Inverse agreement protocols with application to distributed multi-agent dispersion

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Abstract

We propose a distributed inverse agreement control law for multiple kinematic agents that forces the team members to disperse in the workspace. Both the cases of an unbounded and a circular, bounded workspace are considered. In the first case, we show that the closed-loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It is proved that this lower bound coincides with the agents' sensing radius. In the case of a bounded circular workspace, the control law is redefined to force the agents to remain within the workspace boundary. Moreover the proposed control design guarantees collision avoidance between the team members in all cases. The results are supported through relevant computer simulations.

I. Introduction

The emerging use of large-scale multi-robot/vehicle systems in various applications has raised recently the need for the design of control laws that force a team of multiple vehicles/robots (from now on called "agents") to achieve various goals. As the number of agents increases, centralized designs fail to guarantee robustness and are harder to implement than decentralized ones, which also provide a reduce in the computational complexity of the feedback scheme. Among the various objectives of the control design, convergence of the team to a common configuration, also known as the agreement problem, is a design specification that has been extensively pursued. Many distributed control schemes that achieve multi-agent agreement are

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found in literature; see [1],[2],[15],[4],[9], [14],[16],[8] for some recent results. In this paper, we propose an algorithm for swarm dispersion which can be considered as an inverse agreement problem. Each agent follows a flow, whose inverse leads the multi-agent team to agreement. The design is distributed, since each agent only knows the relative positions of agents located within its sensing zone at each time. The sensing zone is a circular area around each agent whose radius is common for all agents. The application of this inverse agreement strategy is dispersion of the team members in the workspace, i.e., convergence to a configuration where the minimum distance between the swarm members is bounded from below by a controllable lower bound. It is shown that this lower bound coincides with the radius of the sensing zone of the agents in the case of an unbounded workspace. Furthermore, the results are extended in order to take into account the workspace boundary for the case of a circular bounded workspace.

Applications of the dispersion algorithm include coverage control [5],[12], [11], and optimal placement of a multi-robot team in small areas [17], [3]. The paper uses inverse agreement for kinematic agents and this results in swarm dispersion. This behavior is similar to [5], however, the technique used here is different than the geometric optimization technique of the aforementioned paper. Moreover, as stated above, the driving force behind our effort was to provide a first step to a framework that uses inverse potential field based control laws for various multi-agent control tasks. In summary, in this paper it is shown that inverse consensus/agreement algorithms can be used to provide solutions to various problems in multi-agent control. This is a topic of future research directions. We finally note that a conference version of the paper appeared in [6], while an extension that takes into account nonholonomic constraints in [7].

The rest of the paper is organized as follows: Section II describes the problem treated in this paper. The swarm dispersion methodology is presented in Section III. The case of a bounded workspace is discussed in Section IV, while relevant computer simulations are included in Section V. Section VI summarizes the results of this paper and indicates current research efforts.

II. System and Problem Description

Consider a system of N point agents operating in the same workspace $W \subset \mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. The configuration space is spanned by $q = [q_1^T, \ldots, q_N^T]^T$. The motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = \{1, \ldots, N\} \tag{1}$$

where u_i denotes the velocity (control input) for each agent.

We assume that each agent has sense of agents within a circle of radius d around the agent. This circle is called the *sensing zone* of each agent i and d its *sensing radius*, assumed common for all agents. We denote by N_i the subset of $\mathcal{N}$ that includes the agents that agent i can sense at each time instant, i.e., $N_i = \{j \in \mathcal{N}, j \neq i : \|q_i - q_j\| \leq d\}$. For the dispersion objective, we equip each agent with a repulsive potential with respect to each other agent within its sensing zone. No global knowledge is imposed to any of the swarm members. The main result states that the closed-loop system converges to a configuration where the sensing zone of each agent is empty, i.e., every agent is located at a distance no less than d from every other. In addition, collision avoidance is guaranteed. We assume first an unbounded and then a bounded workspace.

