems.

In the vein of [6], we consider

$$V = \sum_i V_i$$

as a composite Lyapunov function for the whole subsystem, where each V_i is a suitable positive definite smooth Lyapunov function for each isolated subsystem, i.e. satisfies

$$\frac{\partial V_i}{\partial x_i} f_i(x_i) \leq -a_{ii}\phi_i^2(x_i)$$

where a_{ii} are positive constants, and $\phi(x_i)$ is a continuous positive definite function. Furthermore, assume that the interconnection terms satisfy the following relation:

$$\sum_{j \neq i} \frac{\partial V_i}{\partial x_i} g_{ij}(x_j) \leq -a_{ij}\phi_i(x_i)\phi_j(x_j)$$

for all $i = 1, \ldots, n$. We further assume that the interconnection terms satisfy $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, N$. Please note that no assumptions are made regarding the sign definiteness of the interconnection terms.

Computing now the time derivative of the composite Lyapunov function V we get

$$\begin{aligned} \dot{V} &= \sum_i \left(\tfrac{\partial V_i}{\partial x_i} f_i(x_i) + \sum_{j \neq i} \tfrac{\partial V_i}{\partial x_i} g_{ij}(x_j) \right) \leq \\ &\leq \sum_i \left(-a_{ii}\phi_i^2(x_i) - a_{ij}\phi_i(x_i)\phi_j(x_j) \right) = \\ &= -\phi^T A \varphi \end{aligned}$$

where $A = [a_{ij}]$ and

$$\phi = \begin{bmatrix} \phi_1(x_1) & \ldots & \phi_N(x_N) \end{bmatrix}^T$$

The positive definiteness of A is a sufficient condition for the asymptotic stability of the interconnected system in this case as well and it can be tested via Theorem 5.

V. Numerical Examples

In this section we consider two numerical examples of a matrix who has the form of the A matrices of the previous paragraphs and apply the result of Theorem 5. The results will help us in extracting an intuitive result regarding the stability of interconnected systems based on this Theorem.

We consider an interconnected linear system of the form

$$\begin{aligned} \dot{x}_1 &= -ax_1 - 3x_2 + 10x_3 + 6x_4 \\ \dot{x}_2 &= -3x_1 - 3x_2 - x_3 \\ \dot{x}_3 &= 10x_1 - x_2 - 5x_3 + 3x_4 \\ \dot{x}_4 &= 6x_1 + 3x_3 - 4x_4 \end{aligned}$$

Writing the system in the stack vector form $\dot{x} = -Ax$, we have

$$A = \begin{bmatrix} a & 3 & -10 & -6 \\ 3 & 3 & 1 & 0 \\ -10 & 1 & 5 & -3 \\ -6 & 0 & -3 & 4 \end{bmatrix}$$

Note that the condition

$$\min_{\substack{j=1,\ldots,n \\ j \neq i}} \left\{ \frac{\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}}}{2} \right\} > 0$$

can only hold for $\rho_{ij} > 0$. This yields

$$\rho_{ij} - \sqrt{\rho_{ij}^2 - 4\pi_{ij}} > 0 \Leftrightarrow \pi_{ij} > 0$$

Computing now in the above example the terms π_{1j} for $j = 2, 3, 4$ we get

$$\pi_{12} = 2a - 57, \pi_{13} = a - 190, \pi_{14} = a - 114$$

Based on the result of Theorem 5, a sufficient condition for the positive definiteness of the matrix A is $a > 190$. Indeed for $a = 191$, we have $\lambda_{min}(A) = 0.5071$. Please note that the eigenvalue inclusion regions induced by Theorems 1, 2 do not guarantee that the eigenvalues of A are positive.

The above example provides the following intuition for the result of this paper. First of all, the diagonal element corresponding to the subsystem on which the minimization procedure of Theorem 5 takes place, should be relatively large. Moreover, the absolute values of the interconnection terms of this subsystem with the other subsystems should also be relatively large with respect to the interconnection terms between the other subsystems. In summary, the following statement is derived based on the result of Theorem 5:"if at least one subsystem overcomes the total degree of instability of the other subsystems, then the interconnected system is asymptotically stable".

The second example involves a 3×3 matrix which might represent the interconnection terms of both cases examined in the previous section . Suppose that this matrix has the form

$$A = \begin{bmatrix} a & -3 & 4 \\ -3 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$$

Note again that this matrix does not satisfy the conditions of Gersgorin's Theorems 1,2.

Computing now in the above example the terms π_{1j} for $j = 2, 3$ we get

$$\pi_{12} = a - 21, \pi_{13} = 2a - 28$$

Based on the result of Theorem 5, a sufficient condition for the positive definiteness of the matrix A is $a > 21$. Indeed for $a = 22$, we have $\lambda_{min}(A) = 0.3328$.

VI. Conclusions

In this short paper, a new Gersgorin-type criterion for eigenvalue inclusion sets was used to derive a new sufficient condition for the positive definiteness of real symmetric matrices. The result is less conservative than the classic strict diagonal dominance criterion imposed by Gersgorin's Theorem. We showed that this result can be applied to various classes of interconnected systems, including systems consisting of subsystems with linear and nonlinear interconnection terms. Numerical examples were also provided to support the derived results.

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