The dispersion potential between agents i and j is given by $\gamma_{ij}(\beta_{ij}) = \dfrac{1}{2}\beta_{ij}$, for $0 \leq \beta_{ij} \leq c^2$, $\gamma_{ij}(\beta_{ij}) = \phi(\beta_{ij})$, for $c^2 \leq \beta_{ij} \leq d^2$, and $\gamma_{ij}(\beta_{ij}) = h$, for $d^2 \leq \beta_{ij}$, where $\beta_{ij} = \|q_i - q_j\|^2$ is the squared distance between agents i and j. The positive scalars c, d, h and the function ϕ are chosen in such a way so that γ_{ij} is everywhere continuously differentiable. In this paper, we use a polynomial function: $\phi(x) = a_2 x^2 + a_1 x + a_0$. The parameters of ϕ satisfy the differentiability requirement for γ_{ij}, provided that they fulfil the following relations: $a_2 = \frac{1}{4(c^2-d^2)}, a_1 = \frac{d^2}{2(d^2-c^2)}, a_0 = \frac{c^4}{4(c^2-d^2)}, h = \frac{d^2+c^2}{4}$. The gradient and the partial derivative of γ_{ij} are computed by $\nabla\gamma_{ij} = 2\rho_{ij} D_{ij} q$ and $\frac{\partial\gamma_{ij}}{\partial q_i} = 2\rho_{ij}\left(D_{ij}\right)_i q$, where $\rho_{ij} \triangleq \frac{\partial\gamma_{ij}}{\partial\beta_{ij}}$, and the matrices D_{ij},$(D_{ij})_i$, for $i < j$, are given by $D_{ij} = \tilde{D}_{ij} \otimes I_2$, where $\left(\tilde{D}_{ij}\right)_{ii} = \left(\tilde{D}_{ij}\right)_{jj} = 1$, $\left(\tilde{D}_{ij}\right)_{ij} = \left(\tilde{D}_{ij}\right)_{ji} = -1$ and $\left(\tilde{D}_{ij}\right)_{kl} = 0$ for $k, l \neq i, j$, and $\left(D_{ij}\right)_i = \left[\begin{array}{ccccc} O_{1\times(i-1)} & 1 & O_{1\times(j-i-1)} & -1 & O_{1\times(N-j)} \end{array}\right] \otimes I_2$ where $\otimes$ denotes the Kronecker product between two matrices [10]. The definition of D_{ij},$(D_{ij})_i$, for $i > j$ is straightforward. It can easily be shown that $\rho_{ij} > 0$ for $0 < \beta_{ij} < d^2$ and $\rho_{ij} = 0$ for $\beta_{ij} \geq d^2$. We would like to note that the choice of ϕ is not restricting. Any function ϕ fulfilling the desired properties can be used. Specifically, ϕ should be a continuously differentiable function that renders γ_{ij} continuously differentiable (in particular, it should satisfy $\phi(c^2) = \lim\limits_{\beta_{ij}\to(c^2)^-} \gamma_{ij}(\beta_{ij})$,$\phi'(c^2) = \lim\limits_{\beta_{ij}\to(c^2)^-} \gamma'_{ij}(\beta_{ij})$, and $\phi(d^2) = \lim\limits_{\beta_{ij}\to(d^2)^+} \gamma_{ij}(\beta_{ij}), \phi'(d^2) = \lim\limits_{\beta_{ij}\to(d^2)^+} \gamma'_{ij}(\beta_{ij})$) and should fulfil $\rho_{ij} > 0$ for $c^2 \leq \beta_{ij} < d^2$. The choice of this function was made to provide the means for the design procedure.

III. Swarm Dispersion control design

A. Tools from Matrix Theory

For an undirected graph $\mathcal{G} = (V, E)$ with n vertices we denote by V its set of vertices and by E its set of edges. If there is an edge connecting two vertices i, j, i.e. $(i, j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph $\mathcal{G}$, then $\mathcal{G}$ is called *connected* (otherwise it is called *disconnected*). The *undirected graph* $\mathcal{G} = (V, E)$ *corresponding to a real symmetric* $n \times n$ *matrix* M is a graph with n vertices indexed by $1, \ldots, n$ such that there is an edge between vertices $i, j \in V$ if and only if $M_{ij} \neq 0$, i.e. $(i, j) \in E \Leftrightarrow M_{ij} \neq 0$.

A $n \times n$ real symmetric matrix with non-positive off-diagonal elements and zero row sums is called a *symmetric Metzler* matrix. It is shown in [13] that all the eigenvalues of a symmetric Metzler matrix are non-negative and zero is a trivial eigenvalue. The multiplicity of zero as an eigenvalue of a symmetric Metzler matrix is one if and only if the corresponding undirected graph is connected. The trivial corresponding eigenvector is the vector of ones, $\vec{\mathbf{1}}$.

B. Swarm Dispersion with collision avoidance

We propose the following control law

$$u_i = -\sum_{j \in N_i} \frac{\partial \left(1/\gamma_{ij}\right)}{\partial q_i} \Rightarrow u_i = -\sum_{j \in N_i} \left(-\frac{1}{\gamma_{ij}^2}\right) \frac{\partial \gamma_{ij}}{\partial q_i} = \sum_{j \in N_i} \frac{2\rho_{ij}}{\gamma_{ij}^2} \left(D_{ij}\right)_i q$$

which can be rewritten as

$$u_i = \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} \left(D_{ij}\right)_i q \qquad (2)$$

since $\rho_{ij} = 0$ for $\beta_{ij} > d^2$. It should be noted that each agent takes into account only the agents within its sensing zone at each time instant. We then have $\dot{q} = 2\left(R_2 \otimes I_2\right) q$, where the matrix R_2 is given by $\left(R_2\right)_{ii} = \sum_{j \neq i} \frac{\rho_{ij}}{\gamma_{ij}^2}$, and $\left(R_2\right)_{ij} = -\frac{\rho_{ij}}{\gamma_{ij}^2}$ for $i \neq j$. We consider $V = \sum_i \sum_{j \neq i} \frac{1}{\gamma_{ij}}$ as a candidate Lyapunov function. Its gradient is computed by

$$\nabla V = \sum_i \sum_{j \neq i} \left(-\frac{1}{\gamma_{ij}^2}\right) \nabla \gamma_{ij} = -\sum_i \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} D_{ij} q = -2\left(R_1 \otimes I_2\right) q$$

where the matrix R_1 is given by $(R_1)_{ii} = \sum_{j\neq i} \frac{\rho_{ij}}{\gamma_{ij}^2} + \sum_{j\neq i} \frac{\rho_{ji}}{\gamma_{ji}^2}$, and $(R_1)_{ij} = -\frac{\rho_{ij}}{\gamma_{ij}^2} - \frac{\rho_{ji}}{\gamma_{ji}^2}$ for $i \neq j$. We have $\frac{\rho_{ij}}{\gamma_{ij}^2} = \frac{\rho_{ji}}{\gamma_{ji}^2} \Rightarrow R_1 = 2R_2$. The time derivative of V is now calculated as follows

$$\begin{aligned} &\dot{V} = (\nabla V)^T \cdot \dot{q} = \left(-2\left(R_1 \otimes I_2\right) q\right)^T \cdot 2\left(R_2 \otimes I_2\right) q \\ &\overset{R_1=2R_2}{\Rightarrow} \dot{V} = -8\left\|\left(R_2 \otimes I_2\right) q\right\|^2 \leq 0 \end{aligned} \tag{3}$$

The first result of this section establishes collision avoidance between the swarm members:

Lemma 1: Consider the system of multiple kinematic agents (1) driven by the control law (2) and starting from a feasible set of initial conditions $\mathcal{I}(q) = \{q|\, \|q_i - q_j\| > 0, \forall i,j \in \mathcal{N}, i \neq j\}$. Then the set $\mathcal{I}(q)$ is invariant for the trajectories of the closed-loop system.

Proof: For every $q(0) \in \mathcal{I}(q)$, the time derivative of V remains non-positive for all $t \geq 0$, by virtue of (3). Hence $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$. Since $V \to \infty$ if and only if $\|q_i - q_j\| \to 0$ for at least one pair $i,j \in \mathcal{N}$, we conclude that $q(t) \in \mathcal{I}(q)$, for all $t \geq 0$. $\diamondsuit$

Remark 1: We note here that the control law can be modified to guarantee that the distance between any two agents remains larger than any positive threshold c, i.e., that $\|q_i - q_j\| > c$ for a fixed $c > 0$. To achieve this, the function β_{ij} would have to be redefined as $\beta_{ij} = \|q_i - q_j\| - c^2$. The rest of the analysis would be the same with the set $\mathcal{I}(q)$ defined accordingly.

Thus, collision avoidance is guaranteed. The control design however is also related to the final configurations of the swarm members. The main result of this section is stated as:

Theorem 2: Consider the system (1) driven by the control (2) and starting from a set of initial conditions $\mathcal{I}(q) \cap \mathcal{F}(q)$ where $\mathcal{F}(q) = \{q|\, \|q_i - q_j\| \leq (N-1)\, d^*, \forall i,j \in \mathcal{N}, i \neq j\}$, where $d^* > d$ is chosen arbitrarily, and $\mathcal{I}(q)$ was defined in Lemma 1. Then the agents reach a static configuration (all agents eventually stop) which satisfies $\|q_i - q_j\| \geq d, \forall i,j \in \mathcal{N}, i \neq j$.

Proof: Since the set of initial conditions is included in $\mathcal{I}(q)$, we have $q_i(t) \neq q_j(t)$, for all $i,j \in \mathcal{N}, i \neq j$, and for all $t \geq 0$, by virtue of Lemma 1. We take V as a candidate Lyapunov function. V is continuously differentiable within $\mathcal{I}$. Its time derivative is given by (3): $\dot{V} = -8\left\|\left(R_2 \otimes I_2\right) q\right\|^2 \leq 0$. Pick $d^* > d$. It is easy to see that since $\rho_{ij} = 0$ whenever $\beta_{ij} > d$, the set $\|q_i - q_j\| \leq (N-1)\, d^*$ for all $i,j \in \mathcal{N}$ is positively invariant for the trajectories of the closed loop system. By virtue of Lemma 1, $\mathcal{I}(q) \cap \mathcal{F}(q)$ is also positively invariant. Since this set is closed and bounded, we can apply LaSalle's invariance principle.

By LaSalle's Invariance Principle, the trajectories of the closed loop system converge to the largest invariant subset of the set $S = \left\{q|\dot{V} = 0\right\} = \{q|\left(R_2 \otimes I_2\right) q = 0\}$. Note that within S, we have $\dot{q} = u = 2\left(R_2 \otimes I_2\right) q = 0 \Rightarrow u_i = 0$, for all $i \in \mathcal{N}$, i.e., all agents eventually stop.

We show next that the largest invariant subset of S is the set $S_0 = \{q | \rho_{ij} = 0, \forall i, j \in \mathcal{N}, i \neq j\}$. Clearly, S_0 is a subset of S which is invariant for the trajectories of the closed-loop system. Suppose now that $\rho_{ij} > 0$ for some pairs of the swarm members. We denote the undirected graph corresponding to the matrix R_2 by $\mathcal{G}(R_2)$. The assumption that $\rho_{ij} > 0$ for some pairs of the team members guarantees that $\mathcal{G}(R_2)$ has at least one edge. The graph $\mathcal{G}(R_2)$ can now be decomposed into its connected components, and let m be their number. Since the graph is undirected, no vertex can belong to two different components simultaneously. Ignoring the connected components containing only one vertex (i.e. vertices k for which $\rho_{kj} = 0$ for all $j \neq k$), and rearranging the agent indices accordingly, equation $\left(R_2 \otimes I_2\right) q = 0$ can be decomposed into m different equations, each of which corresponds to a different connected component of $\mathcal{G}(R_2)$. Specifically for the connected component containing agents/vertices $\{i_1, i_2, \ldots, i_l\}, i_j \in \mathcal{N}, j = 1, \ldots l$ with $l \leq n$ we have $\left(\tilde{R}_2 \otimes I_2\right) \tilde{q} = 0$, where $\tilde{q} = \begin{bmatrix} q_{i_1}^T & \ldots & q_{i_l}^T \end{bmatrix}^T$ and the $l \times l$ matrix $\tilde{R}_2$ has the same form as R_2 taking into account the set of agents $\{i_1, i_2, \ldots, i_l\}$. By denoting $\tilde{x}, \tilde{y}$ the stack vectors of $\tilde{q}$ in the x, y directions, we have $\left(\tilde{R}_2 \otimes I_2\right) \tilde{q} = 0 \Rightarrow \tilde{R}_2 \tilde{x} = \tilde{R}_2 \tilde{y} = 0$. The symmetric matrix $\tilde{R}_2$ has zero row sums and non-positive off-diagonal elements, i.e., it is a symmetric Metzler matrix. As mentioned in Section IIIA, the eigenvalues of $\tilde{R}_2$ are nonnegative and zero is the smallest eigenvalue. Furthermore, since $\tilde{R}_2$ corresponds to a connected graph (a connected component of $\mathcal{G}(R_2)$), zero is a simple eigenvalue of $\tilde{R}_2$ with corresponding eigenvector the vector of ones, $\overrightarrow{\mathbf{1}}$. Equations $\tilde{R}_2 \tilde{x} = \tilde{R}_2 \tilde{y} = 0$ now guarantee that both $\tilde{x}, \tilde{y}$ are eigenvectors of $\tilde{R}_2$ belonging to span$\{\overrightarrow{\mathbf{1}}\}$. Hence all elements of $\tilde{q}$ assume the same value, implying that all agents converge to a common configuration at steady state. However this is impossible, since, due to the invariance of $\mathcal{I}(q)$, no trajectory of the closed loop system starting from $\mathcal{I}(q)$ can ever leave this set, i.e., $q_i(t) \neq q_j(t)$ for all $t \geq 0$. We conclude that the largest invariant subset of S is S_0. Since $\rho_{ij} = 0$ only for $\|q_i - q_j\| \geq d$, the proof is complete. $\diamondsuit$

Hence the system converges to a configuration in which each agent is located at a distance no less than d from every other agent in the group. Thus, since any pair of agents is located at least at a distance d from each other, each agent occupies a disc of radius $d/2$ in which no other agent is present. In other words, the agents are dispersed to N disjoint discs of radius $d/2$.

IV. The Bounded Workspace Case

The previous case involved a dispersion algorithm for multiple agents in an unbounded workspace. In practical applications such as coverage control and sensor deployment the problem is to redefine the algorithm in order to take into account the workspace boundary. In this paper, we consider the case of a circular boundary of radius R_W. The purpose is to construct an inverse agreement control law that forces the dispersing agents to remain within the workspace limits.

The same potential to the one for the inter-agent dispersion potential is used for the agent-boundary repulsion potential. Copying with the limited sensing capabilities of the agents, the repulsive potential of each agent with respect to the boundary of the workspace is given by $\gamma_{ib}\left(\beta_{ib}\right)=\frac{1}{2}\beta_{ib}$, for $0\leq\beta_{ib}\leq c_b^2$, $\gamma_{ib}\left(\beta_{ib}\right)=\varphi_b\left(\beta_{ib}\right)$, for $c_b^2\leq\beta_{ib}\leq d_b^2$, and $\gamma_{ib}\left(\beta_{ib}\right)=h_b$, for $d_b^2\leq\beta_{ib}$, where $\beta_{ib}=\left\|q_i-q_{i,\min}\right\|^2$, $d_b<d$ and $q_{i,\min}=\arg\min\limits_{q\in\partial W}\left\|q_i-q\right\|^2$. Note that $q_{i,\min}$ is continuous for all i due to W being circular. The positive scalars h_b, c_b and the function φ_b are defined so that γ_{ib} is rendered everywhere continuously differentiable. Each agent has to have knowledge of the workspace boundary only when located at a distance smaller than d_b from it.

The control law for agent i is now redefined as $u_i=-\sum\limits_{j\in N_i}\dfrac{\partial\left(1/\gamma_{ij}\right)}{\partial q_i}-\dfrac{\partial\left(1/\gamma_{ib}\right)}{\partial q_i}$. Using the notation $\rho_{ib}=\dfrac{\partial\gamma_{ib}}{\partial\beta_{ib}}$, the control law can be rewritten as

$$u_i=\sum_{j\neq i}\frac{2\rho_{ij}}{\gamma_{ij}^2}\left(D_{ij}\right)_i q+2\frac{\rho_{ib}}{\gamma_{ib}}\left(q_i-q_{i,\min}\right)\tag{4}$$

since $\dfrac{\partial\left(1/\gamma_{ib}\right)}{\partial q_i}=-\dfrac{1}{\gamma_{ib}^2}\dfrac{\partial\gamma_{ib}}{\partial q_i}=-2\dfrac{\rho_{ib}}{\gamma_{ib}}\left(q_i-q_{i,\min}\right)$. Note that $\rho_{ib}=0$ for $\beta_{ib}>d_b^2$ and $\rho_{ib}>0$ for $\beta_{ib}\leq d_b^2$. In stack vector form we then have $\dot{q}=2\left(R_3\otimes I_2\right)q-2\left(R_4\otimes I_2\right)q_{\min}$, where $R_3=R_2+\operatorname{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2},\ldots,\frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$ and $R_4=\operatorname{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2},\ldots,\frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$. We also denote by $q_{\min}$ the stack vector of all q_{min}^i. Similarly to the case of an unbounded workspace, using $V_b=\sum\limits_i\sum\limits_{j\neq i}\frac{1}{\gamma_{ij}}+\sum\limits_i\frac{1}{\gamma_{ib}}$ as a candidate Lyapunov function and computing its gradient with respect to q we get$\nabla V_b=-4\left(R_3\otimes I_2\right)q+4\left(R_4\otimes I_2\right)q_{\min}$. The time derivative of V_b is now given by

$$\dot{V}_b=\left(\nabla V_b\right)^T\cdot\dot{q}=-8\left\|\left(R_3\otimes I_2\right)q-\left(R_4\otimes I_2\right)q_{\min}\right\|^2\leq 0\tag{5}$$

We show next that the workspace interior is invariant:

Lemma 3: Consider the system (1) driven by the control law (4) and starting from the set $\mathcal{I}(q)\cap\mathcal{J}(q)$ where $\mathcal{J}\left(q\right)=\left\{q|q\in\operatorname{int}\left(W\right)\triangleq W\backslash\partial W\right\}$ is the interior of W and $\mathcal{I}(q)$ was defined previously. Then $\mathcal{I}(q)\cap\mathcal{J}(q)$ is invariant for the trajectories of the closed-loop system.

Proof: The invariance of $\mathcal{I}(q)$ was shown in Lemma 1. For every $q(0) \in \mathcal{I}(q) \cap \mathcal{J}(q)$, the time derivative of V_b remains non-positive for all $t \geq 0$, by virtue of (5). Hence $V_b(q(t)) \leq V_b(q(0)) < \infty$ for all $t \geq 0$. Since $V_b \to \infty$ whenever $q_i \to q_{i,\min}$ for at least one agent $i \in \mathcal{N}$, and the latter implies $q \to \partial W$, we conclude that $q(t) \in \mathcal{J}(q)$, for all $t \geq 0$. $\diamondsuit$

Thus, if the agents start within $\operatorname{int}(W)$, they are forced to remain within it. Also, Lemma 1 holds and hence collisions are avoided. Similar results can now be derived from the analysis held previously. We first state that the agents reach a configuration where $u_i = 0$ for all i:

Corollary 4: Consider the system of multiple kinematic agents (1) driven by the control law (4) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$. Then the system reaches a configuration in which $u = 0$, where u is the stack vector of u_i's, i.e., $u_i = 0$ for all $i \in \mathcal{N}$.

Proof: By Lemma 3, the set $\mathcal{J}(q)$ is closed and bounded for the trajectories of the closed-loop system. Equation (5) guarantees that $\dot{V}_b$ is negative semidefinite. By LaSalle's Invariance Principle, the trajectories of the closed-loop system reach the largest invariant subset of the set $S_b = \left\{ q | \dot{V}_b = 0 \right\} = \{ q | \left(R_3 \otimes I_2\right) q - \left(R_4 \otimes I_2\right) q_{\min} = 0 \}$. Within S_b, we have $\dot{q} = u = 2\left(R_3 \otimes I_2\right) q - 2\left(R_4 \otimes I_2\right) q_{\min} = 0$. Hence $u_i = 0$ for all $i \in \mathcal{N}$. $\diamondsuit$

From the proof of Corollary 4, the system converges to the largest invariant subset of the set S_b. Note that Lemma 3 holds for arbitrarily small c_b, d_b. We now show that the control law is related to the final relative positions of the agents in a manner similar to the unbounded case, whenever the parameters c_b, d_b tend to zero. For $c_b, d_b \to 0$, we have that either $q_i \to q_{i,\min}$, or $\rho_{ib} \to 0$, for those agents that do not satisfy the condition $q_i \to q_{i,\min}$. Thus, in this case

$$
\begin{aligned}
&\left(R_3 \otimes I_2\right) q - \left(R_4 \otimes I_2\right) q_{\min} = \\
&= \left(R_2 \otimes I_2\right) q - \left(R_4 \otimes I_2\right) \left(q - q_{\min}\right) \\
&= \left(R_2 \otimes I_2\right) q - \left(\left(\operatorname{diag} \left\{ \tfrac{\rho_{1b}}{\gamma_{1b}^2}, \ldots, \tfrac{\rho_{Nb}}{\gamma_{Nb}^2} \right\} \right) \otimes I_2 \right) \left(q - q_{\min}\right) \\
&= \left(R_2 \otimes I_2\right) q - \begin{bmatrix} \tfrac{\rho_{1b}}{\gamma_{1b}^2} \left(q_1 - q_{1,\min}\right) & \ldots & \tfrac{\rho_{1b}}{\gamma_{1b}^2} \left(q_N - q_{N,\min}\right) \end{bmatrix}^T \\
&= \left(R_2 \otimes I_2\right) q
\end{aligned}
$$

since for each $i \in \mathcal{N}$, we have either $q_i \to q_{i,\min}$, or $\rho_{ib} \to 0$, for $c_b, d_b \to 0$ as discussed above.

Therefore the set S_b coincides with the set S of the proof of Theorem 2. As proved in that Theorem, the largest invariant subset within S is the set $S_0 = \{ q | \rho_{ij} = 0, \forall i, j \in \mathcal{N}, i \neq j \} = \{ q | \left\| q_i - q_j \right\| \geq d, \forall i, j \in \mathcal{N}, i \neq j \}$. Hence the system reaches a configuration it which all agents remain within the workspace bounds and each agent is located at a distance no less than d from every other agent in the group, *provided that such configuration exists within the workspace bounds*. This result is stated as follows:

Theorem 5: Consider the system (1) driven by (4) and starting from $\mathcal{I}(q) \cap \mathcal{J}(q)$ and assume that $B(q) = \{q \in \mathrm{int}(W) \mid \|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j\} \neq \emptyset$. Then the system reaches a configuration in which all agents remain within $\mathrm{int}(W)$, and $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$.

Remark 2: Similarly to the unbounded case, $B(q)$ being non-empty corresponds to a situation where each agent occupies a $d/2$-disc at steady state. Whenever $B(q)$ is empty, i.e. there does not exist a configuration in the interior of the workspace such that $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$, the workspace is not large enough to fulfill the above condition, and the system converges to a configuration that minimizes V_b, respecting the constraint that agents remain within $\mathrm{int}(W)$. Thus, some of the $d/2$-discs may overlap. This is visualized via an example in the next section.

Remark 3: The equilibrium property of Theorem 5 holds only as c_b, d_b tend to zero. However, the effect of the boundary is included in the control, even for arbitrarily small values of c_b, d_b, and provides the geometric property of Theorem 5. Thus, although in realistic situations c_b, d_b have larger values, the analysis presented is still important since it provides the geometric result taking into account the workspace boundary, albeit in the limiting case when c_b, d_b tend to zero.

V. Simulations

To support the results of the previous paragraphs, we provide a series of computer simulations.

In the first simulation, nine agents navigate under the control law (2). Screenshots I-III in Figure 1 show the evolution of the closed-loop system in time. The agents are located at their initial positions in the first screenshot. Collision avoidance is fulfilled, due to the proposed control design. The agents disperse in the workspace and eventually stop in screenshot III. Screenshot IV depicts the final positions of the swarm members. Each agent occupies a disc of radius $d/2$. These discs are visualized in the last screenshot by the large discs whose center is the corresponding agent. By virtue of Theorem 2, the large discs are disjoint.

In the second simulation of Figure 2, agents navigate under the control (4). The workspace radius is $R_W = 18d$. The agents start from an initial condition where they are aggregated near the workspace center. Some agents approach the workspace boundary and are forced to remain within it due to the existence of the repulsive potential on the boundary. The workspace is large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state, i.e., the set

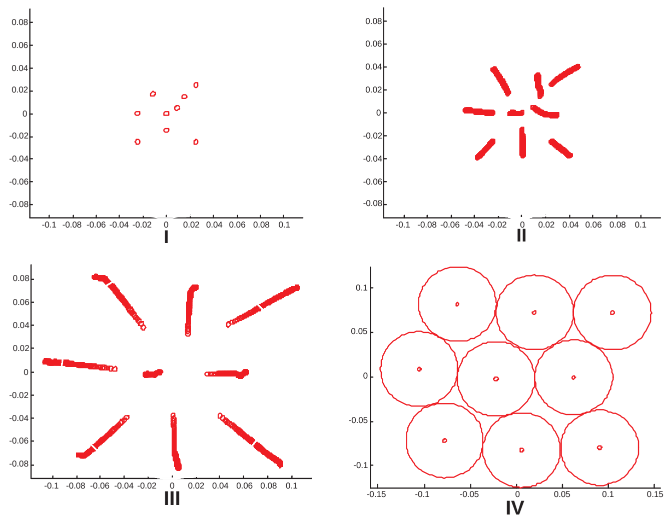

Fig. 1. Swarm dispersion for nine single integrator agents. The agents disperse in the workspace and eventually occupy nine disjoint discs of radius $d/2$, one for each agent.

B of Theorem 5 is nonempty. This is depicted in the last screenshot of Figure 2. By reducing the

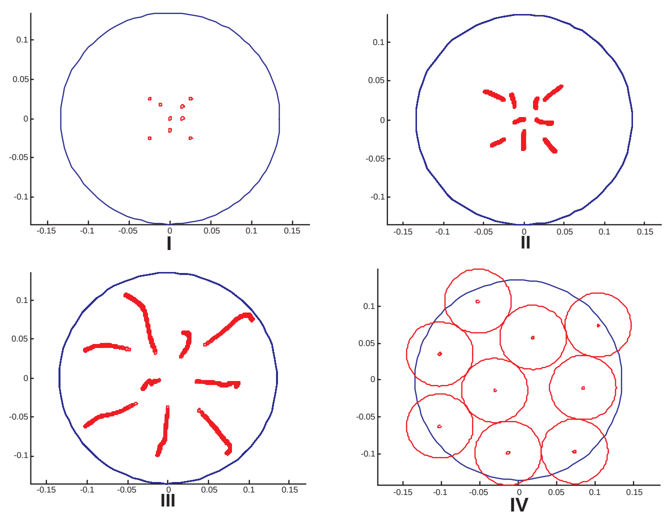

Fig. 2. Swarm dispersion for nine single integrator agents in a bounded workspace. The workspace is large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state. Agents are forced to remain within the workspace boundary.

workspace radius of the previous example, the set B of Theorem 5 is rendered empty, i.e. there does not exist a configuration in $\text{int}(W)$ such that the condition $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$

is fulfilled. This is the case in Figure 3, where we have $R_W = 16d$. The agents disperse again within the limits of the workspace, avoiding collisions with each other. In the last screenshot, some of the circles of radius $d/2$ surrounding the agents overlap, since the set B is now empty.

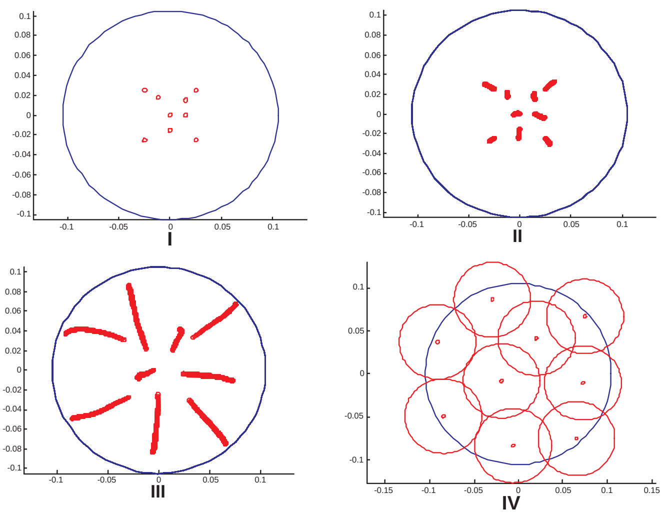

Fig. 3. Swarm dispersion for nine single integrator agents in a bounded workspace. The workspace is not large enough to allow the agents to occupy nine disjoint discs of radius $d/2$ at steady state. These discs are overlapping in screenshot IV.

VI. Conclusions

We proposed a distributed inverse agreement control strategy for multiple kinematic agents that forces the agents to disperse. Both the cases of an unbounded and a circular, bounded workspace were considered. In the first case, we showed that the closed-loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. This lower bound was proven to coincide with the agents' sensing radius. In the case of a circular workspace, the control law was redefined to force the agents to remain within the workspace boundary throughout the closed-loop system evolution. Moreover the control design guaranteed collision avoidance. The results were supported through computer simulations.

Current research involves exploring the relation of the sensing radius, the number of agents and the radius of the workspace with the emptiness of the set B. Considering arbitrary convex boundaries and not only the circular ones treated here is another current research endeavor. The results should also be extended to tackle with dynamic and nonholonomic constraints.

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